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NUMERICAL ANALYSIS VIA CHEBYSHEV PSEUDOSPECTRAL METHOD FOR NONLINEAR INITIAL/BOUNDARY VALUE PROBLEMS

NADER Y. ABD ELAZEM, ABDELHALIM EBAID*

Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia

Abstract. In applied science, the physical models are usually described by nonlinear initial/boundary value problems. The exact solutions for such nonlinear models are not always available, the reason that many authors resort to the numerical methods. One of these numerical methods is the Chebyshev pseudospectral method. This method is applied in the current paper to solve some nonlinear initial and boundary value problems of particular interest in applied sciences and engineering. In order to explore the effectiveness and the validity of the present method, many physical models of nonlinear type such as generalized nonlinear oscillator, relativistic oscillator, and Bratu's equations have been solved numerically. The obtained results are compared with other published works through tables and graphs where good accuracy has been achieved.

Keywords: Chebyshev collocation method; Nonlinear oscillator; Initial and boundary value problems; Bratu's problem.

2010 AMS Subject Classification: 35G31, 65L05.

1. Introduction

Physical models in applied mathematics and engineering sciences are usually formulated as nonlinear initial or boundary value problems. The exact solutions of such models can not be obtained in the most cases, especially, when the considered model is of complex nonlinearity.

*Corresponding author

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Because of the difficulty of obtaining the exact solution for nonlinear differential equations with complex nonlinearities, many mathematicians resort to one of the efficient numerical methods. Chebyshev have proven successfully in the numerical solution of various boundary value problems [1,2] and in computational fluid problems [3,4]. The spectral method distinguishes itself from the finite-difference and finite-element methods by the fact that global information is incorporated in computing a spatial derivative. The spectral method can yield greater accuracy for a smooth solution with far fewer nodes and therefore less computational time than the finite-difference and finite-element schemes [5].

Chebyshev pseudospectral methods are widely used in the numerical approximation of many types of ordinary and partial differential equations which arise from the engineering problems [6-9]. Therefore, when many decimal places of accuracy are needed, the contest between pseudospectral algorithms and finite difference is not an even battle but a rout: pseudospectral methods win hands-down. Moreover, engineers and mathematicians who need accurate many decimal places have always preferred spectral methods [10]. Elbarbary and El-Sayed [11-13] has recently introduced a new pseudospectral differentiation matrix to decrease the round off error, specially on increasing N (the number of degrees of freedom) or number of equations.

Hence, on solving nonlinear equations in the present results the error becomes nearly zero [11-13]. Hua Chen et al. [14] discussed notes on a conservative nonlinear oscillator. Ebaid [15] investigated analytical periodic solution to a generalized nonlinear oscillator: application of He's frequency-amplitude formulation. [16] studied He's frequency formulation for the relativistic harmonic oscillator. He's frequency–amplitude formulation for the Duffing harmonic oscillator was investigated by Fan [17]. Fang Liu [18] studied He's variational approach for nonlinear oscillators with high nonlinearity. Zhao [19] studied He's frequency amplitude formulation for nonlinear oscillators with an irrational force. Li Zhang [20] investigated periodic solutions for some strongly nonlinear oscillations by He's energy balance method. Shawagfeh [21] studied analytic approximate solution for a nonlinear oscillator equation. Moreover, modification of Lesnic's approach and new analytic solutions for some nonlinear second-order boundary value problems with Dirichlet boundary conditions was investigated by Ebaid [22].

Wazwaz [23] discussed A domian decomposition method for a reliable treatment of the Bratu-type equations. An efficient computational method for second order boundary value problems was studied by Zhou et al. [24].

The purpose of this work is to study the applicability of the Chebyshev collocation method to solve this kind of issues; see [10, 15, 16, 18, 21, 22, 23]. The absolute errors of the present results are compared with those obtained by other methods. The nonlinear oscillator with both initial and boundary conditions is investigated in this paper utilizing Chebyshev collocation method. It is hoped that the results obtained will not only provided useful information for applications but also serve as a complement to the previous studies.

2. Analysis

A numerical solution based on Chebyshev collocation approximations seems to be a very good choice in many practical problems (as described in the literature review and for example (Canuto et al. [3] and Peyret [7]). Accordingly, Chebyshev collocation method will be applied for the presented model. The derivatives of the function $f(x)$ at the Gauss-Lobatto points, $x_k = \cos\left(\frac{k\pi}{L}\right)$, which are the linear combination of the values of the function $f(x)$ [13]

$$f^{(n)} = D^{(n)} f,$$

where,

$$\underline{f} = [f(x_0), f(x_1), ..., f(x_L)]^T,$$

$$\underline{f}^{(n)} = [f^{(n)}(x_0), f^{(n)}(x_1), ..., f^{(n)}(x_L)]^T,$$

$$D^{(n)} = [d_{k,j}^{(n)}],$$

or

$$f^{(n)}(x_k) = \sum_{j=0}^{L} d_{k,j}^{(n)} f(x_j),$$

where,

$$d_{k,j}^{(n)} = \frac{2\gamma_j^*}{L} \sum_{l=n}^{L} \sum_{\substack{m=0 \\ (m+l-n) even}}^{l-n} \gamma_l^* a_{m,l}^n \, (-1)^{[\frac{lj}{L}]+[\frac{mk}{L}]} x_{lj-L[\frac{lj}{L}]} x_{mk-L[\frac{mk}{L}]},$$

$$a_{m,l}^n = \frac{2^n l}{(n-1)! c_m} \frac{(s-m+n-1)! \, (s+n-1)!}{(s)! \, (s-m)!},$$

such that $2s = l+m-n$ and $c_0 = 2, c_i = 1, i \geq 1$,where $k,\ j = 0,1,2,...,L$ and $\gamma_0^* = \gamma_l^* = \frac{1}{2},\ \gamma_j^* = 1$ for $j = 1,2,3,...,L-1$. The round off errors incurred during computing differentiation matrices $D^{(n)}$ are investigated in [13].

