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GENERALIZED QUASI-VARIATIONAL-LIKE INCLUSIONS WITH (A, η) AND RELAXED COCOERCIVE MAPPINGS

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Abstract. In this paper, we study an existence theorem of solutions for generalized quasi-variational-like inclusions involving (A, η) and relaxed cocoercive mappings. We have shown that the approximate solutions obtained by proposed algorithm converge to the exact solutions of generalized quasi-variational-like inclusions. As an application, we have shown that generalized quasi-variational-like inclusions include optimization problems and also an equivalence with A-resolvent equations is given.

Keywords: (A, η)-accretive mappings; algorithm; quasi-variational-like inclusions; resolvent operator.

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1. Introduction

In 1979, Robinson [16] studied variational inclusion problem, that is for each $x \in \mathbb{R}^n$, find $y \in \mathbb{R}^m$ such that $0 \in g(x, y) + Q(x, y)$, where $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a single-valued mapping and $Q : \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^p$ is a multi-valued mapping. In the last decade, various

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classes of variational (-like) and quasi-variational (-like) inclusions have been extensively studied and generalized in different directions as they have potential and significant applications in optimization theory, structural analysis and economics etc. (see for example [1, 2, 5, 7]. Recently Lin [14] studied generalized quasi-variational inclusion problems and applied them to solve simultaneous equilibrium problems and optimization problems.

Huang and Fang [11] introduced generalized m-accretive mappings and defined resolvent operator for generalized m-accretive mappings. Lan et al. [12] introduced a new concept of (A,η)-accretive mappings and their resolvent operator.

Inspired and motivated by recent research going on in this fascinating and interesting field, in this paper, we study generalized quasi-variational-like inclusions involving (A,η) and relaxed cocoercive mappings. An iterative algorithm is suggested for finding the approximate solutions of generalized quasi-variational-like inclusions. Convergence analysis is also discussed. Some applications are given.

2. Preliminaries

Throughout the paper, unless otherwise specified,we assume that E is a real Banach space with its norm $\|.\|$, E^* is the topological dual of E, $\langle\cdot,\cdot\rangle$ is the pairing between E and E^*, d is the metric induced by the norm $\|.\|$, 2^E (respectively $CB(E)$) is the family of nonempty (respectively, nonempty closed and bounded) subsets of E and $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$H(P,Q) = \max\{\sup_{x\in P}\ d(x,Q),\ \sup_{y\in Q}\ d(P,y)\},$$

where $d(x,Q) = \inf_{y\in Q}\ d(x,y)$ and $d(P,y) = \inf_{y\in P}\ d(x,y)$.

The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q,\ \|f\| = \|x\|^{q-1}\},\ \text{for all } x \in E,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$, and J_q is single-valued if E^*

is strictly convex. If $E = X$ is a Hilbert space, then J_2 becomes the identity mapping on X. In what follows we shall denote the single-valued generalized duality mapping by j_q.

The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) \;=\; \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \; \|x\| \leq 1, \; \|y\| \leq t \right\}.$$

A Banach space E is called uniformly smooth if $\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$. E is called q-uniformly smooth if there exists a constant $C > 0$ such that $\rho_E(t) \leq Ct^q, \; q > 1$.

Lemma 2.1.[17]**.** Let E be a real uniformly smooth Banach space. Then E is called q-uniformly smooth if and only if there exists a constant $C_q > 0$ such that for all $x, y \in E$

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + C_q\|y\|^q.$$

Definition 2.1. Let E be a q-uniformly smooth Banach space and $\eta : E \times E \to E$ be a single-valued mapping. Then

(i) the single-valued mapping $A : E \to E$ is said to be r-strongly η-accretive, if there exists a constant $r > 0$ such that

$$\langle A(x) - A(y), j_q(\eta(x,y))\rangle \geq r\|x-y\|^q, \text{ for all } x, y \in E;$$

(ii) the set-valued mapping $M : E \to 2^E$ is said to be m-relaxed η-accretive, if there exists a constant $m > 0$ such that

$$\langle u - v, j_q(\eta(x,y))\rangle \geq -m\|x-y\|^q, \text{ for all } x, y \in E, \; u \in M(x), \; v \in M(y).$$

Remark 2.1.

