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OSCILLATION THEOREMS FOR SECOND ORDER QUASILINEAR NEUTRAL DIFFERENCE EQUATIONS

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Abstract. Some new oscillation results are established for the difference equation

$$\Delta\left(a_n(\Delta(x_n + p_n x_{\tau(n)}))^{\alpha}\right) + q_n x^{\beta}_{\sigma(n)} = 0$$

via comparison theorems. Examples are provided to illustrate the main results.

Keywords: Quasilinear neutral difference equations, Second order, Comparison theorems, Oscillation

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1. Introduction

In this paper, we study the oscillatory behavior of second order quasilinear neutral difference equation of the form

$$\Delta\left(a_n(\Delta(x_n + p_n x_{\tau(n)}))^{\alpha}\right) + q_n x^{\beta}_{\sigma(n)} = 0,\ n \geq n_0 \in \mathbb{N}, \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and $\mathbb{N} = \{0, 1, 2, \dots\}$, subject to the following hypotheses:

(H_1) $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences with $\{q_n\}$ not identically zero for infinitely many values to n;

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(H_2) $\{a_n\}$ is a positive real sequence such that $\sum\limits_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} < \infty$;

(H_3) α and β are ratios of odd positive integers;

(H_4) $\{\tau(n)\}$ and $\{\sigma(n)\}$ are nondecreasing sequences of positive integers such that $\lim\limits_{n\to\infty} \sigma(n) = \lim\limits_{n\to\infty} \tau(n) = \infty$ and $\tau \circ \sigma = \sigma \circ \tau$;

(H_5) there is a constant p such that $0 \le p_n \le p < \infty$.

By a solution of equation (1.1) we mean a real sequence $\{x_n\}$ defined and satisfies equation (1.1) for all $n \ge n_0 \in \mathbb{N}$. We consider only those solutions $\{x_n\}$ of equation (1.1) which satisfy $\sup\{|x_n|;\ n \ge N\} > 0$ for all $N \ge n_0$. We assume that equation (1.1) possesses such a solution. A solution $\{x_n\}$ of equation (1.1) is oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In recent years, there has been much research activities concerning the oscillation of delay and neutral type difference equations, see for example [1, 2, 5], and the references cited therein. In [7, 8], the authors considered difference equation of the type (1.1) and obtained oscillation criteria when $\sum\limits_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} < \infty$ and $0 \le p_n \le p < 1$ or $\sum\limits_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty$ and $0 \le p_n \le p < \infty$, respectively.

Motivated by this observation in this paper, we establish sufficient conditions for the oscillation of all solutions of equation (1.1) when $\sum\limits_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} < \infty$ and $0 \le p_n \le p < \infty$. In Section 2, we present some basic lemmas and in Section 3, we establish oscillation results for the equation (1.1). In Section 4, we present some examples to illustrate the main results. Thus our results extend and complement to those obtained in [7, 8].

2. **Some Basic Lemmas**

In this section, we present some basic lemmas which will be used to prove the main results.

Lemma 2.1. *Let $A \ge 0$, $B \ge 0$ and $\gamma \ge 1$. Then*

$$(A + B)^{\gamma} \le 2^{\gamma-1}(A^{\gamma} + B^{\gamma}). \tag{2.1}$$

Proof. The proof can be found in [3, pp.292], and also in [4, Remark 2.1]. □

Lemma 2.2. *Let* $A \geq 0$, $B \geq 0$ *and* $0 < \gamma \leq 1$. *Then*

$$(A+B)^\gamma \leq A^\gamma + B^\gamma. \tag{2.2}$$

Proof. The proof can be found in [9].

Next we present the structure of positive solution of equation (1.1) since the opposite case is similar. □

Lemma 2.3. *If* $\{x_n\}$ *is a positive solution of equation* (1.1), *then* $z_n = x_n + p_n x_{\tau(n)}$ *satisfies the following two cases eventually:*

(*I*) $z_n > 0,\ \Delta z_n > 0,\ \Delta\left(a_n(\Delta z_n)^\alpha\right) \leq 0$;

(*II*) $z_n > 0,\ \Delta z_n < 0,\ \Delta\left(a_n(\Delta z_n)^\alpha\right) \leq 0$.