3. Applications

In this section, the proposed method, ChCM [25-30] is applied to solve various nonlinear initial and boundary value problems that have been analyzed by the authors [10, 15, 16, 18, 21, 22, 23]. The grid points (x_i, x_j) in this situation are given as $x_i = \cos\left(\frac{i\pi}{L_1}\right)$, $x_j = \cos\left(\frac{j\pi}{L_2}\right)$ for $i = 1,...,L_1-1$ and $j = 1,...,L_2-1$.

The domain in the x-direction is $[0, x_{\max}]$ where $x_{\max}$ is the length of the dimensionless axial coordinate and the domain in the η-direction is $[0, \eta_{\max}]$ where $\eta_{\max}$ corresponds to η_∞. The domain $[0, x_{\max}] \times [0, \eta_{\max}]$ is mapped into the computational domain $[0, x_{\max}] \times [-1, 1]$. Chebyshev pseudospectral method will be introduced for the following nonlinear initial and boundary value problems. The computer programs of the numerical method was executed in Mathematica. The numerical results of the considered models are discussed in section 4.

3.1. *Nonlinear oscillator.*

The general nonlinear oscillator was introduced by Fang Liu [18] in the form:

$$u'' + f(u) = 0, \tag{1}$$

where,

$$f(u) = u + a\,u^3 + b\,u^5 + c\,u^7,$$

with the initial conditions:

$$u(0) = A,\ u'(0) = 0. \tag{2}$$

The approximate solution can be readily obtained by Fang Liu [18]:

$$u(\eta) = A\ Cos\left(\sqrt{1 + \frac{3}{4}a\,A^2 + \frac{5}{8}b\,A^4 + \frac{35}{64}c\,A^6}\,\eta\right). \tag{3}$$

The nonlinear oscillator equation (1), with initial conditions (2) are approximated by using Chebyshev collocation method and the equation (1) is transformed into the following equation:

$$\left(\frac{2}{\eta_{\max}}\right)^2\left(\sum_{l=0}^{L^*} d_{j,l}^{(2)} u_l\right)+u_j+a\, u_j^3+b\, u_j^5+c\, u_j^7=0, \tag{4}$$

This system of equations for the unknowns u_j where $j=1(1)L^*$ (take $L^*=32$) is solved by Newton-Raphson iteration technique [30]. This technique shall be also used for next problems.

3.2. *The relativistic oscillator.*

The nonlinear differential equation of a relativistic oscillator was discussed by Chu Cai and Ying Wu [16] in the form:

$$\frac{d^2u}{d^2\eta}+\frac{u}{\sqrt{1+u^2}}=0, \tag{5}$$

with initial conditions

$$u(0)=B,\ u'(0)=0, \tag{6}$$

where the approximate solution was obtained by Chu Cai and Ying Wu [16] as

$$u(\eta)=A\ Cos(\sqrt{\frac{1}{\sqrt{1+\frac{3}{4}\,B^2}}}\eta). \tag{6}$$

As described in the previous problem, equation (5) is transformed into the following equations:

$$\left(\frac{2}{\eta_{\max}}\right)^2\left(\sum_{l=0}^{L^*} d_{j,l}^{(2)} u_l\right)+\frac{u_j}{\sqrt{1+u_j^2}}=0. \tag{7}$$

3.3. *The generalized nonlinear oscillator.*

The nonlinear oscillator was generalized by Ebaid [15] in the form:

$$u''+\Omega\, u^p+\frac{\lambda\, u^q}{\mu+\sigma\, u^r}=0, \tag{8}$$

under the initial conditions

$$u(0)=A,\ u'(0)=0. \tag{9}$$

The approximate solution was obtained by Ebaid [15] as

$$u(\eta) = A\ Cos\ (\omega\ \eta), \tag{11}$$

where,

$$\omega^2 = \frac{\frac{\mu\ \Omega\ A^{p-1}\ \Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p\ +3}{2})} + \frac{\sigma\ \Omega\ A^{p+r-1}\ \Gamma(\frac{p+r}{2}+1)}{\Gamma(\frac{p\ +r\ +3}{2})} + \frac{\lambda\ A^{q-1}\ \Gamma(\frac{q}{2}+1)}{\Gamma(\frac{q\ +3}{2})}}{\frac{\sqrt{\pi}}{2}\mu + \frac{\sigma\ A^r\ \Gamma(\frac{r\ +3}{2})}{\Gamma(\frac{r}{2}+2)}}.$$

According to the current method of solution, the nonlinear oscillator equation (9) is transformed into the following equation:

$$\left(\frac{2}{\eta_{\max}}\right)^2 \left(\sum_{l=0}^{L^*} d_{j,l}^{(2)} u_l\right) + \Omega\ u_j^p + \frac{\lambda\ u_j^q}{\mu + \sigma\ u_j^r}\ = 0. \tag{12}$$

3.4. *Oscillator with open nonlinearity.*

The nonlinear oscillator equation is solved by Ebaid [22] in the form:

$$u'' + \omega^2\ u = \lambda\ u^m, \tag{13}$$

with the boundary conditions

$$u(0) = 0, u(1) = \alpha,\ \alpha > 0, \tag{14}$$

where,

$$\omega^2 = \frac{5}{4},\ \lambda = \frac{1}{2}, \alpha = sn\ (1 \mid \frac{1}{4}),$$

using only three terms of the ADM decomposition series and the approximate solutions $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ were compared with the exact solution:

$$u = sn\ (\eta\ \mid \frac{1}{4}), \tag{15}$$

where, *sn*.is the Jacobi-elliptic funcation [21]. Equation (13) becomes

$$\left(\frac{2}{\eta_{\max}}\right)^2 \left(\sum_{l=0}^{L^*} d_{j,l}^{(2)} u_l\right) + \omega^2\ u_j - \lambda\ u_j^m\ = 0. \tag{16}$$

3.5. *Bratu's model*

Bratu's model has application as such as the fuel ignition of the thermal combustion theory Buckmire [31] and in the Chandrasekhar model of the expansion of the universe. It stimulates a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction [32]. The exact solutions of the initial and bounary value problems of Bratu-type was discussed by Wazwaz [23].