(i) If $r = 0$ and equality holds if and only if $x = y$, then (i) of Definition 2.1 reduces to the definition of strictly η-accretive mappings.

(ii) If $\eta(x,y) = x - y$, then (i) of Definition 2.1 reduces to the Definition of r-strongly accretive mappings.

Example 2.1. Let $E = (-\infty, \infty)$, $M(x) = \sqrt{x}$, $\eta(x,y) = (-2\sqrt{x}) - (-2\sqrt{y})$, for all $x, y \geq 0 \in E$. Then M is 2-relaxed η-accretive.

Definition 2.2. Let $A : E \to E$, $\eta : E \times E \to E$ be the single-valued mappings. Then a multi-valued mapping $M : E \to 2^E$ is called (A, η)-accretive if M is m-relaxed η-accretive and $(A + \rho M)(E) = E$, for every $\rho > 0$.

Example 2.2. Let $E = (-\infty, \infty)$, $A(x) = x^5$, $M(x) = x^4$, $\eta(x, y) = (-2x^4) - (-2y^4)$, for all $x, y \in E$. Then M is (A, η)-accretive.

Remark 2.2.

(i) If $m = 0$, then Definition 2.2 reduces to the definition of (H, η)-accretive operators [8] which includes generalized m-accretive operators [11], H-accretive operators [6] and classical m-accretive operators.

(ii) When $m = 0$ and $E = X$ (Hilbert space), then Definition 2.2 reduces to the Definition of (H, η)-monotone operators [9, 10], which includes classical maximal monotone operators [18].

Definition 2.3. A mapping $g : E \to E$ is said to be (b, ξ)-relaxed cocoercive, if there exist constants $b, \xi > 0$ such that

$$\langle g(x) - g(y), j_q(x - y)\rangle \geq -b\|g(x) - g(y)\|^q + \xi\|x - y\|^q, \text{ for all } x, y \in E.$$

Definition 2.4. Let $A : E \to E$ be a strictly η-accretive mapping and $M : E \to 2^E$ be an (A, η)-accretive mapping. Then the resolvent operator $J^{\rho,A}_{\eta,M} : E \to E$ is defined by

$$J^{\rho,A}_{\eta,M}(u) = (A + \rho M)^{-1}(u), \text{ for all } u \in E.$$

Proposition 2.1[12]**.** Let E be a q-uniformly smooth Banach space and $\eta : E \times E \to E$ be τ-Lipschitz continuous, $A : E \to E$ be an r-strongly η-accretive mapping and $M : E \to 2^E$ be an (A, η)-accretive mapping. Then the resolvent operator $J^{\rho,A}_{\eta,M} : E \to E$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous, i.e.,

$$\|J^{\rho,A}_{\eta,M}(u) - J^{\rho,A}_{\eta,M}(v)\| \leq \frac{\tau^{q-1}}{r - \rho m}\|u - v\|, \text{ for all } u, v \in E,$$

where $\rho \in (0, \frac{r}{m})$ is a constant.

3. Main results

Let $N, W, \eta : E \times E \to E$, $g, m, A : E \to E$ be the single-valued mappings, $B, C, D, F, G : E \to 2^E$ be the multi-valued mappings. Let $M : E \times E \to 2^E$ be an (A, η)-accretive mapping in the first argument such that $g(u) - m(w) \in \mathrm{dom}(M(\cdot, u))$, $\forall\, u, w \in E$. We consider the following generalized quasi-variational-like inclusion problem:

Find $u \in E$, $x \in B(u)$, $y \in C(u)$, $z \in D(u)$, $v \in F(u)$ and $w \in G(u)$ such that

$$0 \in N(x, y) - W(z, v) + m(w) + M(g(u) - m(w), u). \tag{3.1}$$

Below are some special cases of problem (3.1).