Proof. The proof is similar to that of in [7] and hence details are omitted.

Next we state two lemmas given in [6]. □

Lemma 2.4. *Let* $\gamma > 1$ *be a quotient of odd positive integers. Assume that* k *is a positive integer,* $\{d_n\}$ *is a positive sequence defined for all* $n \geq n_0 \in \mathbb{N}$, *and there exists* $\lambda > \frac{1}{k}\log\gamma$ *such that*

$$\liminf_{n\to\infty}\left[d_n \exp(e^{-\lambda n})\right] > 0. \tag{2.3}$$

Then all the solutions of the difference equation

$$\Delta y_n + d_n y_{n-k}^\gamma = 0 \tag{2.4}$$

are oscillatory.

Lemma 2.5. *Let* $0 < \gamma < 1$ *be a quotient of odd positive integers. Assume that* k *is a positive integer and* $\{d_n\}$ *is a positive real sequence defined for all* $n \geq n_0 \in \mathbb{N}$ *such that*

$$\sum_{n=n_0}^{\infty} d_n = \infty.$$

Then all solutions of equation (2.4) *are oscillatory.*

Next we state a result given in [5].

Lemma 2.6. *Assume that* $\{d_n\}$ *is a positive sequence defined for all* $n \geq n_0 \in \mathbb{N}$, *and*

$$\lim_{n\to\infty} \inf \sum_{s=n-k}^{n-1} d_s > \left(\frac{k}{k+1}\right)^{k+1}$$

where k *is a positive integer. Then all solutions of equation* (2.4) *with* $\gamma = 1$ *are oscillatory.*

We conclude this section with the following lemmas proved in [8] *and* [9].

Lemma 2.7. *Assume the difference inequality*

$$\Delta y_n + d_n y_{n-k}^{\gamma} \leq 0$$

has an eventually positive solution. Then the difference equation (2.4) *also has an eventually positive solution.*

Lemma 2.8. *Let* $\{x_n\}$ *be an eventually positive solution of equation* (1.1), *and suppose case (II) of Lemma 2.3 holds. Then there exists an integer* $N \geq n_0 \in \mathbb{N}$ *such that* $\{x_n\}$ *is nonincreasing for all* $n \geq N$.

3. **Oscillation Theorems**

In this section, we establish some new oscillation criteria for the equation (1.1). We use the following notations throughout this paper without further mention;

$$R_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s^{\frac{1}{\alpha}}}, \; B_n = \sum_{s=n}^{\infty} \frac{1}{a_s^{\frac{1}{\alpha}}},$$

$$Q_n = \min\{q_n, q_{\tau(n)}\}, Q_n^* = \frac{Q_n}{a_{\sigma(n)-1}^{\frac{\beta}{\alpha}}}.$$

We begin with the following theorem.

Theorem 3.1. *Let* $\alpha = \beta = 1$ *in equation* (1.1). *Assume that* $\sigma(n) \leq \tau(n) \leq n$ *and*

(i) *the difference inequality*

$$\Delta(y_n + p y_{\tau(n)}) + Q_n (R_{\sigma(n)} - R_{n_1}) y_{\sigma(n)} \leq 0 \tag{3.1}$$

has no positive solution, and

(ii) *for all large* $n_1 \in \mathbb{N}$

$$\sum_{n=n_1}^{\infty}\left[B_{n+1}q_n\left(\frac{1}{1+p_n}\right)-\frac{1}{a_nB_n}\right]=\infty. \tag{3.2}$$

Then every solution of equation (1.1) *is oscillatory.*

Proof. Assume that $\{x_n\}$ is a positive solution of equation (1.1). Then the corresponding function $z_n = x_n+p_nx_{\tau(n)}$ satisfies the two cases of Lemma 2.3 for all $n \geq n_1 \geq n_0 \in \mathbb{N}$.