$$u'' - 2e^{u} = 0,\ 0 < x < 1, \tag{17}$$

The initial value problem of Bratu-type in the form:

$$u(0) = u'(0) = 0. \tag{18}$$

The exact solution is given by

$$u(x) = -2\ \ln(\cos(x)). \tag{19}$$

It is followed from equation (17) that

$$\left(\frac{2}{\eta_{\max}}\right)^{2}\left(\sum_{l=0}^{L^{*}} d_{j,l}^{(2)} u_l\right) - 2\ e^{u_j}\ = 0. \tag{20}$$

An additional boundary value problem of Bratu-type is in the form [23]

$$u'' + \pi^{2} e^{-u} = 0,\ 0 < x < 1, \tag{21}$$

with the boundary conditions

$$u(0) = u(1) = 0, \tag{22}$$

where the exact solution is given by

$$u(x) =\ \ln(1+\sin(\pi x)). \tag{23}$$

We have from equation (21), that

$$\left(\frac{2}{\eta_{\max}}\right)^{2}\left(\sum_{l=0}^{L^{*}} d_{j,l}^{(2)} u_l\right) + \pi^{2} e^{-u_j}\ = 0. \tag{24}$$

3.6. *Fluid flow over a stretching surface.*

In this final problem we consider the following boundary value problem which describes the fluid flow over a stretching surface with variable heat flux[33]

$$f'''(\eta)+f(\eta)f''(\eta)+(f'(\eta))^2=0,$$
$$f'(0)=1,\ f(0)=0,\ f'(\infty)=0. \tag{25}$$

The exact solution is given by

$$f(\eta)=\sqrt{2}\ \tanh(\frac{\eta}{\sqrt{2}}). \tag{26}$$

Here, equation (25) is transformed into the following equations:

$$\left(\frac{2}{\eta_{\max}}\right)^3\left(\sum_{l=0}^{L^*} d_{j,l}^{(3)} f_l\right)+\left(\frac{2}{\eta_{\max}}\right)^2 f_j\ \left(\sum_{l=0}^{L^*} d_{j,l}^{(2)} f_l\right)+\left(\frac{2}{\eta_{\max}}\right)^2\left(\sum_{l=0}^{L^*} d_{j,l}^{(1)} f_l\right)^2=0. \tag{27}$$

4. Results and discussion

In this section, the numerical results obtained by using Chebyshev collocation method for the present physical models shall be validated through comparisons with the available exact or approximate solutions. Such comparisons are reported in terms of tables and graphs.

The present numerical results have been compared with the approximate solution given by Eq. (3) , where good agreement has been achieved in most cases as displayed in Fig. 1. In addition, the approximate solution given in (7) for the relativistic oscillator has been also compared with the current results in Fig. 2 and in this case the current numerical results can viewed as effective.

Regarding the initial value problem describing the generalized oscillator, the solution given by Eq. (11) is depicted in Fig. 3 with the *ChCM* solution at $(\Omega=\omega^2=\frac{5}{4},\ p=1,\mu=-1,\sigma=0,q=3)$ with different values for λ and A . It can be concluded form this figure that when the amplitude A takes high values, λ should be very small to ensure the accuracy.

In Fig. 4 the present numerical results for Eqs. (13-14) have been compared with the approximate solutions $\phi_1(x),\ \phi_2(x)$ and $\phi_3(x)$ obtained by [22] along with the exact solution given in Eq. (15). Furthermore, Table 1 displays the values of $u(\eta)$ obtained through Mathematica using the (*NDSolve*)-command with the present method (*ChCM*) at $m=(2$ and $8.5),\omega^2=5/4,\lambda=0.5,\alpha=sn(1\mid(\frac{1}{4}))$ are represented in Table $(2-a)$ and Table $(2-b)$. Also, the present results have been compared with the two terms approximate solution whose obtained by Shawagfeh [21] and the the exact solution in equation (15) at $m = 3$ as shown in Table $(2-c)$. These comparisons declare that the present numerical method is accurate and effective.

For the fifth model, the obtained numerical results have been compared with initial and boundary value problems of Bratue-type which have been discussed by Wazwaz [23] as shown in Fig. 5 $(a-b)$. Finally, for the sixth model 6 Table $(3-a)$ and Fig. 6 represent comparisons of the values of $f(\eta)$ for the exact solution, the shooting method and the present method $(ChCM)$. The maximum absolute error of Shootting method $(E_{e,Shooting})$ and (the error $E_{e,ChCM}$) of the present method is given in Table $(3-b)$. From the obtained results for these fifth and sixth models we can also conclude that the current approach is accurate and therefore it can be used to analyze similar physical models.

5. Conclusion

A comprehensive numerical study is conducted for a class of nonlinear initial and boundary value problems. The results are reported in terms of tables and graphs. This is done in order to illustrate special features of the solutions. So, the obtained results using the present method indicate that it is an adequate scheme for the solution of the present problems. Therefore, the current approach may be useful to analyze similar nonlinear problems.

Conflict of Interests

The authors declare that there is no conflict of interests.