(1) If $m = 0, M(g(u) - m(w), u) = M(g(u))$ and $W, D, F = 0$, then problem (3.1) reduces to the problem of finding $u \in E$, $x \in B(u)$, $y \in C(u)$ such that

$$0 \in N(x, y) + M(g(u)). \tag{3.2}$$

Problem (3.2) is considered by Peng [15].

(2) If B and C are single-valued mappings, then problem (3.2) can be replaced by finding $u \in E$ such that

$$0 \in N(B(u), C(u)) + M(g(u)). \tag{3.3}$$

Similar problem to (3.3) is considered by Lan [13].

(3) If $C = 0$ and $B, g = I$, the identity mapping, then (3.3) reduces to the problem of finding $u \in E$ such that

$$0 \in N(u) + M(u), \tag{3.4}$$

which is considered by Bi et al. [4].

Lemma 3.1. $u \in E$, $x \in B(u)$, $y \in C(u)$, $z \in D(u)$, $v \in F(u)$ and $w \in G(u)$ is the solution of problem (3.1) if and only if (u, x, y, z, v, w) satisfies the relation:

$$g(u) = m(w) + J^{\rho,A}_{\eta,M(\cdot,u)}\big[A(g(u) - m(w)) - \rho(N(x, y) - W(z, v) + m(w))\big], \tag{3.5}$$

where $J^{\rho,A}_{\eta,M(\cdot,u)} = (A + \rho M(\cdot, u))^{-1}$ and $\rho \in (0, \frac{r}{m})$ is a constant.

Proof. The proof follows directly from the Definition 2.4.

Algorithm 3.1. For any given $u_o \in E$, we choose $x_o \in B(u_o)$, $y_o \in C(u_o)$, $z_o \in D(u_o)$, $v_o \in F(u_o), w_0 \in G(u_0)$ and compute $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}, \{v_n\}$ and $\{w_n\}$ by the following iterative schemes:

$$g(u_{n+1}) = m(w_n) + J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_n) - m(w_n)) - \rho(N(x_n,y_n) - W(z_n,v_n) + m(w_n))\big], \quad (3.6)$$

$$x_{n+1} \in B(u_{n+1}),\ \|x_{n+1} - x_n\| \le H(B(u_{n+1}), B(u_n)), \quad (3.7)$$

$$y_{n+1} \in C(u_{n+1}),\ \|y_{n+1} - y_n\| \le H(C(u_{n+1}), C(u_n)), \quad (3.8)$$

$$z_{n+1} \in D(u_{n+1}),\ \|z_{n+1} - z_n\| \le H(D(u_{n+1}), D(u_n)), \quad (3.9)$$

$$v_{n+1} \in F(u_{n+1}),\ \|v_{n+1} - v_n\| \le H(F(u_{n+1}), F(u_n)), \quad (3.10)$$

$$w_{n+1} \in G(u_{n+1}),\ \|w_{n+1} - w_n\| \le H(G(u_{n+1}), G(u_n)), \quad (3.11)$$

$n = 0, 1, 2, 3......$, $\rho \in (0, \frac{r}{m})$ is a constant.

Theorem 3.1. Let E be a q-uniformly smooth Banach space and $\eta : E \times E \to E$ be Lipschitz continuous mapping with constant τ. Let $A : E \to E$ be r-strongly η-accretive and Lipschitz continuous mapping with constant $\lambda_A, m : E \to E$ be Lipschitz continuous mapping with constants λ_m and $M : E \times E \to 2^E$ be (A, η)-accretive mapping in the first argument such that $g(u) - m(w) \in \ \mathrm{dom}(M(\cdot, u))$, $\forall\ u, w \in E$. Suppose $N, W : E \times E \to E$ be Lipschitz continuous mappings in both arguments with constants λ_{N_1}, λ_{N_2}, λ_{W_1} and λ_{W_1}, respectively and B, C, D, F and $G : E \to CB(E)$ be H-Lipschitz continuous mappings with constants $\alpha, \beta, \gamma, \mu$ and δ, respectively. Let $g : E \to E$ be (b, ξ)-relaxed cocoercive, Lipschitz continuous mapping with constant λ_g and strongly accretive with constant l.

