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INTEGRAL INEQUALITY OF GRONWALL TYPE WITH AN APPLICATION

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Abstract: The aim of the present paper is to obtain some new proofs of integral inequalities of Gronwall type in two independent variables which provide explicit bounds on unknown functions. The inequalities given here can be used as tools in the qualitative theory of certain partial differential equations.

Keywords: Integral inequalities, two independent variables, partial differential equations.

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1. Introduction

The differential and integral inequalities occupy a privileged position in the theory of differential and integral equations. In recent years, these inequalities have been greatly enriched by the recognition of their potential and intrinsic worth in many applications of the applied sciences. Since the appearance of Gronwalls fundamental paper[13] in 1919, an enormous amount of effort has been devoted to the discovery of new type of inequalities and to the application of inequalities in many parts of analysis. Integral inequalities were motivated by certain applications in the theory of partial differential and integral equations [1-12] of two independent variables.

2. Main Results

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In what follows, R denotes the set of real numbers and $R_+ = [0,\infty]$ is the subset of R . $I_1 = [x_0, x)$, $I_2 = [y_0, y)$, are the given subsets of R_+, and $D = I_1 \times I_2$ .The first order partial derivatives of a function

$z(x, y)$ defined for $x, y \in R$ with respect to x and y are denoted by $z_x(x, y)$ and $z_y(x, y)$ respectively. Throughout in this paper we use summation convention ,all the functions and their partial derivatives appear in the inequalities are assumed to be real valued and all the integrals involved are of positive values and exist on the respective domains of their definitions.

Theorem 2.1: Let $u(x, y)$ and $g(x, y)$ be nonnegative real valued continuous functions defined for D such that $x_0 \le s \le x$, $y_0 \le t \le y$. Suppose that $H(x, y) > 1$ and $H_x(x, y)$, $H_y(x, y)$ and $H_{xy}(x, y)$ be nonnegative and continuous functions defined for $x, y \in R$. If

$$u(x, y) \le H(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)u^p(s,t)dtds, \tag{2.1}$$

for $x, y, x_0, y_0 \in R_+$, $0 < p < 1$, $p + q = 1, q > 0$,then

$$u(x, y) \le E_0(x, y) + \left[(1-p)\left[H(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)dtds\right]\right]^{\frac{1}{1-p}} \tag{2.2}$$

where
$$E_0(x, y) = H(x_0, y)\left[1 - qH^{\frac{p}{q}}(x_0, y)\right] + H(x, y_0)\left[1 - qH^{\frac{p}{q}}(x, y_0)\right] - H(x_0, y_0)\left[1 - qH^{\frac{p}{q}}(x_0, y_0)\right], \tag{2.3}$$

Proof: Define a function $z(x, y)$ by the right-hand side of (2.1).Then

$$z(x, y) = H(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)u^p(s,t)dtds, \tag{2.4}$$

where $z(x_0, y) = H(x_0, y), \quad z(x, y_0) = H(x, y_0), \quad z(x_0, y_0) = H(x_0, y_0)$ (2.5)

Then $z(x,y)>1$.From (2.1) and (2.4),we have

$$u(x,y)\le z(x,y),\quad \text{for } x,y\in D \tag{2.6}$$

Differentiating (2.4) with respect to x and y, we get

$$z_{xy}(x,y)=H_{xy}(x,y)+g(x,y)u^{p}(x,y) \tag{2.7}$$

Using (2.6) in (2,7) and since $H(x,y)>1$, then the above equation can be written in the form

$$\frac{z_{xy}(x,y)}{z^{p}(x,y)}\le H_{xy}(x,y)+g(x,y) \tag{2.8}$$

Keeping x fixed in (2.8),set $y=t$ and then integrating with respect to t from y_0 to y ,we have

$$\frac{z_{x}(x,y)}{z^{p}(x,y)}-\frac{z_{x}(x,y_0)}{z^{p}(x,y_0)}\le H_{x}(x,y)-H_{x}(x,y_0)+\int_{y_0}^{y}g(x,t)dt \tag{2.9}$$

Now keeping y fixed in (2.9),set $x=s$ and then integrating with respect to s from x_0 to x ,we get

$$\frac{1}{1-p}\left[z^{1-p}(x,y)-z^{1-p}(x_0,y)-z^{1-p}(x,y_0)+z^{1-p}(x_0,y_0)\right]\le$$

$$H(x,y)-H(x_0,y)-H(x,y_0)+H(x_0,y_0)+\int_{x_0}^{x}\int_{y_0}^{y}g(s,t)dtds \tag{2.10}$$