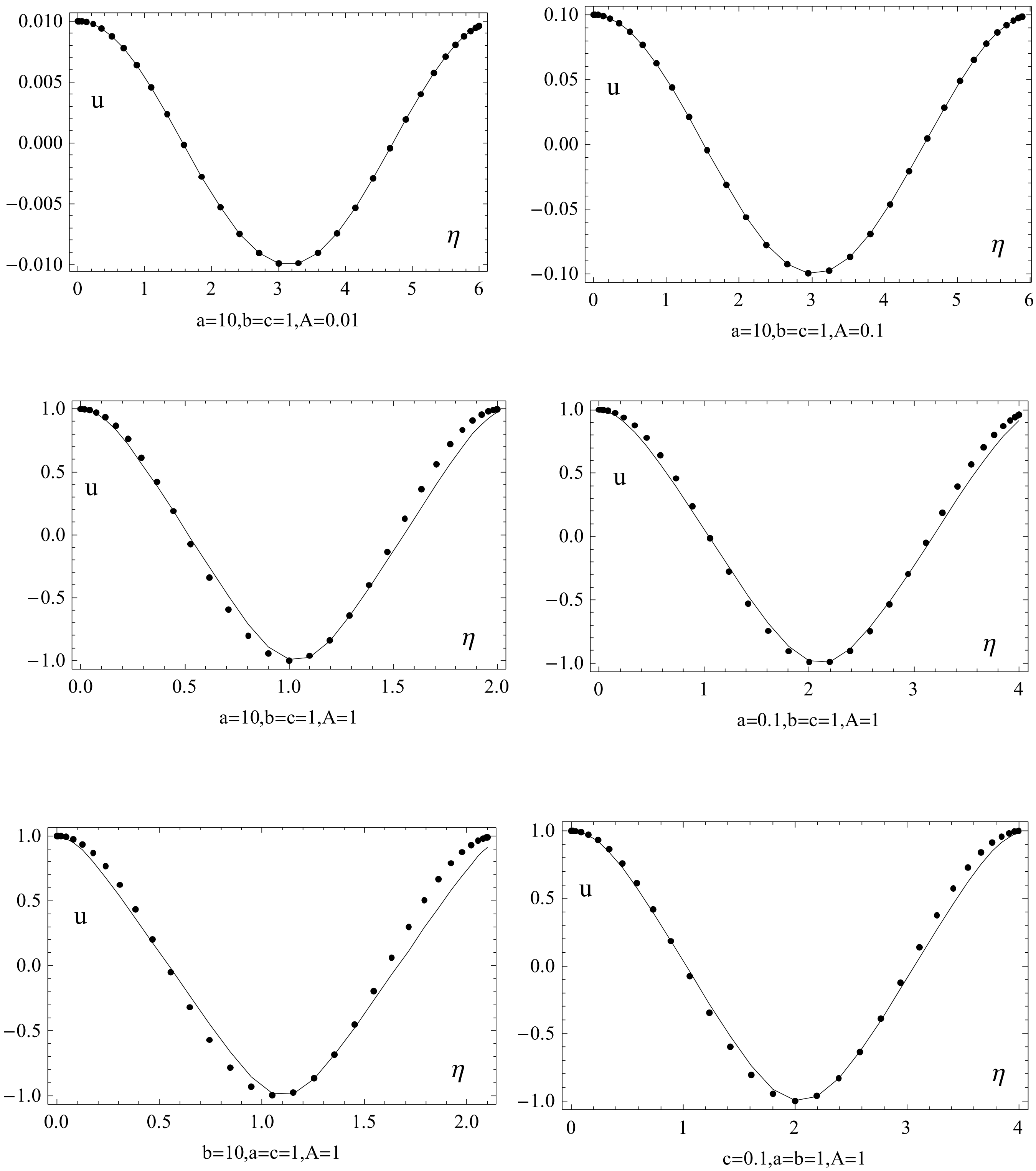


Fig.1. Comparison of the present results with the approximate solution Fang Liu [18] : dots : the approximate solution Fang Liu [18] and solid line : the present results.

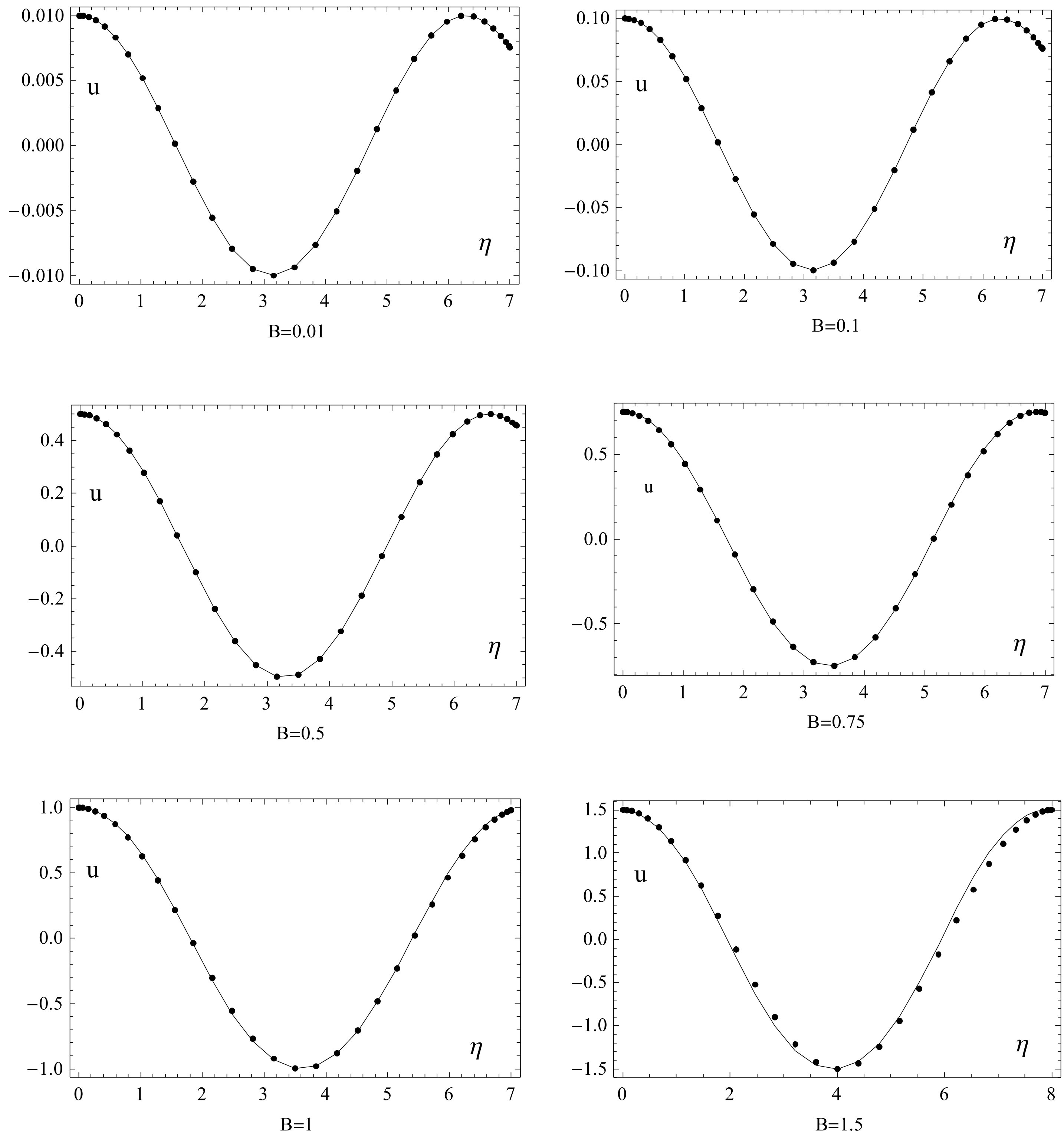


Fig. 2. Comparison of the present results with the approximate solution [16] :
dots : the approximate solution [16] and solid line : the present results