Suppose that there exist $\rho \in (0, \frac{r}{m})$ and $t > 0$ such that the following conditions hold:

$$\|J^{\rho,A}_{\eta,M(\cdot,u_n)}(x) - J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}(x)\| \le t\|u_n - u_{n-1}\|, \text{ for all } u_n, u_{n-1} \in E, \quad (3.12)$$

and

$$0 < \lambda_m\delta(\rho + \lambda_A) + \lambda_A \sqrt[q]{(1 - q\xi + (qb + C_q)\lambda_g{}^q} + \rho\sqrt[q]{(\lambda_{N_1}\alpha + \lambda_{N_2}\beta)^q - (q - C_q)(\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q}$$

$$+\lambda_A < \frac{[l - (\lambda_m\delta + t)](r - \rho m)}{\tau^{q-1}},\ l > (\lambda_m\delta + t),\ \frac{r}{\rho} > m, \quad (3.13)$$

where C_q is the constant as in Lemma 2.1, then the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$,$\{v_n\}$ and $\{w_n\}$ generated by Algorithm 3.1 converge strongly to u, x, y, z, v and w, respectively and (u, x, y, z, v, w) is a solution of problem (3.1).

Proof. From Algorithm 3.1, Proposition 2.1 and (3.12),we have

$$\begin{aligned}
\|g(u_{n+1}) - g(u_n)\| = \|&m(w_n) + J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_n) - m(w_n)) - \rho(N(x_n, y_n) \\
&- W(z_n, v_n) + m(w_n))\big] - \big\{m(w_{n-1}) + J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}\big[A(g(u_{n-1}) - m(w_{n-1})) \\
&- \rho(N(x_{n-1}, y_{n-1}) - W(z_{n-1}, v_{n-1}) + m(w_{n-1}))\big]\big\}\| \\
\le \|&m(w_n) - m(w_{n-1})\| + \|J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_n) - m(w_n)) - \rho(N(x_n, y_n) \\
&- W(z_n, v_n) + m(w_n))\big] - J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_{n-1}) - m(w_{n-1})) \\
&- \rho(N(x_{n-1}, y_{n-1}) - W(z_{n-1}, v_{n-1}) + m(w_{n-1}))\big]\| \\
&+ \|J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_{n-1}) - m(w_{n-1})) - \rho(N(x_{n-1}, y_{n-1}) - W(z_{n-1}, v_{n-1}) \\
&+ m(w_{n-1}))\big] - J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}\big[A(g(u_{n-1}) - m(w_{n-1})) - \rho(N(x_{n-1}, y_{n-1}) \\
&- W(z_{n-1}, v_{n-1}) + m(w_{n-1}))\big]\| \\
\le \|&m(w_n) - m(w_{n-1})\| + \frac{\tau^{q-1}}{r - \rho m}\big[\|A(g(u_n) - m(w_n)) - A(g(u_{n-1}) \\
&- m(w_{n-1})) - \rho\big\{(N(x_n, y_n) - W(z_n, v_n) + m(w_n) - (N(x_{n-1}, y_{n-1}) \\
&- W(z_{n-1}, v_{n-1}) - m(w_{n-1}))\big\}\|\big] + t\|u_n - u_{n-1}\| \\
\le (1 &+ \frac{\rho\tau^{q-1}}{r - \rho m})\|m(w_n) - m(w_{n-1})\| + \frac{\tau^{q-1}}{r - \rho m}\|A(g(u_n) - m(w_n)) \\
&- A(g(u_{n-1}) + m(w_{n-1}))\| + \frac{\tau^{q-1}}{r - \rho m}\rho\|N(x_n, y_n) \\
&- N(x_{n-1}, y_{n-1}) - (W(z_n, v_n)) - W(z_{n-1}, v_{n-1}))\| + t\|u_n - u_{n-1}\|. \qquad (3.14)
\end{aligned}$$