Case(I): From the definition of z_n, we have

$$z_{\sigma(n)} = x_{\sigma(n)} + p_{\sigma(n)}x_{\tau(\sigma(n))} \leq x_{\sigma(n)} + px_{\sigma(\tau(n))}$$

where we have used the hypothesis (H_4) and (H_5). From the equation (1.1), we have

$$\Delta(a_n\Delta z_n) + q_nx_{\sigma(n)} = 0, \tag{3.3}$$

and

$$p\Delta(a_{\tau(n)}\Delta z_{\tau(n)}) + pq_{\tau(n)}x_{\tau(\sigma(n))} = 0. \tag{3.4}$$

Combining (3.3) and (3.4), we are led to

$$\Delta(a_n\Delta z_n + pa_{\tau(n)}\Delta z_{\tau(n)}) + Q_nz_{\sigma(n)} \leq 0. \tag{3.5}$$

It follows from Lemma 2.3 that $y_n = a_n\Delta z_n > 0$ is decreasing, and then

$$z_n \geq \sum_{s=n_1}^{n-1}\frac{1}{a_s}(a_s\Delta z_s) \geq y_n(R_n - R_{n_1}). \tag{3.6}$$

Therefore, (3.6) together with (3.5) ensures that $\{y_n\}$ is a positive solution of the inequality (3.1), which is a contradiction.

Case(II): Define a function v_n by

$$v_n = \frac{a_n\Delta z_n}{z_n},\ n \geq n_1 \in \mathbb{N}. \tag{3.7}$$

Then $v_n < 0$ for $n \geq n_1$. Since $\{a_n\Delta z_n\}$ is nonincreasing, we have

$$\Delta z_s \leq \frac{a_n\Delta z_n}{a_s} \text{ for } s \geq n.$$

Summing the last inequality from n to $l-1$, we obtain

$$z_l \leq z_n + a_n \Delta z_n \sum_{s=n}^{l-1} \frac{1}{a_s}.$$

Letting $l \to \infty$, we obtain

$$0 \leq z_n + a_n \Delta z_n B_n, \;\; n \geq n_1,$$

or

$$-1 \leq v_n B_n \leq 0, \;\; n \geq n_1. \tag{3.8}$$

From (3.7), we have

$$\Delta v_n \leq \frac{-q_n x_{\sigma(n)}}{z_n} - \frac{a_{n+1}\Delta z_{n+1}}{\Delta z_n \Delta z_{n+1}} \Delta z_n. \tag{3.9}$$

From Lemma 2.8, $\Delta x_n \leq 0$ for $n \geq n_1$ and by $\sigma(n) \leq \tau(n) \leq n$, we have

$$\frac{x_{\sigma(n)}}{z_n} = \frac{x_{\sigma(n)}}{x_n + p_n x_{\tau(n)}} \geq \frac{x_{\sigma(n)}}{x_{\sigma(n)} + p_n x_{\sigma(n)}} \geq \frac{1}{1+p_n}. \tag{3.10}$$

From (3.9) and (3.10), we have

$$\Delta v_n + q_n \left(\frac{1}{1+p_n}\right) \leq 0, \;\; n \geq N \geq n_1. \tag{3.11}$$

Multiplying (3.11) by B_{n+1} and summing it from N to $n-1$, we obtain

$$\sum_{s=N}^{n-1} B_{s+1}\Delta v_s + \sum_{s=N}^{n-1} q_s B_{s+1} \left(\frac{1}{1+p_s}\right) \leq 0$$

or

$$B_n v_n - B_N v_N + \sum_{s=N}^{n-1} \frac{v_s}{a_s} + \sum_{s=N}^{n_1} B_{s+1} q_s \left(\frac{1}{1+p_s}\right) \leq 0. \tag{3.12}$$

From (3.8) and (3.12),we obtain

$$B_n v_n \leq B_N v_N - \sum_{s=N}^{n-1} \left[B_{s+1} q_s \left(\frac{1}{1+p_s}\right) - \frac{1}{B_s a_s} \right].$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (3.2). This completes the proof. □

Corollary 3.2. *Let $\alpha = \beta = 1$, $\sigma(n) = n - m$, $\tau(n) = n - k$ in equation (1.1) where m and k are positive integers with $m > k$. If condition (3.2) and*

$$\lim_{n\to\infty} \inf \sum_{s=n-m+k}^{n-1} Q_s R_{s-m} > (1+p)\left(\frac{m-k}{1+m-k}\right)^{1+m-k} \tag{3.13}$$

hold, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1),. Proceeding as in the proof of Theorem 3.1, we have two cases. For the case (I), we have the inequality (3.1). Since $y_n = a_n \Delta z_n > 0$ is decreasing and $\tau(n) = n - k$, we have