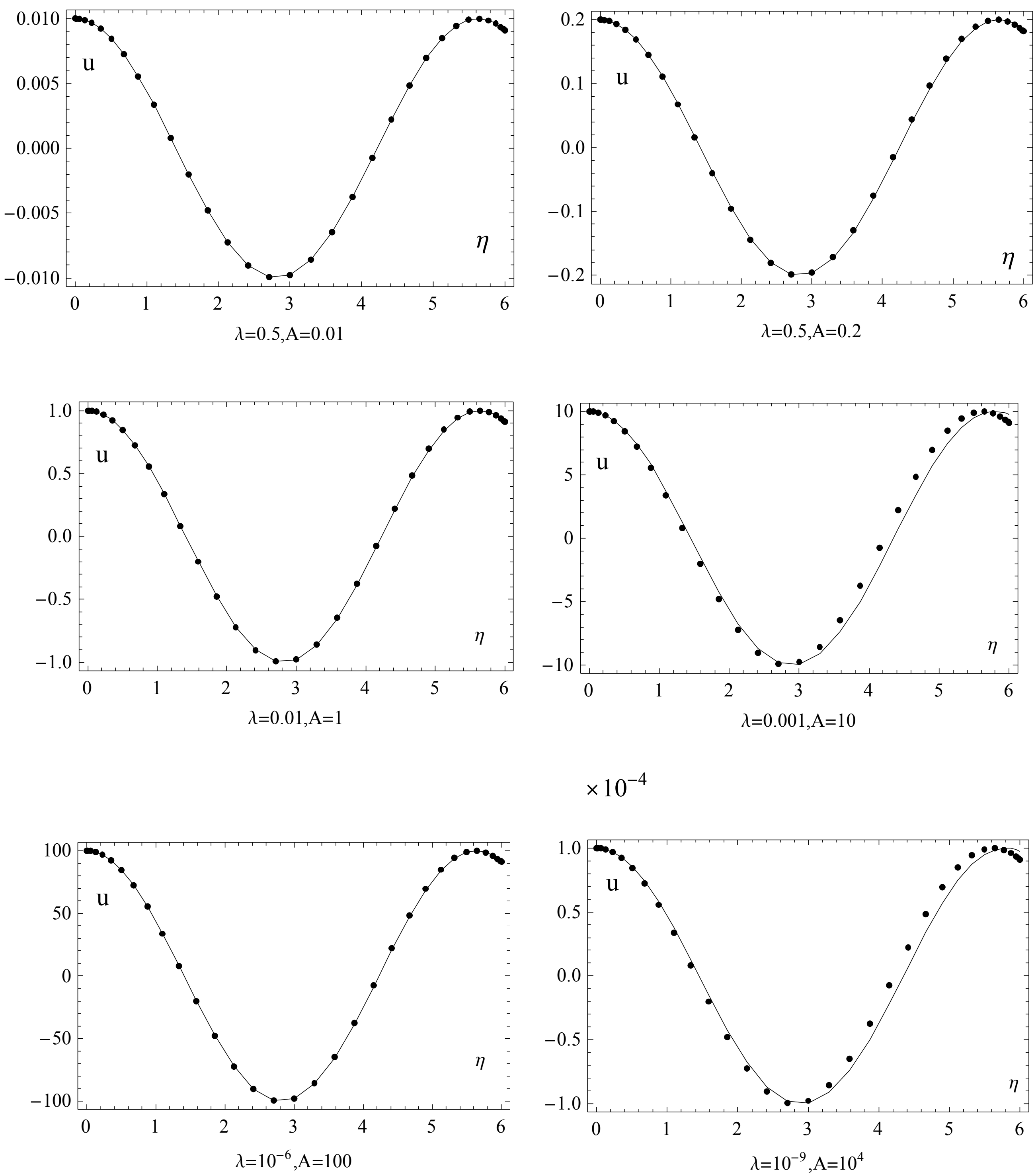


Fig. 3. Comparison of the present results with the approximate solution A. Ebaid [15] at $\omega = \sqrt{\frac{5}{4}}$ and $q = 3$: dots : the approximate solution [15] and solid line : the present results.