Since A is λ_A-Lipschitz continuous, we have

$$\begin{aligned}
\|g(u_{n+1}) - g(u_n)\| \le \big[1 &+ \frac{\tau^{q-1}}{r - \rho m}(\rho + \lambda_A)\big]\|m(w_n) - m(m_{n-1})\| \\
&+ \frac{\tau^{q-1}}{r - \rho m}\lambda_A\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \\
&+ \frac{\tau^{q-1}}{r - \rho m}\rho\|N(x_n, y_n) - N(x_{n-1}, y_{n-1})
\end{aligned}$$

$$
\begin{aligned}
&- (W(z_n, v_n) - W(z_{n-1}, v_{n-1}))\| \\
&+ \Big(\frac{\tau^{q-1}}{r-\rho m}\lambda_A + t\Big)\|u_n - u_{n-1}\|. \qquad (3.15)
\end{aligned}
$$

Since m is Lipschitz continuous with constant λ_m and G is H-Lipschitz continuous with constant δ, we have

$$
\begin{aligned}
\|m(w_n) - m(w_{n-1})\| \leq \lambda_m\|w_n - w_{n-1}\| &\leq \lambda_m H(G(u_n), G(u_{n-1})) \\
&\leq \lambda_m\delta\|u_n - u_{n-1}\|. \qquad (3.16)
\end{aligned}
$$

Since g is (b,ξ)-relaxed cocoercive and λ_g-Lipschitz continuous, we have

$$
\begin{aligned}
&\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\|^q \\
&\quad\leq \|u_n - u_{n-1}\|^q - q\langle g(u_n) - g(u_{n-1}), j_q(u_n - u_{n-1})\rangle + C_q\|g(u_n) - g(u_{n-1})\|^q \\
&\quad\leq \|u_n - u_{n-1}\|^q + qb\|g(u_n) - g(u_{n-1})\|^q - q\xi\|u_n - u_{n-1}\|^q + C_q\lambda_g^q\|u_n - u_{n-1}\|^q \\
&\quad\leq \|u_n - u_{n-1}\|^q + qb\lambda_g^q\|u_n - u_{n-1}\|^q - q\xi\|u_n - u_{n-1}\|^q + C_q\lambda_g^q\|u_n - u_{n-1}\|^q \\
&\quad= (1 - q\xi + (qb + C_q)\lambda_g^q)\|u_n - u_{n-1}\|^q.
\end{aligned}
$$

Thus, we have

$$
\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \leq \sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q}\|u_n - u_{n-1}\|. \qquad (3.17)
$$

Also

$$
\begin{aligned}
&\|N(x_n, y_n) - N(x_{n-1}, y_{n-1}) - (W(z_n, v_n) - W(z_{n-1}, v_{n-1}))\|^q \\
&\quad\leq \|N(x_n, y_n) - N(x_{n-1}, y_{n-1})\|^q - (q - C_q)\|W(z_n, v_n) - W(z_{n-1}, v_{n-1})\|^q. \qquad (3.18)
\end{aligned}
$$

By using Lipschitz continuity of N with constant λ_{N_1} for the first argument and λ_{N_2} for the second argument and H-Lipschitz continuity of B and C with constant α and β, respectively, we have

$$\begin{aligned}\|N(x_n,y_n)-N(x_{n-1},y_{n-1})\| &= \|N(x_n,y_n)-N(x_n,y_{n-1})+N(x_n,y_{n-1})\\ &\quad -N(x_{n-1},y_{n-1})\|\\ &\le \|N(x_n,y_n)-N(x_n,y_{n-1})\|+\|N(x_n,y_{n-1})\\ &\quad -N(x_{n-1},y_{n-1})\|\\ &\le \lambda_{N_2}\|y_n-y_{n-1}\|+\lambda_{N_1}\|x_n-x_{n-1}\|\\ &\le \lambda_{N_2}\beta\|u_n-u_{n-1}\|+\lambda_{N_1}\alpha\|u_n-u_{n-1}\|\\ &= (\lambda_{N_1}\alpha+\lambda_{N_2}\beta)\|u_n-u_{n-1}\|. \end{aligned} \tag{3.19}$$

