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WEAK COMPATIBLE MAPPINGS OF TYPE (A) IN METRIC SPACES

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Abstract: We prove a common fixed point theorem for two pair of weak compatible mappings of type (A) in a metric space. Presented work generalizes the results due to Fisher [1], Jungck [3] and Lohani and Badshah [8].

Keywords: Compatible mappings, weak compatible mappings of type (A).

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1. Introduction

In 1976, Jungck [2] proved a common fixed point theorem for commuting maps by generalizing the Banach's fixed point theorem, which states that, 'Let (X, d) be a complete metric space. If T satisfies $d(Tx, Ty) \leq kd(x, y)$ for each $x, y \in X$ where $0 \leq k < 1$, then T has a unique fixed point in X'. This result was further generalized and extended in various directions by different authors. On the other hand, Sessa [10] defined weak commutativity and proved common fixed point theorem for weakly commuting maps. Further, Jungck [3] introduced more generalized commutativity, the so-called compatibility, which is more general than that of weak commutativity. Since then, various fixed point theorems for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings, have been obtained by many authors. It has been known from the paper of Kannan [6] that there exists maps that have a discontinuity in the domain but which have fixed points, moreover, the maps involved in every case were continuous at the fixed

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point.

Further Jungck-Murthy and Cho [5] introduced the concept of compatible mappings of type (A) in metric space and improved the results of various authors. We use the idea of weak compatible mappings of type (A) in metric space as used by Pathak-Kang-Baek [9] in Menger and 2-metric spaces respectively which is equivalent to concept of compatible and compatible mappings of type (A) under some conditions. In this paper, we prove a common fixed point theorem which generalizes the result of Fisher [1], Jungck [2] and Lohani and Badshah [8].

2. Preliminaries

Now we give preliminaries and basic definitions which are used throughout the paper.

Definition 2.1. [3] The pair (A, S) is said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 2.2. [5] The pair (A, S) is said to be compatible of type (A) if $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$ and $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} AX_n = t$ for some in X.

Definition 2.3. [7] The pair (A, S) is said to be weak compatible of type (A) if $\lim_{n\to\infty} d(ASx_n, SSx_n) \leq \lim_{n\to\infty} d(SAx_n, SSx_n)$ and $\lim_{n\to\infty} d(SAx_n, AAx_n) \leq \lim_{n\to\infty} d(ASx_n, AAx_n)$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$ for some $t \in X$.

Proposition 2.4. [7] Let S and T be weak compatible of mappings of type (A) from a metric space (X, d) into itself. If one of S and T is continuous, then S and T are compatible.

Proposition 2.5. [7] Let S and T be continuous mappings from a metric space (X, d) into itself. Then

i) S and T are compatible of type (A) if and only if they are weak compatible of type (A)

ii) S and T are compatible if and only if they are weak compatible of type (A)

3. Main results

In 1998, Lohani and Badshah [7] proved the following:

Theorem 3.1. *Let A, B, S and T be self mappings from a metric space (X, d) into itself satisfying*

(3.1) $A(X) \subset T(X)$, $B(X) \subset S(X)$

(3.2) $$d(Ax, By) \leq \alpha \frac{d(Ty, By)[1 + d(Sx, Ax)]}{[1 + d(Sx, Qy)]} + \beta[d(Sx, Ax) + d(Ty, By)]$$

$$+ \gamma[d(Sx, By) + d(Ty, Ax)] \text{ for all } x, y \text{ in } X,$$

where $\alpha, \beta, \gamma, \geq 0$, $0 \leq \alpha + 2\beta + 2\gamma < 1$.

(3.3) One of A, B S, and T is continuous

(3.4) Pairs (A, S) and (B, T) are compatible on X.

Then A, B S, and T have a unique common fixed point in X.