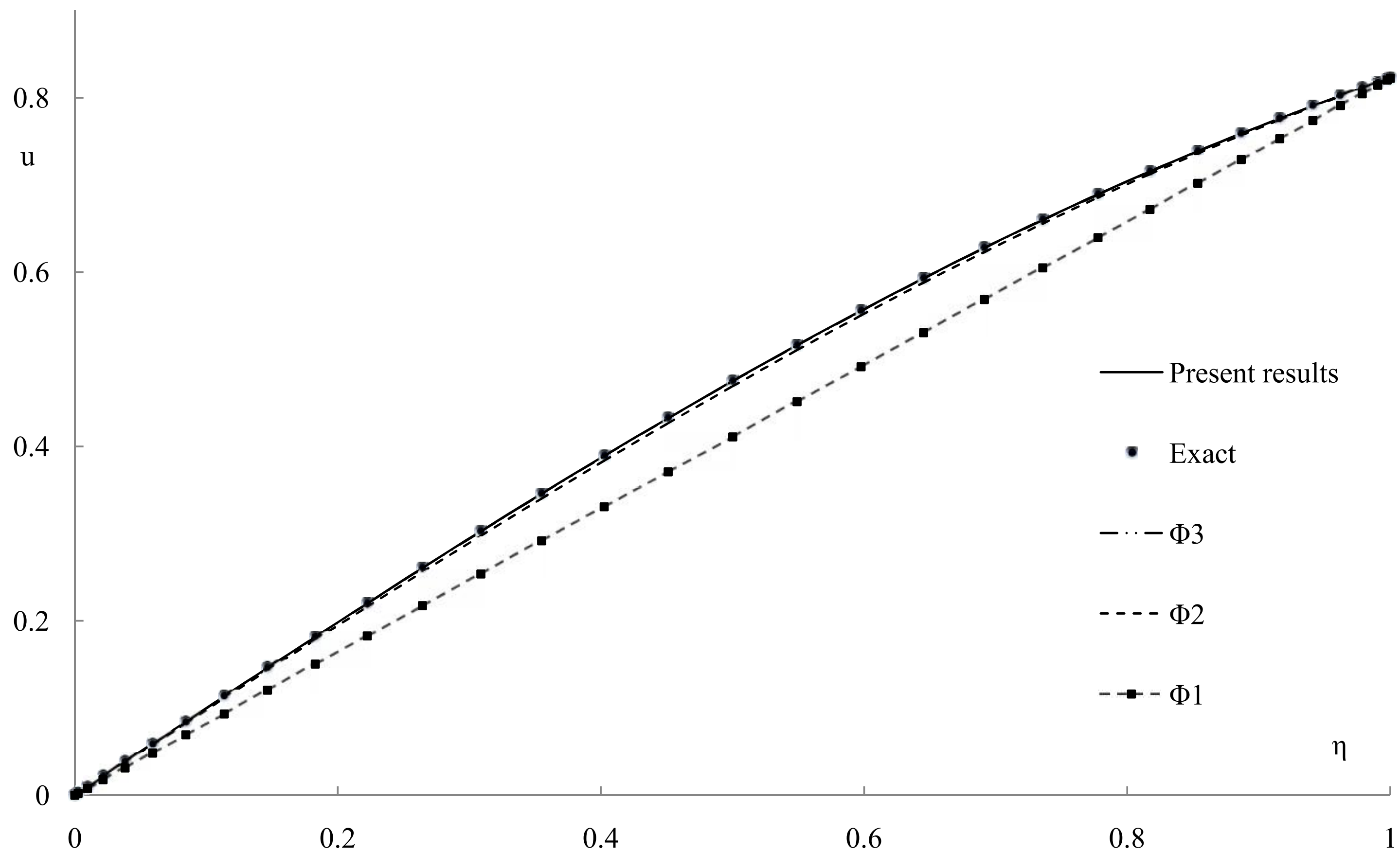
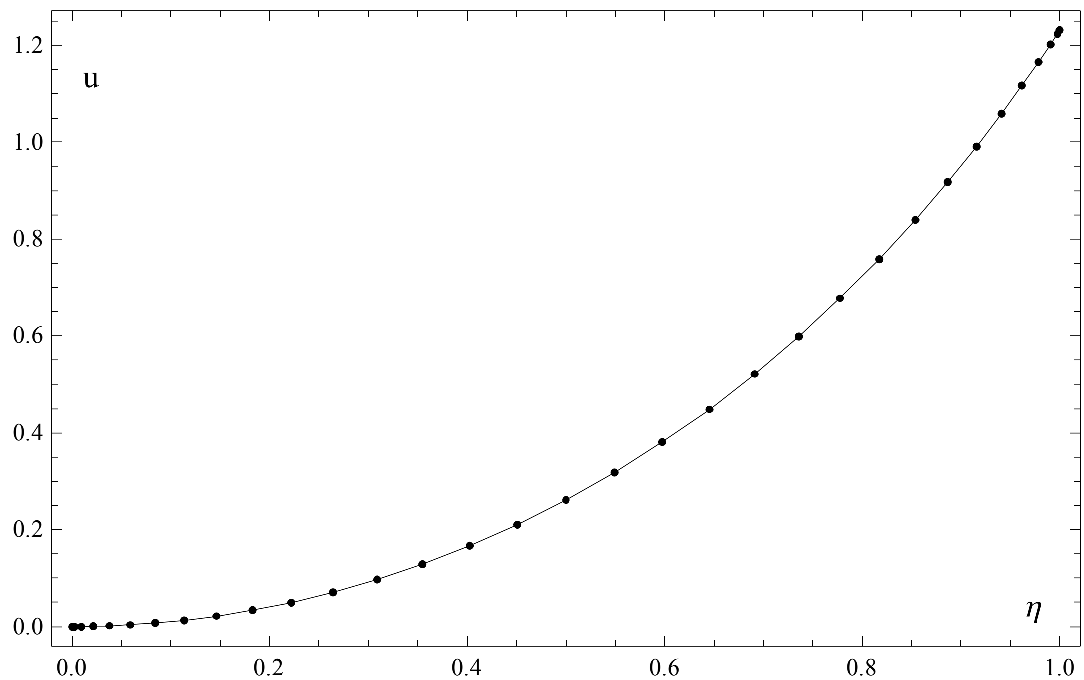
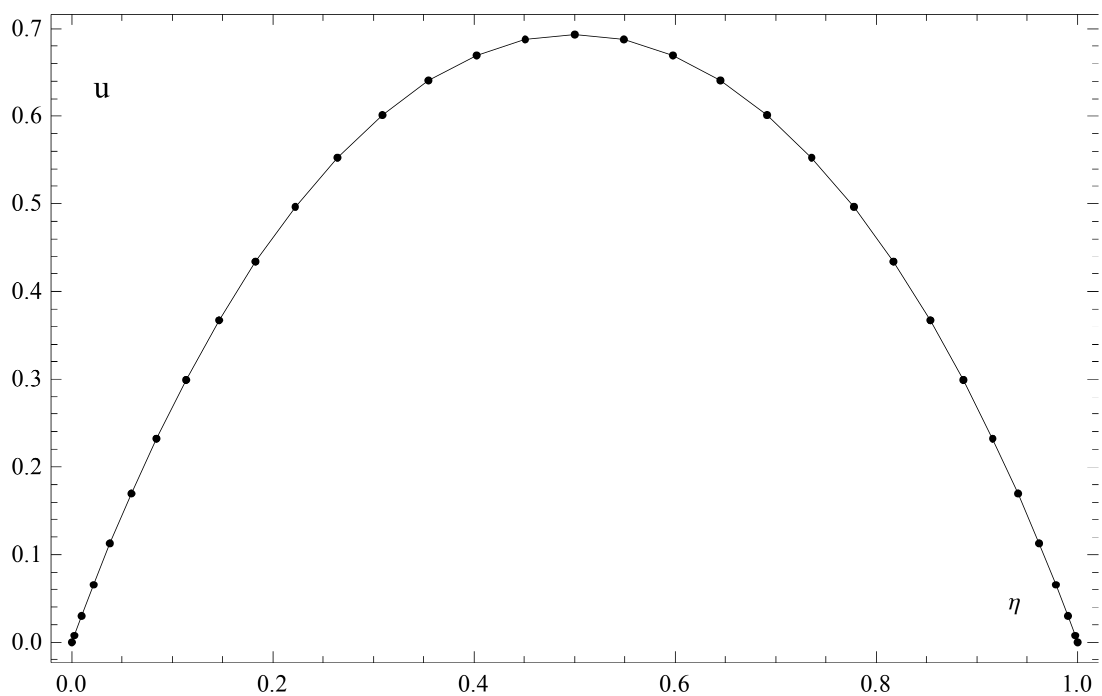