$$w_n = y_n + p y_{\tau(n)} \leq (1+p) y_{n-k}.$$

Using this in (3.1), we obtain

$$\Delta w_n + \frac{Q_n}{(1+p)}(R_{n-m} - R_{n_1}) w_{n-m+k} \leq 0.$$

By Lemma 2.7, the difference equation

$$\Delta w_n + \frac{Q_n}{(1+p)}(R_{n-m} - R_{n_1}) w_{n-m+k} = 0 \tag{3.14}$$

has a positive solution. By Lemma 2.6 the condition (3.13) implies that all solutions of equation (3.14) are oscillatory, which is a contradiction. For the case (II), proceeding as in Theorem 3.1, we obtain again a contradiction to condition (3.2). This completes the proof. □

Next we consider the case $0 < \beta < 1$ and $\sigma(n) \leq \tau(n) \leq n$ in equation (1.1).

Theorem 3.3. *Let $0 < \beta < 1$, $\beta \leq \alpha$ and $\sigma(n) \leq \tau(n) \leq n$ for all $n \geq n_0 \in \mathbb{N}$. If*

(i) *the difference inequality*

$$\Delta(w_n + p^\beta w_{\tau(n)}) + Q_n^* w_{\sigma(n)}^{\frac{\beta}{\alpha}} \leq 0 \tag{3.15}$$

has no positive solution, and

(ii) *for large $n_1 \in \mathbb{N}$ and for some $L > 0$*

$$\sum_{n=n_1}^{\infty} \left[B_{n+1} q_n \left(\frac{1}{1+p_n}\right)^\beta - \frac{L^{\alpha-\beta}}{B_n^\alpha a_n^{\frac{1}{\alpha}}} \right] = \infty, \tag{3.16}$$

then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1). Then z_n satisfies two cases of Lemma 2.3 for all $n \geq n_1 \geq n_0 \in \mathbb{N}$.

Case(I) From equation (1.1), we have

$$0 = \Delta(a_n(\Delta z_n)^{\alpha}) + q_n x^{\beta}_{\sigma(n)}, \tag{3.17}$$

and

$$0 = p^{\beta}\Delta(a_{\tau(n)}(\Delta z_{\tau(n)})^{\alpha}) + p^{\beta} q_{\tau(n)} x^{\beta}_{\sigma(\tau(n))}. \tag{3.18}$$

Combining (3.17) and (3.18), we obtain

$$\Delta(a_n(\Delta z_n)^{\alpha} + p^{\beta} a_{\tau(n)}(\Delta z_{\tau(n)})^{\alpha}) + Q_n \left(x^{\beta}_{\sigma(n)} + p^{\beta} x^{\beta}_{\sigma(\tau(n))}\right) \leq 0. \tag{3.19}$$

By Lemma 2.2, we have

$$\begin{aligned} z^{\beta}_{\sigma(n)} &= (x_{\sigma(n)} + p_{\sigma(n)} x_{\sigma(\tau(n))})^{\beta} \\ &\leq x^{\beta}_{\sigma(n)} + p^{\beta} x^{\beta}_{\sigma(\tau(n))}. \end{aligned} \tag{3.20}$$

Using (3.20) in (3.19), we have

$$\Delta(a_n(\Delta z_n)^{\alpha} + p^{\beta} a_{\tau(n)}(\Delta z_{\tau(n)})^{\alpha}) + Q_n z^{\beta}_{\sigma(n)} \leq 0. \tag{3.21}$$

It follows from Lemma 2.3 that $w_n = a_n(\Delta z_n)^{\alpha} > 0$ is decreasing and so

$$z_n \geq \sum_{s=n_1}^{n-1} \frac{(a_s(\Delta z_s)^{\alpha})^{\frac{1}{\alpha}}}{a_s^{\frac{1}{\alpha}}} \geq \frac{w_n^{\frac{1}{\alpha}}}{a_{n-1}^{\frac{1}{\alpha}}}.$$

Using the last inequality in (3.21), we see that $\{w_n\}$ is a positive solution of

$$\Delta(w_n + p^{\beta} w_{\tau(n)}) + Q^{*}_n w^{\frac{\beta}{\alpha}}_{\sigma(n)} \leq 0$$

which is a contradiction.

