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GENERALIZED K_4-FUNCTION AND ITS APPLICATION IN SOLVING KINETIC EQUATION OF FRACTIONAL ORDER

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Abstract. This paper is devoted to introduce a new generalized K_4- function in terms of some special functions. The differ-integration of this function is also investigated. A method for deriving the solution of the generalized fractional kinetic equation in term of the generalized K_4- function defined, generalized M-Series $\overset{\alpha,\beta}{\underset{p,q,\mathrm{m,n}}{M}}(z)$ and generalized Mittag – Leffler function $E_{\alpha,\beta,q}^{\gamma,\delta,p}(z)$ is investigated. The applied method depends on the fractional differ-integral operator techniques.

Keywords: generalized K_4-function; generalized M-series; generalized Mittag Leffler function; fractional kinetic equation; fractional differintegral operator.

2010 AMS Subject Classification: 82B40.

1. Introduction

Fractional calculus is a field that deals with derivative and integral of arbitrary orders which almost used at every field of mathematics namely special functions. The Mittag – Leffler function has gained importance during the last century due to its applications in the solution of fractional order differential and integral equations, that function is introduced by Mittag – Leffler [8] in terms of power series

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$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \qquad \alpha>0 \tag{1.1}$$

Many authors defined and studied in their research papers different generalization of Mittag – Leffler type function like $E_{\alpha,\beta}(z)$ defined by Wiman[25], $E_{\alpha,\beta}^{\gamma}(z)$ studied by Prabahaker [11], $E_{\alpha,\beta}^{\gamma,q}(z)$ introduced by Shukla and Prajapati [24] and $E_{\alpha,\beta}^{\gamma,\delta}(z)$ investigated by Salim [14].

A new generalization of Mittag – Leffler type function introduced by Salim and Faraj [15] as

$$E_{\alpha,\beta,q}^{\gamma,\delta,p}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_{pk}\quad z^k}{\Gamma(\alpha k+\beta)(\delta)_{qk}} \tag{1.2}$$

where $z,\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\gamma),\mathrm{Re}(\delta)\}>0$, $p,q>0$

The authors [1] introduced in a recent paper a new generalization of M-Series $\underset{p,q,\mathrm{m,n}}{\overset{\alpha,\beta}{M}}(z)$ as

$$\underset{p,q,\mathrm{m,n}}{\overset{\alpha,\beta}{M}}(z)=\sum_{k=0}^{\infty}\frac{(a_1)_{km},\ldots,(a_p)_{km}}{(b_1)_{kn},\ldots,(b_q)_{kn}}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{1.3}$$

where $z,\alpha,\beta\in\mathbb{C}$, m,n nonnegative real number, also of the parameter b_j,s is negative or zero.

The Series in (1.3) is defined depending on the M-Series $\underset{p,q}{\overset{\alpha}{M}}(z)$ introduced by Sharma [22] and its generalized M-Series $\underset{p,q}{\overset{\alpha,\beta}{M}}(z)$ studied by Sharma and Jain [23].

The new generalization of the M-Series (1.3) is interesting because the ${}_pF_q(z)$ hypergeometric function and generalized Mittag – Leffler function (1.2) follow as its particular cases [2,15].

The interest R and G -function defined by Lorenzo and hartely [5], [6], and their populanty have sharply increased in view of their importance role and applications in fractional calculus.

$$R_{\alpha,\beta}[a,c,z]=\sum_{k=0}^{\infty}\frac{(a)^k\quad(z-c)^{(k+1)\alpha-\beta-1}}{\Gamma(\alpha(k+1)-\beta)} \tag{1.4}$$

and

$$G_{\alpha,\beta,\gamma}[a,c,z]=\sum_{k=0}^{\infty}\frac{(\gamma)_k\quad(a)^k\quad(z-c)^{(k+\gamma)\alpha-\beta-1}}{k!\quad\Gamma((k+\gamma)\alpha-\beta)} \tag{1.5}$$