Fig. 4. Comparsion of the exact solution, ADM series solutions [22] and the present method at m=3, ω^2=5/4, λ=1/2, α=sn (1/(1/4)).



a) initial value problem



b) boundary value problem

Fig. 5 (a – b). Comparison of the present results with intial and boundary value problem of the Bratue – type [23] : dots : the exact solution of intial and boundary of the Bratue – type [23] and solid line : the present results

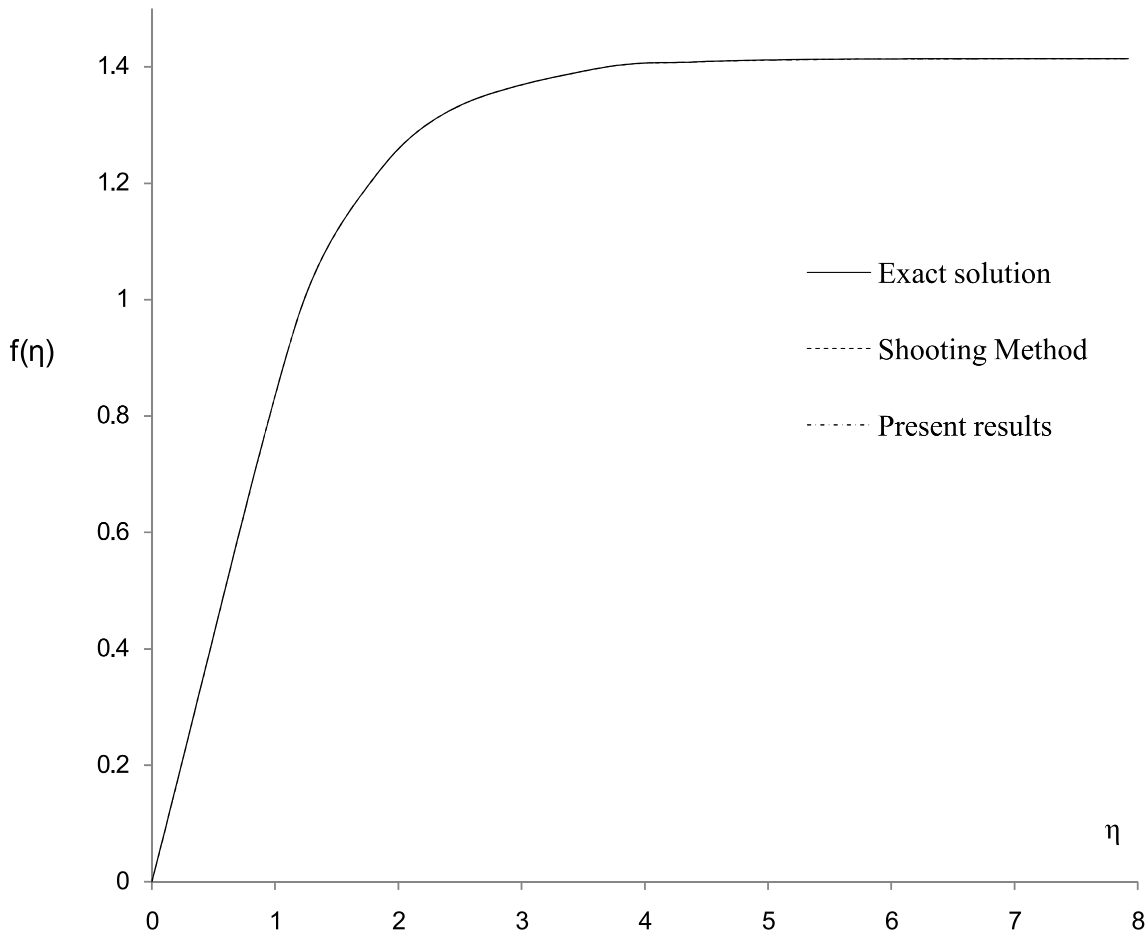


Fig. 6. Comparison of values $f(\eta)$ for the exact solution, the shooting method and the present results.

Table 1. Values of $u(\eta)$ for the exact solution, the Adomian decomposition method (ADM) [22] and the present method ($ChCM$) at $m = 3$.