Thus

$$\|N(x_n,y_n)-N(x_{n-1},y_{n-1})\|^q \le (\lambda_{N_1}\alpha+\lambda_{N_2}\beta)^q\|u_n-u_{n-1}\|^q. \tag{3.20}$$

Using the similar arguments as for (3.19), we have

$$\|W(z_n,v_n)-W(z_{n-1},v_{n-1})\|^q \le (\lambda_{W_1}\gamma+\lambda_{W_2}\mu)^q\|u_n-u_{n-1}\|^q. \tag{3.21}$$

Using (3.19) and (3.20), (3.17) becomes

$$\begin{aligned}&\|N(x_n,y_n)-N(x_{n-1},y_{n-1})-[W(z_n,v_n)-W(z_{n-1},v_{n-1})]\|^q\\ &\qquad = \left[(\lambda_{N_1}\alpha+\lambda_{N_2}\beta)^q-(q-C_q)(\lambda_{W_1}\gamma+\lambda_{W_2}\mu)^q\right]\|u_n-u_{n-1}\|^q. \end{aligned} \tag{3.22}$$

It follows that

$$\begin{aligned}&\|N(x_n,y_n)-N(x_{n-1},y_{n-1})-(W(z_n,v_n)-W(z_{n-1},v_{n-1}))\|\\ &\qquad \le \sqrt[q]{(\lambda_{N_1}\alpha+\lambda_{N_2}\beta)^q-(q-C_q)(\lambda_{W_1}\gamma+\lambda_{W_2}\mu)^q}\|u_n-u_{n-1}\|^q. \end{aligned} \tag{3.23}$$

Combining (3.16),(3.17),(3.23) with (3.15), we obtain

$$\begin{aligned}\|g(u_{n+1})-g(u_n)\| &\le (\lambda_m\delta+\frac{\tau^{q-1}\lambda_m\delta}{r-\rho m}(\rho+\lambda_A))\|u_n-u_{n-1}\|\\ &\quad +\frac{\tau^{q-1}\rho}{r-\rho m}\lambda_A\sqrt[q]{(1-q\xi+(qb+C_q)\lambda_g{}^q}\|u_n-u_{n-1}\|\\ &\quad +\frac{\tau^{q-1}\rho}{r-\rho m}\sqrt[q]{(\lambda_{N_1}\alpha+\lambda_{N_2}\beta)^q-(q-C_q)(\lambda_{W_1}\gamma+\lambda_{W_2}\mu)^q}\|u_n-u_{n-1}\|\\ &\quad +(\frac{\tau^{q-1}\lambda_A}{r-\rho m}+t)\|u_n-u_{n-1}\| \end{aligned}$$

$$\begin{aligned} &= \Bigg[\lambda_m\delta + \frac{\tau^{q-1}\lambda_m\delta}{r-\rho m}(\rho+\lambda_A) + \frac{\tau^{q-1}\rho}{r-\rho m}\lambda_A\sqrt[q]{(1-q\xi+(qb+C_q)\lambda_g{}^q} \\ &\quad + \frac{\tau^{q-1}}{r-\rho m}\rho\sqrt[q]{(\lambda_{N_1}\alpha+\lambda_{N_2}\beta)^q-(q-C_q)(\lambda_{W_1}\gamma+\lambda_{W_2}\mu)^q} \\ &\quad + \frac{\tau^{q-1}}{r-\rho m}\lambda_A + t\Bigg]\|u_n-u_{n-1}\|. \end{aligned} \tag{3.24}$$