Case(II) Define a function v_n by

$$v_n = \frac{a_n(\Delta z_n)^{\alpha}}{z_n^{\beta}}, \ n \geq n_1, \tag{3.22}$$

then $v_n < 0$ for $n \geq n_1$. From (3.22), we have

$$\begin{aligned}\Delta v_n &= \frac{-q_n x^{\beta}_{\sigma(n)}}{z_n^{\beta}} - \frac{a_{n+1}(\Delta z_{n+1})^{\alpha}}{z^{\beta}_{n+1} z^{\beta}_n} \Delta z_n^{\beta} \\ &\leq -q_n \left(\frac{x_{\sigma(n)}}{x_n + p_n x_{\tau(n)}} \right)^{\beta}.\end{aligned}$$

Since $\sigma(n) \leq \tau(n) \leq n$ and $\Delta x_n \leq 0$, we have

$$\Delta v_n \leq \frac{-q_n}{(1+p_n)^{\beta}}, \ n \geq n_1.$$

Multiplying the last inequality by B_{n+1} and then summing from $N \geq n_1$ to $n-1$, we have

$$B_n v_n - B_N v_N + \sum_{s=N}^{n-1} \frac{v_s}{a_s^{\frac{1}{\alpha}}} + \sum_{s=N}^{n-1} \frac{B_{s+1} q_s}{(1+p_s)^{\beta}} \leq 0. \tag{3.23}$$

Since $\{a_n(\Delta z_n)^{\alpha}\}$ is nonincreasing, we have

$$\Delta z_s \leq \frac{a_n^{\frac{1}{\alpha}} \Delta z_n}{a_s^{\frac{1}{\alpha}}} \text{ for } s \geq n.$$

Summing the last inequality from n to $l-1$ and then letting $l \to \infty$, we obtain

$$0 \leq z_n + a_n^{\frac{1}{\alpha}} \Delta z_n B_n, \ n \geq n_1$$

or

$$-z_n^{\alpha-\beta} \leq B_n^{\alpha} v_n, \ n \geq n_1. \tag{3.24}$$

Since $z_n > 0$ is decreasing and $\alpha \geq \beta$, we have $z_n^{\alpha-\beta} \leq L^{\alpha-\beta}$ for some $L > 0$. Using this in (3.24), we obtain

$$-1 \leq \frac{B_n^{\alpha} v_n}{L^{\alpha-\beta}}, \ n \geq n_1. \tag{3.25}$$

From (3.23) and (3.25), we obtain

$$B_n v_n \leq B_N v_N - \sum_{s=N}^{n-1} \left[\frac{B_{s+1} q_s}{(1+p_s)^{\beta}} - \frac{L^{\alpha-\beta}}{B_s^{\alpha} a_s^{\frac{1}{\alpha}}} \right].$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (3.16). This completes the proof. □

Theorem 3.4. *Let $0<\beta<1$, and $\beta\ge\alpha$ and $\sigma(n)\le\tau(n)\le n$ for all $n\ge n_0\in\mathbb{N}$. If the difference inequality* (3.15) *has no positive solution and*

$$\sum_{n=n_1}^{\infty}\left[\frac{L^{\beta-\alpha}B_{n+1}q_n}{(1+p_n)^{\beta}}-\frac{a_n^{\frac{-1}{\alpha}}}{(\alpha+1)^{\alpha+1}}\right]=\infty \tag{3.26}$$

for some constant $L>0$, then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1). Then $\{z_n\}$ satisfies the two cases of Lemma 2.3 for all $n\ge n_1\ge n_0\in\mathbb{N}$. The proof of case (I) is same as that of case (I) of Theorem 3.3. Next we consider case (II). Define a function v_n by

$$v_n=\frac{a_n(\Delta z_n)^{\alpha}}{z_n^{\alpha}},\ n\ge n_1. \tag{3.27}$$

Then $v_n<0$, for $n\ge n_1$. From (3.27), we have

$$\begin{aligned}\Delta v_n &= \frac{-q_nx^{\beta}_{\sigma(n)}}{z_n^{\alpha}}-\frac{a_{n+1}(\Delta z_{n+1})^{\alpha}}{z^{\alpha}_{n+1}z_n^{\alpha}}\Delta z_n^{\alpha}\\ &\le -q_n\left(\frac{x_{\sigma(n)}}{x_n+p_nx_{\tau(n)}}\right)^{\beta}L^{\beta-\alpha}-\frac{\alpha v_n^{1+\frac{1}{\alpha}}}{a_n^{\frac{1}{\alpha}}}\end{aligned} \tag{3.28}$$