η	Exact solution	Ebaid [22]	Present results
0	0	0	0
0.0215	0.021527753184314	0.021496840449489	0.021527753184314
0.0381	0.038048750666070	0.037994258467514	0.038048750666070
0.0590	0.058996522078775	0.058912488704097	0.058996522078775
0.0843	0.084140701923113	0.084022009055905	0.084140701923113
0.1135	0.113190917768906	0.113033703017937	0.113190917768906
0.2643	0.260503473299994	0.260193408528946	0.260503473299994
0.3087	0.302636337275760	0.302300382303633	0.302636337275760
0.4025	0.389261774816383	0.388902811746843	0.389261774816383
0.5000	0.475082936028531	0.474741879621621	0.475082936028531
0.6913	0.627920287048594	0.627693211433239	0.627920287048594
0.7357	0.660039308172239	0.659845026792322	0.660039308172239
0.8865	0.759023161243265	0.758940502736828	0.759023161243265
0.9157	0.776316925238970	0.776255547519948	0.776316925238970
1	0.822635578129862	0.822635578129862	0.822635578129862

Table 2. Represents the values of $u(\eta)$ for Mathematica (NDSolve) and the present method ($ChCM$) at $m = 2$.

η	NDSolve	$ChCM$	$E_{NDSolve,ChCM}$
0	2.216844×10^{-21}	0	$2.2168E-21$
0.0215	0.02112836209	0.02112837356	$1.14668E-08$
0.0381	0.03734293177	0.03734294319	$1.14200E-08$
0.0590	0.05790246281	0.05790246281	$1.10838E-08$
0.0843	0.08258160883	0.08258161968	$1.08447E-08$
0.1828	0.17821000669	0.17821001951	$1.28265E-08$
0.2643	0.25583270380	0.25583271758	$1.37779E-08$
0.3549	0.33982975211	0.33982977033	$1.82279E-08$
0.4510	0.42570059687	0.42570060781	$1.09385E-08$
0.5000	0.46794941921	0.46794942643	$7.22079E-09$
0.6913	0.62130146059	0.62130146166	$1.06673E-09$
0.7778	0.68379903935	0.68379903789	$1.46641E-09$
0.8865	0.75579078097	0.75579077351	$7.46462E-09$
0.9904	0.81731628075	0.81731627326	$7.48671E-09$
1	0.82263558573	0.82263557812	$7.60120E-09$

Table 3. Represents the values of $u(\eta)$ for Mathematica (NDSolve) and the present method $(ChCM)$ at $m = 8.5$.

η	NDSolve	$ChCM$	$E_{NDSolve,ChCM}$
0	-5.41969×10^{-21}	0	$5.41969E-21$
0.0215	0.02197144187461	0.02197145295276	$1.10781E-08$
0.0381	0.03883294543526	0.03883295616852	$1.07333E-08$
0.0590	0.06021244.58721	0.06021245745128	$9.86407E-09$
0.0843	0.08587477504774	0.08587478403168	$8.98394E-09$
0.1828	0.18527486029162	0.18527486790503	$7.61341E-09$
0.2643	0.26584043801347	0.26584044369167	$5.6782E-09$
0.3549	0.35274481836500	0.35274481998803	$1.62303E-09$
0.4510	0.44103016433725	0.44103015713759	$7.19966E-09$
0.5000	0.48414214397530	0.48414213161335	$1.23619E-08$
0.6913	0.63747018932352	0.63747015608099	$3.32425E-08$
0.7778	0.69768758507902	0.69768754859829	$3.64807E-08$
0.8865	0.76438852645325	0.76438849111688	$3.53364E-08$
0.9904	0.81816684814189	0.81816681635677	$3.17851E-08$
1	0.82263560971081	0.82263557812986	$3.15809E-08$

Table 4. Comparison of the values of u(η) for the exact solution,
the two terms approximation(Shawagfeh) [21]
and the present method ($ChCM$) at $m = 3$.

η	Exact solution	Shawagfeh [21]	$ChCM$
0	0	0	0
0.1	0.099792	0.0996758	0.09979204601963
0.2	0.198345	0.1981148	0.19834539121447
0.3	0.294466	0.294127	0.29446555154955
0.4	0.3875042	0.386609	0.38704242323395
0.5	0.475083	0.474580	0.47508293602853
0.6	0.557734	0.557202	0.55773380237106
0.7	0.634293	0.633792	0.63429327633511
0.8	0.704212	0.703813	0.70421214154716
0.9	0.767085	0.766860	0.76708523758272
1	0.822636	0.822636	0.82263600000000

Table 5. Values of $f(\eta)$ for the exact solution, the shooting method and the present method ($ChCM$).

η	The exact solution	$ChCM$	Shootting method
0	0	0	3.1102×10^{-21}
1.17157	0.961141183786619	0.9611070045979400	0.9611054342686112
1.77772	1.202428601606520	1.2023643183102284	1.2023645509982483
2.46927	1.330665896574968	1.3305704365379722	1.330570838696424
3.60793	1.397114413277153	1.3969685880488372	1.396968522682298
4.39207	1.4085494708083601	1.4083673080884052	1.408367284942106
5.16114	1.4123021434700513	1.4120834942495526	1.4120834612222437
5.88559	1.4135271302075076	1.4132736020214456	1.4132735693434024
6.53757	1.4139405233760458	1.4136553770920586	1.4136553408479557
7.09204	1.4140889100927603	1.4137767754240098	1.4137767398762842
7.52769	1.4141462431225706	1.4138128682103317	1.4138128330749593
7.92314	1.4141750797928212	1.4138224084323320	1.4138223729952500
$\eta_\infty = 8$	1.4141790433479533	1.4138226201158337	1.4138225847075525

$Table\ 6.\ The maximum absolute error$

$E_{e,ChCM}$	$E_{e,Shooting}$
0	3.1102×10^{-21}
3.41792×10^{-5}	3.57495×10^{-5}
6.42833×10^{-5}	6.40506×10^{-5}
9.546×10^{-5}	9.50579×10^{-5}
1.45825×10^{-4}	1.45891×10^{-4}
1.82163×10^{-4}	1.82186×10^{-4}
2.18649×10^{-4}	2.18682×10^{-4}
2.53528×10^{-4}	2.53561×10^{-4}
2.85146×10^{-4}	2.85183×10^{-4}
3.12135×10^{-4}	3.1217×10^{-4}
3.33375×10^{-4}	3.3341×10^{-4}
3.52671×10^{-4}	3.52707×10^{-4}
3.56423×10^{-4}	3.56459×10^{-4}

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