By the strong accretivity of g with constant l, we have

$$\|g(u_{n+1})-g(u_n)\|.\|u_{n+1}-u_n\|^{q-1} \geq \langle g(u_{n+1})-g(u_n), j_q(u_{n+1}-u_n)\rangle \geq l\|u_{n+1}-u_n\|^q,$$

which implies that

$$\|u_{n+1}-u_n\| \leq \frac{1}{l}\|g(u_{n+1})-g(u_n)\|. \tag{3.25}$$

Combining (3.24) and (3.25), we have

$$\|u_{n+1}-u_n\| \leq \theta\|u_n-u_{n-1}\|, \tag{3.26}$$

where $\theta = \Big[\lambda_m\delta+\frac{\tau^{q-1}\lambda_m\delta}{r-\rho m}(\rho+\lambda_A)+\frac{\tau^{q-1}\rho}{r-\rho m}\lambda_A\sqrt[q]{(1-q\xi+(qb+C_q)\lambda_g{}^q}$
$+\frac{\tau^{q-1}}{r-\rho m}\rho\sqrt[q]{(\lambda_{N_1}\alpha+\lambda_{N_2}\beta)^q-(q-C_q)(\lambda_{W_1}\gamma+\lambda_{W_2}\mu)^q}+\frac{\tau^{q-1}}{r-\rho m}\lambda_A+t\Big]\Big/l.$

By (3.13), we know that $\theta < 1$ and so (3.26) implies that $\{u_n\}$ is a Cauchy sequence. Thus, there exists $u \in E$ such that $u_n \to u$ as $n \to \infty$. The Lipschitz continuity of multi-valued mappings B, C, D, F and G implies that $x_n \to x$, $y_n \to y$, $z_n \to z$, $v_n \to v$ and $w_n \to w$.

As $A, \eta, M, N, W, B, C, D, F, G, m, g$ and $J^{\rho,A}_{\eta,M}$ are all continuous and by Algorithm 3.1, it follows that u, x, y, z, v, w satisfy the following relation:

$$g(u) = m(w) + J^{\rho,A}_{\eta,M(\cdot,u)}\big[A(g(u)-m(w))-\rho(N(x,y)-W(z,v)+m(w))\big].$$

It follow that (u, x, y, z, v, w) is a solution of generalized quasi-variational-like inclusion problem (3.1). This completes the proof.

4. Applications

(1) We show that generalized quasi- variational-like inclusion problem (3.1) includes optimization problem.

If $E = X$, a Hilbert space ; $B, C, D, F, G, W, m = 0; g = I$, identity mapping; $N : X \to 2^X$ is a multi-valued mapping and $M(u,u) = M(u) = \partial\varphi(.)$, where $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper function and $\partial\varphi$ denotes the η-subdifferential of φ. If in addition $\partial\varphi = \delta_k$, the indicator function on a nonempty closed convex set $K \subset X$, then generalized quasi- variational-like inclusion problem (3.1) reduces to the problem finding $u \in K, \xi \in N(u)$ such that

$$\langle \xi, \eta(a,u) \rangle \geq 0, \text{ for all } a \in K. \tag{4.1}$$

Let $\psi : K \to \mathbb{R}$ be a given function, then the optimization problem is to find $u \in K$ such that

$$\psi(a) - \psi(u) \geq 0, \text{ for all } a \in K. \tag{4.2}$$

By using the definition of Clarke generalized subdifferential of ψ and invexity, the equivalence of (4.1) and (4.2) can be shown easily. See for example, Ansari and Yao [3] and references therein.

(2) The resolvent operator technique is very important from the point of view that it is used to established an equivalence between variational inequalities and resolvent equations. The resolvent equations are used to develop powerful and numerical techniques for solving variational inequalities and related optimization problems. Due to this fact, here we show that generalized quasi-variational-like inclusion problem (3.1) is equivalent to an A-resolvent equation. We consider the following problem.