By using (2.5) in (2.10) and from (2.10),we observe that

$$z(x,y)\le H(x_0,y)+H(x,y_0)-H(x_0,y_0)+(1-p)\left[-H^{\frac{1}{1-p}}(x_0,y)-H^{\frac{1}{1-p}}(x,y_0)+H^{\frac{1}{1-p}}(x_0,y_0)\right]$$

$$+\left[(1-p)\left[H(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}g(s,t)dtds\right]\right]^{\frac{1}{1-p}}$$

Which can be rewritten as

$$z(x,y)\le H(x_0,y)\left[1-qH^{\frac{p}{q}}(x_0,y)\right]+H(x,y_0)\left[1-qH^{\frac{p}{q}}(x,y_0)\right]$$

$$-H(x_0,y_0)\left[1-qH^{\frac{p}{q}}(x_0,y_0)\right]+\left[(1-p)\left[H(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}g(s,t)dtds\right]\right]^{\frac{1}{p-1}}$$

Or
$$z(x,y) \le E_0(x,y) + \left[(1-p)\left[H(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)dtds\right]\right]^{\frac{1}{p-1}} \tag{2.12}$$

From (2.6) and (2.12).we get

$$u(x,y) \le E_0(x,y) + \left[(1-p)\left[H(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)dtds\right]\right]^{\frac{1}{1-p}}$$

Where $E_0(x,y)$ is defined as (2.3).

Theorem 2.2: Let $u(x,y)$ and $g(x,y)$ be nonnegative real valued continuous functions defined for D such that $x_0 \le s \le x$, $y_0 \le t \le y$. Suppose that $H(x,y) > 1$ and $H_x(x,y)$, $H_y(x,y)$ and $H_{xy}(x,y)$ be nonnegative and continuous functions defined for $x, y \in R$. If

$$u^p(x,y) \le H(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)u(s,t)dtds, \tag{2.13}$$

for $x, y, x_0, y_0 \in R_+$, $p > 1, p + q = 1, q > 0$,then

$$u(x,y) \le E_1^{\frac{1}{p}}(x,y) + \left[\frac{(p-1)}{p}\left[H(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)dtds\right]\right]^{\frac{1}{p-1}} \tag{2.14}$$

where
$$\begin{aligned} E_1(x,y) = H(x_0,y)\left[1 + \frac{q}{p}H^{-\frac{1}{q}}(x_0,y)\right] + H(x,y_0)\left[1 + \frac{q}{p}H^{-\frac{1}{q}}(x,y_0)\right] \\ - H(x_0,y_0)\left[1 + \frac{q}{p}H^{-\frac{1}{q}}(x_0,y_0)\right], \end{aligned} \tag{2.15}$$

Proof: Define a function $z(x,y)$ by the right-hand side of (2.13).Then

$$z(x,y) = H(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)u(s,t)dtds, \tag{2.16}$$

where $\quad z(x_0, y) = H(x_0, y), \qquad z(x, y_0) = H(x, y_0), \qquad z(x_0, y_0) = H(x_0, y_0) \qquad (2.17)$

Then $z(x, y) > 1$.From (2.13) and (2.16),we have

$$u^p(x, y) \le z(x, y) \Rightarrow u(x, y) = z^{\frac{1}{p}}(x, y) \quad \text{for } x, y \in D \tag{2.18}$$

Differentiating (2.16) with respect to x and y, we get

$$z_{xy}(x, y) = H_{xy}(x, y) + g(x, y)u(x, y) \tag{2.19}$$

Using (2.18) in (2,19) and since $H(x, y) > 1$, then the above equation can be written in the form

$$\frac{z_{xy}(x, y)}{z^{\frac{1}{p}}(x, y)} \le H_{xy}(x, y) + g(x, y) \tag{2.20}$$

Keeping x fixed in (2.8),set $y = t$ and then integrating with respect to t from y_0 to y ,we have

$$\frac{z_x(x, y)}{z^{\frac{1}{p}}(x, y)} - \frac{z_x(x, y_0)}{z^{\frac{1}{p}}(x, y_0)} \le H_x(x, y) - H_x(x, y_0) + \int_{y_0}^{y} g(x, t)dt \tag{2.21}$$

Now keeping y fixed in (2.21),set $x = s$ and then integrating with respect to s from x_0 to x ,we get

$$\frac{p}{p-1}\left[z^{\frac{p-1}{p}}(x, y) - z^{\frac{p-1}{p}}(x_0, y) - z^{\frac{p-1}{p}}(x, y_0) + z^{\frac{p-1}{p}}(x_0, y_0)\right] \le$$

$$H(x, y) - H(x_0, y) - H(x, y_0) + H(x_0, y_0) + \int_{x_0}^{x}\int_{y_0}^{y} g(s, t)dtds \tag{2.22}$$