where we have used $z_n>0$ is decreasing and $\beta\ge\alpha$. Since $\sigma(n)\le\tau(n)\le n$ and $\Delta x_n\le 0$, we have

$$\frac{x_{\sigma(n)}}{x_n+p_nx_{\tau(n)}}\ge\frac{1}{1+p_n}. \tag{3.29}$$

From (3.28) and (3.29) we have

$$\Delta v_n\le\frac{-L^{\beta-\alpha}q_n}{(1+p_n)^{\beta}}-\frac{\alpha v_n^{1+\frac{1}{\alpha}}}{a_n^{\frac{1}{\alpha}}},\ n\ge n_1.$$

Multiply the last inequality by B_{n+1} and then summing it from $N\ge n_1$ to $n-1$, we have

$$B_nv_n-B_Nv_N+\sum_{s=N}^{n-1}\frac{B_{s+1}q_sL^{\beta-\alpha}}{(1+p_s)^{\beta}}+\sum_{s=N}^{n-1}\left(\frac{v_s}{a_s^{\frac{1}{\alpha}}}+\frac{\alpha}{a_s^{\frac{1}{\alpha}}}v_s^{1+\frac{1}{\alpha}}\right)\le 0. \tag{3.30}$$

Let $u_n=-v_n$. Then $u_n>0$ and $u^{\alpha+\frac{1}{\alpha}}=v_n^{\alpha+\frac{1}{\alpha}}$, since α is a ratio of odd positive integers. From (3.30) we have

$$B_nv_n-B_Nv_N+\sum_{s=N}^{n-1}\frac{B_{s+1}q_sL^{\beta-\alpha}}{(1+p_s)^{\beta}}-\sum_{s=N}^{n-1}\left(\frac{u_s}{a_s^{\frac{1}{\alpha}}}-\frac{\alpha}{a_s^{\frac{1}{\alpha}}}u_s^{\frac{\alpha+1}{\alpha}}\right)\le 0. \tag{3.31}$$

Using the inequality $Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \dfrac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \dfrac{B^{\alpha+1}}{A^\alpha}$ with $B = \dfrac{1}{a_n^{\frac{1}{\alpha}}}$ and $A = \dfrac{1}{a_n^{\frac{1}{\alpha}}}$, we have from (3.31), that

$$B_n v_n \leq B_N v_N - \sum_{s=N}^{n-1} \left(\frac{L^{\beta-\alpha} B_{s+1} q_s}{(1+p_s)^\beta} - \frac{1}{(\alpha+1)^{\alpha+1} a_s^{\frac{1}{\alpha}}} \right).$$

Letting $n \to \infty$ in the above inequality, we obtain a contradiction to (3.26). The proof is now complete.

From Theorems 3.3 and 3.4, we obtain the following corollaries. □

Corollary 3.5. *Let* $0 < \beta < 1$, $\beta < \alpha$, $\sigma(n) = n - m$ *and* $\tau(n) = n - k$ *with* $m > k$ *where* m *and* k *are positive integers. If condition* (3.16) *and*

$$\sum_{n=n_1}^{\infty} Q_n^* = \infty \tag{3.32}$$

hold, then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1). Proceeding as in the proof of Theorem 3.3, we have two cases to consider for the sequence $\{z_n\}$. For the case (I), we have the inequality (3.15). Since $w_n = a_n(\Delta z_n)^\alpha > 0$ is decreasing and $\tau(n) = n - k$, we have

$$y_n = w_n + p^\beta w_{\tau(n)} \leq (1 + p^\beta) w_{n-k}.$$

Using this in (3.15), we obtain

$$\Delta y_n + \frac{Q_n^*}{(1+p^\beta)^{\frac{\beta}{\alpha}}} y_{n-m+k}^{\frac{\beta}{\alpha}} \leq 0.$$