Find $s, u \in E$, $x \in B(u), y \in C(u), z \in D(u), v \in F(u)$ and $w \in G(u)$ such that

$$N(x,y) - W(z,u) + m(w) + \rho^{-1} R^{\rho,A}_{\eta,M(.,u)}(s) = 0, \tag{4.3}$$

where $R^{\rho,A}_{\eta,M(.,u)} = I - A(J^{\rho,A}_{\eta,M(.,u)}), A\big[J^{\rho,A}_{\eta,M(.,u)}(s)\big] = \big[A(J^{\rho,A}_{\eta,M(.,u)})\big](s)$ and I is the identity operator, $J^{\rho,A}_{\eta,M(.,u)}$ is the resolvent operator and $\rho \in (0, \frac{r}{m})$ is a constant. We call (4.3) as A-resolvent equation, which is new and different from those resolvent equations given in the literature.

Proposition 4.1. The generalized quasi-variational-like inclusion problem (3.1) has a solution (u,x,y,z,v,w) with $u\in E, x\in B(u), y\in C(u), z\in D(u), v\in F(u)$ and $w\in G(u)$ if and only if A-resolvent equation problem (4.3) has a solution (s,u,x,y,z,v,w) with $s,u\in E, x\in B(u), y\in C(u), z\in D(u), v\in F(u)$ and $w\in G(u)$ where

$$g(u)=m(w)+J^{\rho,A}_{\eta,M(.,u)}(s) \tag{4.4}$$

and

$$s=A(g(u)-m(w))-\rho\{N(x,y)-W(z,v)+m(w)\}, \tag{4.5}$$

where $\rho\in(0,\frac{r}{m})$ is a constant.

Proof. Let u,x,y,z,v and w is a solution of problem (3.1). Then by Lemma 3.1, it is the solution of following equation

$$g(u)=m(w)+J^{\rho,A}_{\eta,M(.,u)}\big[A(g(u)-m(w))-\rho\{N(x,y)-W(z,v)+m(w)\}\big]. \tag{4.6}$$

Let $s=A(g(u)-m(w))-\rho\{N(x,y)-W(z,v)+m(w)\}$, then above equation (4.6), becomes

$$g(u)=m(w)+J^{\rho,A}_{\eta,M(.,u)}(s),$$

using the fact that $R^{\rho,A}_{\eta,M(.,u)}=I-A(J^{\rho,A}_{\eta,M(.,u)})$, where $A\big[J^{\rho,A}_{\eta,M(.,u)}(s)\big]=\big[A(J^{\rho,A}_{\eta,M(.,u)})\big](s)$, we obtain

$$\begin{aligned}
s=A(m(w)+J^{\rho,A}_{\eta,M(.,u)}(s)-m(w))-\rho\{N(x,y)-W(z,v)+m(w)\}\\
\Leftrightarrow s-A(J^{\rho,A}_{\eta,M(.,u)}(s))=-\rho\{N(x,y)-W(z,v)+m(w)\}\\
\Leftrightarrow \big[I-A(J^{\rho,A}_{\eta,M(.,u)})\big](s)=-\rho\{N(x,y)-W(z,v)+m(w)\}\\
\Leftrightarrow R^{\rho,A}_{\eta,M(.,u)}(s)=-\rho\{N(x,y)-W(z,v)+m(w)\}.
\end{aligned}$$

Hence $N(x,y)-W(z,v)+m(w)+\rho^{-1}R^{\rho,A}_{\eta,M(.,u)}(s)=0.$

Concluding Remark:

In this paper, we have introduced generalized quasi- variational-like inclusion problem which contains many variational (-like), quasi variational (-like) inequalities as special cases. Based on Lemma 3.1, we develop a general frame work for an iterative algorithm approximating solution of problem (3.1) while discussing the convergence analysis for the iterative procedure.

In section 4, we have shown that problem (3.1) includes classical optimization problem. The equivalence of (3.1) and (4.1) is available vastly in literature. Further we have shown that the problem (3.1) is equivalent to an A-resolvent equation, which is useful to solve variational inequality problems and related optimization problems. Our results are new and different from those given in the literature and have less assumptions but in new general setting.

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