By using (2.17) in (2.22) and from (2.22),we observe that

$$z(x, y) \le H(x_0, y) + H(x, y_0) - H(x_0, y_0) - \frac{q}{p}\left[-H^{-\frac{p}{q}}(x_0, y) - H^{-\frac{p}{q}}(x, y_0) + H^{-\frac{p}{q}}(x_0, y_0)\right]$$

$$+\left[(\frac{p-1}{p})\left[H(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s, t)dtds\right]\right]^{\frac{p}{p-1}}$$

Which can be rewritten as

$$z(x,y) \le H(x_0,y)\left[1+\frac{q}{p}H^{-\frac{1}{q}}(x_0,y)\right]+H(x,y_0)\left[1+\frac{q}{p}H^{-\frac{1}{q}}(x,y_0)\right]$$

$$-H(x_0,y_0)\left[1+\frac{q}{p}H^{-\frac{1}{q}}(x_0,y_0)\right]+\left[(\frac{p-1}{p})\left[H(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}g(s,t)dtds\right]\right]^{\frac{p}{p-1}} \qquad (2.23)$$

From (2.18) and (2.23).we get

$$u(x,y) \le E_1^{\frac{1}{p}}(x,y)+\left[(\frac{p-1}{p})\left[H(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}g(s,t)dtds\right]\right]^{\frac{1}{p-1}}$$

Where $E_1(x,y)$ is defined as (2.15).

Theorem 2.3: Let $u(x,y)$ and $g(x,y)$ be nonnegative real valued continuous functions defined for D such that $x_0 \le s \le x$, $y_0 \le t \le y$. Suppose that $H(x,y) > 1$ and $H_x(x,y)$, $H_y(x,y)$ and $H_{xy}(x,y)$ be nonnegative and continuous functions defined for $x,y \in R$. If

$$u^p(x,y) \le H(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}g(s,t)u^p(s,t)dtds, \qquad (2.24)$$

for $x,y,x_0,y_0 \in R_+$, $p>1$, $p+q=1$,then

$$u^p(x,y) \le \frac{H(x_0,y)H(x,y_0)}{H(x_0,y_0)}\exp\left[H(x,y)-H(x_0,y)-H(x,y_0)+H(x_0,y_0)+\int_{x_0}^{x}\int_{y_0}^{y}g(s,t)dtds\right]$$

(2.25)

Proof: The proof of Theorem 2.3 is the same as proof of Theorem 2.1 with suitable modifications.

3. Application

As an application,we obtain the bound on the solution of a nonlinear partial differential equation

$$u_{xy}(x,y) = f(x,y,u(x,y)) \qquad (2.26)$$

with the given boundary conditions

$$u(x,y_0)=a_1(x)\ ,u(x_0,y)=a_2(y),\ \ a_1(x_0)=a_2(y_0)=0 \qquad (2.27)$$

where $u \in C[D, R]$, $f \in C[D \times R, R]$, and $D = \{(x, y), x \geq 0, y \geq 0\}$ such that

$$|f(x, y, u)| \leq g(x, y)\left|u^{p}\right| \tag{2.28}$$

where $0 < p < 1, p + q = 1$ and $g(x, y)$ is nonnegative continuous real valued functions defined on a domain D.The equation (2.26) with (2.27) is equivalent to the integral equation

$$u(x, y) = a_1(x) + a_2(y) + \int_{x_0}^{x} \int_{y_0}^{y} f(s, t, u(s, t)) dt ds \tag{2.29}$$

Let $u(x, y)$ be any solution of (2.26) with (2.27),we get

$$|u(x,y)| = |a_1(x) + a_2(y)| + \int_{x_0}^{x} \int_{y_0}^{y} |f(s, t, u(s, t)| dt ds,$$

Using (2.28) in (2.29) and assuming that $|a_1(x)| + |a_2(y)| \leq H(x, y)$, where $H(x, y) > 1$, we have

$$|u(x,y)| \leq H(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} g(s, t)\left|u^{p}(s, t)\right| dt ds, \tag{2.30}$$

The remaining proof will be the same as the proof of Theorem 2.1 with suitable modifications. We note that Theorem 2.1 can be used to study the stability, boundedness and continuous dependence of the solutions of (2.26).

Conflict of Interests

The author declares that there is no conflict of interests.

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