Now for any arbitrary point x_0 in X, by (3.1), there exists a point $x_1 \in X$ such that $Tx_1 = Ax_0$ and for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

(3.5) $$y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = S_{2n+2} = Bx_{2n+1} \text{ for } n = 0, 1, 2, \ldots .$$

To prove our main results we need the following Lemmas:

Lemma 3.1 *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the condition (3.1) and (3.2) and (3.5). Then the sequence* $\{y_n\}$ *defined by (3.5) is a Cauchy sequence in X*

Proof: From (3.2), we have

$$d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1})$$

$$\leq \alpha \frac{d(Tx_{2n+1}, Bx_{2n+1})[1 + d(Sx_{2n}, Ax_{2n})]}{[1 + d(Sx_{2n}, Tx_{2n+1})]}$$

$$+ \beta [d(Sx_{2n}, Ax_{2n}) + d(Tx_{2n+1}, Bx_{2n+1})]$$

$$+ \gamma[d(Sx_{2n}, Bx_{2n+1}) + d(Tx_{2n+1}, Ax_{2n})]$$

$$= \alpha \frac{d(y_{2n}, y_{2n+1})[1 + d(y_{2n-1}, y_{2n})]}{[1 + d(y_{2n-1}, y_{2n})]} + \beta[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]$$

$$+ \gamma[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})]$$

$$= \alpha\, d(y_{2n}, y_{2n+1}) + \beta d(y_{2n-1}, y_{2n}) + \beta d(y_{2n}, y_{2n+1}) + \gamma d(y_{2n-1}, y_{2n})$$

$+ \gamma d(y_{2n}, y_{2n+1})$

$$d(y_{2n}, y_{2n+1}) \le \frac{(\beta+\gamma)d(y_{2n}, y_{2n-1})}{1-\alpha-\beta-\gamma}$$

$$d(y_{2n}, y_{2n+1}) \le h\, d(y_{2n}, y_{2n-1}) \text{ where } h = \frac{\beta+\gamma}{1-\alpha-\beta-\gamma} < 1.$$

In general, $d(y_n, y_{n+1}) \le h\, d(y_{n-1}, y_n) \le \dots \le h^n d(y_0, y_1)$

For each positive integer p, we get

$$d(y_n, y_{n+p}) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots + d(y_{n+p-1}, y_{n+p})$$

$$\le (1 + h + h^2 + \dots + h^{p-1})\, d(y_n, y_{n+1})$$

$$d(y_n, y_{n+p}) \le \frac{h^{n+p}}{1-h} d(y_0, y_1)$$

Letting $n\to\infty$, we have $d(y_n, y_{n+p})\to 0$. Therefore, $\{y_n\}$ is a Cauchy sequence in X.

Lemma 3.2 *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the condition (3.1) and (3.2) and (3.5).*

(3.6) $S(X) \cap T(X)$ is a complete subspace of X. Then pairs (A, S) and (B, T) have a coincidence point in X.

Proof. By Lemma 3.1, the sequence $\{y_n\}$ defined by (3.5) is a Cauchy sequence in $S(X)\cap T(X)$. Since $S(X)\cap T(X)$ is a complete subspace of X, so $\{y_n\}$ converges to a point w, (say) in $S(X)\cap T(X)$. On the other hand, since the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of $\{y_n\}$ are also Cauchy sequence in $S(X)\cap T(X)$, so they also converges to the same limit w. Hence there exist two points u, v in X such that Su = w and Tv = w, respectively.

By (3.2), we have

$$d(Su, y_{2n+1}) = d(Au, Bx_{2n+1})$$

$$\le \alpha \frac{d(Tx_{2n+1}, Bx_{2n+1})[1 + d(Su, Au)]}{[1 + d(Su, Tx_{2n+1})]}$$

$$+ \beta[d(Su, Au) + d(Tx_{2n+1}, Bx_{2n+1})]$$

$$+ \gamma[d(Su, Bx_{2n+1}) + d(Tx_{2n+1}, Au)]$$

Letting limit as $n\to\infty$, we have

$$d(Au, w) \le \beta d(Su, Au) + \gamma[d(Su, w) + d(w, Au)]$$

$$= \beta d(Au, w) + \gamma d(w, Au)$$

$= (\beta + \gamma)\, d(Au, w)$ a contradiction. Hence Au = w = Su.

Similarly, we can show that v is also coincidence point of B and T.

Lemma 3.3 *Let A and S be weak compatible mappings of type (A) from a metric space (X, d) into itself, If Au = Su for some u $\in$ X, then ASu = AAu = SSu = SAu.*

Proof. Let $\{x_n\}$ be a sequence in X defined by $x_n = u$, n = 1, 2,…, and Au = Su. Now, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = Su = Au.$$

Since A and S are weak compatible mapping of type (A), we have

$$d(ASu, SSu) = \lim_{n\to\infty} d(ASx_n, SSx_n) \le \lim_{n\to\infty} d(SAx_n, SSx_n) = 0.$$

Hence ASu = SSu.