By Lemma 2.7, the corresponding equation

$$\Delta y_n + \frac{Q_n^*}{(1+p^\beta)^{\frac{\beta}{\alpha}}} y_{n-m+k}^{\frac{\beta}{\alpha}} = 0. \tag{3.33}$$

has a positive solution. From condition (3.32) and Lemma 2.5 we obtain a contradiction. The proof for the case (II) is similar to that of Theorem 3.3. This completes the proof. □

Corollary 3.6. *Let* $0 < \beta < 1$, $\beta > \alpha$, $\sigma(n) = n - m$, $\tau(n) = n - k$ *with* $m > k$ *where* m *and* k *are positive integers. If there exists a* $\lambda > \dfrac{1}{(m-k)} \log \frac{\beta}{\alpha}$ *such that*

$$\liminf_{n\to\infty} \left[Q_n^* \exp\left(e^{-\lambda n}\right) \right] > 0 \tag{3.34}$$

and condition (3.16) *hold, then every solution of equation* (1.1) *is oscillatory.*

Proof. The proof is similar to that of Corollary 3.5 by using Lemma 2.4 instead of Lemma 2.5. □

Theorem 3.7. *Let* $\beta > 1$, $\beta \leq \alpha$ *and* $\sigma(n) \leq \tau(n) \leq n$ *for all* $n \geq n_0 \in \mathbb{N}$. *If*

(i) *the difference inequality*

$$\Delta(w_n + p^\beta w_{\tau(n)}) + \frac{Q_n^*}{2^{\beta-1}} w_{\sigma(n)}^{\frac{\beta}{\alpha}} \leq 0 \tag{3.35}$$

has no positive solution, and

(ii) *for all* $n_1 \geq n_0 \in \mathbb{N}$ *and for some* $L > 0$

$$\sum_{n=n_1}^{\infty} \left[B_{n+1} q_n \left(\frac{1}{1+p_n} \right)^\beta - \frac{L^{\alpha-\beta}}{B_n^\alpha a_n^{\frac{1}{\alpha}}} \right] = \infty, \tag{3.36}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof is similar to that Theorem 3.3 using Lemma 2.1 instead of Lemma 2.2 and hence the details are omitted.

Similar to that of Theorem 3.4 and Lemma 2.1 we have the following theorem. □

Theorem 3.8. *Let* $\beta > 1$, $\beta \geq \alpha$ *and* $\sigma(n) \leq \tau(n) \leq n$ *for all* $n \geq n_0 \in \mathbb{N}$. *If the difference inequality* (3.35) *has no positive solution and*

$$\sum_{n=n_1}^{\infty} \left[\frac{L^{\beta-\alpha} B_{n+1} q_n}{(1+p_n)^\beta} - \frac{a_n^{\frac{-1}{\alpha}}}{(\alpha+1)^{\alpha+1}} \right] = \infty \tag{3.37}$$

for some constant $L > 0$, *then every solution of equation* (1.1) *is oscillatory.*

From Theorem 3.7, and Lemmas 2.7 and 2.5 we have the following corollary.

Corollary 3.9. *Let* $\beta > 1$, $\beta < \alpha$ *and* $\sigma(n) = n - m$ *and* $\tau(n) = n - k$ *with* $m > k$. *If* (3.32) *and* (3.36) *hold then every solution of equation* (1.1) *is oscillatory.*

Finally from Theorem 3.8 and Lemmas 2.7 and 2.4, we obtain the following corollary.

Corollary 3.10. *Let* $\beta > 1$, $\beta > \alpha$ *and* $\sigma(n) = n - m$ *and* $\tau(n) = n - k$ *with* $m > k$. *If there exists a* $\lambda > \frac{1}{(m-k)} \log \frac{\beta}{\alpha}$ *such that* (3.34) *and* (3.37) *hold then every solution of equation* (1.1) *is oscillatory.*

4. **Examples**

In this section we provide some examples to illustrate the main results.

Example 4.1. *Consider the difference equation*

$$\Delta((n+3)(n+4)\Delta(x_n+2x_{n-1}))+\frac{4(n+2)^2}{(n+1)}x_{n-2}=0,\ n\geq 1. \tag{4.1}$$

Here $a_n=(n+3)(n+4)$, $p=2$, $m=2$, $k=1$, $q_n=\dfrac{4(n+2)^2}{(n+1)}$, *and* $\alpha=\beta=1$.