Therefore, ASu = AAu = SSu = SAu. This completes the proof.

Now, we generalize Theorem 3.1 by using weak compatible mappings of type (A).

Theorem 3.2 *Let A, B, S and T be mapping from a metric space (X, d) into itself satisfying the conditions (3.1), (3.2), (3.3), (3.4) and (3.5). The pairs (A, S) and (B, T) are weak compatible of type (A). Then A, B, S and T have a unique common fixed point in X .*

Proof. Now, by Lemma 3.2, the pairs (A, S) and (B, T) have a coincidence point in X, i.e., there exist two points u, v in X such that Au = Su = w and Bv = Tv = w, respectively. Since A and S are weak compatible of type (A), then by Lemma 3.3, ASu = AAu = SAu, which implies that Aw = Sw. Similarly, we have Bw = Tw. Now, we prove that Aw = w. If Aw $\neq$ w, then by (3.2), we have

$$d(Aw, y_{2n+1}) = d(Aw, Bx_{2n+1}) \le \alpha \frac{d(Tx_{2n+1}, Bx_{2n+1})[1 + d(Sw, Aw)]}{[1 + d(Sw, Tx_{2n+1})]}$$

$$+ \beta[d(Sw, Aw) + d(Tx_{2n+1}, Bx_{2n+1})]$$

$$+ \gamma[d(Sw, Bx_{2n+1}) + d(Tx_{2n+1}, Aw)]$$

Proceeding limit as $n\to\infty$, we have

$$d(Aw, w) \le \gamma[d(Sw, w) + d(w, Aw)]$$

$$= 2\gamma\, d(Aw, w), \text{ a contradiction.}$$

Hence Aw = w = Sw.

Similarly, we have Bw = w = Tw. This mean that w is a common fixed point of A, B, S and T.

Uniqueness. Let w′ and z two different common fixed point of A, B, S and T. Then we have

$$d(w', z) = d(Aw', Bz) = \alpha \frac{d(Tz, Bz)[1 + d(Sw', Aw')]}{[1 + d(Sw', Tz)]}$$

$$+ \beta[d(Sw', Aw') + d(Tz, Bz)]$$

$$+ \gamma[d(Sw', Bz) + d(Tz, Aw')]$$

$= 2\gamma\ d(w', z)$, a contradiction. Hence $w' = z$. This proves uniqueness.

The following example illustrates our Theorem.

Example 3.1 Let X = [0, 1], with Euclidean metric d, i.e.,

$$d(x, y) = |x - y|.$$

Define maps A, B, S, T: $X \to X$ as following:

$$Ax = 0,\ Bx = x/32,\ Sx = x,\ Tx = x \text{ for all } x \in X.$$

Then we observe that $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$. Let us consider sequence $\{x_n\}$ be in [0, 1] such that $x_n \to 0$ as $n \to \infty$.

$BTx_n = x_n/32$, $TBx_n = x_n/32$, $BBx_n = x_n/(32)^2$ and $TTx_n = x_n$.

Also, we have

$$\lim_{n\to\infty} (BTx_n, TTx_n) \le \lim_{n\to\infty} d(TBx_n, TTx_n)$$

$\lim_{n\to\infty} (TBx_n, BBx_n) \le \lim_{n\to\infty} d(BTx_n, BBx_n)$ and $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = 0$, $0 \in X$.

Hence pair (B, T) is weak compatible of type (A). Similarly, we can show that pair (A, S) is weak compatible of type (A).

Now

$$d(Ax, By) = |y/32| \le [x/2 + 59y/384 + 31xy/192] \text{ for all } x, y \in X.$$

$$\le 1/6 \left[31/32 \frac{|y|(1 + |x|)}{1 + |x - y|}\right] + 1/6\,[|x| + 31/32\,|y|]$$

$$+ 1/12\,[|x - y/32| + |y|]$$

$$= 1/6 \frac{d(Ty, By)[1 + d(Sx, Ax)]}{[1 + d(Sx, Ty)]} + 1/6\,[d(Sx, Ax)\,d(Ty, By)]$$

$$+ 1/12\,[d(Sx, By) + d(Ty, Ax)]$$

Hence all the assumptions of the theorem are satisfied with $\alpha = 1/6$, $\beta = 1/6$, $\gamma = 1/12$ and zero is unique common fixed point of A, B, S and T.

Conflict of Interests

The author declares that there is no conflict of interests.

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