Then $R_n=\dfrac{(n-1)}{4(n+3)}$, $Q_n=4\left(\dfrac{n+2}{n+1}\right)$ *and clearly condition* (3.13) *holds. Further*

$$\sum_{n=1}^{\infty}\left[B_{n+1}\frac{q_n}{(1+p_n)}-\frac{1}{a_nB_n}\right]=\sum_{n=1}^{\infty}\left(\frac{n^2+n+1}{3(n+1)(n+4)}\right)=\infty$$

and therefore condition (3.2) *holds. Hence, by Corollary 3.2 every solution of equation* (4.1) *is oscillatory.*

Example 4.2. *Consider the difference equation*

$$\Delta(2^{3n}(\Delta(x_n+3x_{n-1}))^3)+2^{4n}x_{n-2}^{\frac{1}{3}}=0,\ n\geq 1 \tag{4.2}$$

Here $a_n=2^{3n}$, $p=3$, $q_n=2^{4n}$, $\alpha=3$, $\beta=\dfrac{1}{3}$, $k=1$, $m=2$. *Further* $B_n=\dfrac{1}{2^{n-1}}$, $Q_n^*=2^{\frac{11n}{3}-\frac{29}{9}}$. *It is easy to verify that all conditions of Corollary 3.5 are satisfied and hence every solution of equation* (4.2) *is oscillatory.*

Example 4.3. *Consider the difference equation*

$$\Delta(2^n(\Delta(x_n+2x_{n-1}))^{\frac{1}{3}})+\exp(e^{2n})x_{n-2}^3=0,\ n\geq 1. \tag{4.3}$$

Here $a_n=2^n$, $p=2$, $q_n=\exp(e^{2n})$, $\alpha=\frac{1}{3}$, $\beta=3$, $k=1$ *and* $m=2$. *Choose* $\lambda=2$, *then it is easy to see that all conditions of Corollary 3.10 are satisfied and hence every solution of equation* (4.3) *is oscillatory.*

We conclude the paper with the following remarks.

Remark 4.4. *1. The results of this paper may be extended to forced equation of the form*

$$\Delta(a_n(\Delta(x_n+p_nx_{\tau(n)}))^{\alpha})+q_nx_{\sigma(n)}^{\beta}=e_n$$

where $\{e_n\}$ is a sequence of real numbers.

2. The results of this paper are extendable to equations of form

$$\Delta(a_n(\Delta(x_n + p_n x_{\tau(n)}))^{\alpha}) + q_n x^{\beta}_{\sigma(n)} + r_n x^{\gamma}_{\delta(n)} = 0$$

where $\{\delta(n)\}$ is a sequence of integers and $\lim_{n\to\infty} \delta(n) = 0$, α, β and γ are ratio of odd positive integers.

The details are left to the reader.

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Second Edition, Marcel Dekker, New York, 2000.

[2] R. P. Agarwal, M. Bohner , S. R.Grace,and D. O. Regan, Discrete Oscillation Theory, Hindawi Publishing Corporation, New York, 2005.

[3] T. H. Hildebrandt, Introduction to the Theory of Integration, Academic Press, New York, 1963.

[4] B. Karpuz, R. N. Rath and S. K. Rath, On oscillation and asymptotic behavior of higher order functional difference equation of neutral type, Int. J. Diff. Eqn., 4(2009), No 1, 69-96.

[5] G. Ladas, Ch. G. Philos and Y. G. Sficas, Sharp condition for the oscillation of delay difference equations, J. Math. Simulation, 2(1989), 101-112.

[6] X. H. Tang and Y. Liu, Oscillation for nonlinear delay difference equations, Tamkang J. Math., 32(2001), 275-280.

[7] E. Thandapani and K. Mahalingam, Oscillation and nonoscillation of second order neutral delay difference equations, Czech, Math. J., 53(128)(2003), 935-947.

[8] E. Thandapani and S. Selvarangam, Oscillation of second order Emden-Fowler type neutral differene equations, Dynm. Cont. Dis. Imp. Sys.(to appear)

[9] E. Thandapani and S. Selvarangam, Oscillation theorems for second order nonlinear netural difference equations, Arch. Math. (to appear)