Recently Sharma [20] defined K_4-function

$$K_4^{(\alpha,\beta,\gamma),(a,c),(p,q)}(z)=\sum_{k=0}^{\infty}\frac{(a_1)_k,\ldots,(a_p)_k}{(b_1)_k,\ldots,(b_q)_k}\frac{(\gamma)_k\,(a)^k\,(z-c)^{(k+\gamma)\alpha-\beta-1}}{\Gamma((k+\gamma)\alpha-\beta)} \tag{1.6}$$

which is closely related to another special functions especially the R and G -function and M-Series defined by Sharma and Jain [23].

On the other hand, Fractional kinetic equation have gained importance due to their occurrence in science and engineering, the generalized fractional kinetic equation in term of Mittag – Leffler function studied by Sexcena, Mathai and Haubold [19], they introduced the solution of the generalized fractional kinetic equation associated with generalized Mittag – Leffler function and the R -function, for more result one can refer to the work of Sharma [21], Saichev and Zaslavsky [13], Sexcena [18], Zaslavsky [26], and Sexcena, Kalla [17].

Recently, Gupta and Parihar [3], introduced an alternative method for solving generalized fractional kinetic equation involving the generalized functions for the fractional calculus based on fractional differ-integral operator technique which differ from Laplace transform operator method.

This paper is divided to:

- Define a new generalized K_4-function and its relation to the other special functions.
- Investigate the differ-integration properity of the new function.
- Solving Fractional and general fractional kinetic equation in terms of the new generalized K_4-function, the generalized M-Series and the generalized Mittag – Leffler function.

2. A new special function

The generalized K_4-function introduced by the authors is defined as follows.

$$K_{4(m,n)}^{(\alpha,\beta,\gamma);(a,c),(p,q)}(a_1,\ldots,a_p,b_1,\ldots,b_a;z)=K_{4(m,n)}^{(\alpha,\beta,\gamma),(a,c),(p,q)}(z)$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_{km},\ldots,(a_p)_{km}}{(b_1)_{kn},\ldots,(b_q)_{kn}}\frac{(\gamma)_k\,(a)^k\,(z-c)^{(k+\gamma)\alpha-\beta-1}}{k!\;\Gamma((k+\gamma)\alpha-\beta)} \tag{2.1}$$

where $\mathrm{Re}(\alpha\gamma-\beta)>0$ and $(a_i)_k$ $(i=1,2,...,p)$ and $(b_j)_k$ $(j=1,2,...,q)$ are the Pochhammer symbols.

The series (2.1) is defined when non of the parameters b_j 's is a negative integer or zero. If any numerator parameter a_i is anegative integer or zero, then the series terminate to a polynomial of z .

From the ratio test it is evident that the series is convergent for all z if $pm<qn+\mathrm{Re}(\alpha)$, also when $pm=qn+\mathrm{Re}(\alpha)$ it is convergent in some cases, let $\xi=\sum_{j=1}^{pm}a_j-\sum_{j=1}^{qn}b_j$. It can be shown that when $pm=qn+\mathrm{Re}(\alpha)$, the series is absolutely convergent for $|z|=1$ if $\mathrm{Re}(\xi)<0$, conditionally convergent for $z=-1$ if $0\le\mathrm{Re}(\xi)<1$ and divergent for $|x|=1$ if $\mathrm{Re}(\xi)\ge 1$.

Relation with another special functions:

(i) Putting $m=n=1$ in the generalized K_4-function (2.1) becomes the K_4-function (1.6) defined by Sharma [21]

$$K_{4(1,1)}^{(\alpha,\beta,\gamma),(a,c),(p,q)}(z)=K_4^{(\alpha,\beta,\gamma),(a,c),(p,q)}(z) \tag{2.2}$$

(ii) When there is no upper and lower parameters of (2.1) , we get

$$K_{4(m,n)}^{(\alpha,\beta,\gamma),(a,c),(0,0)}(-,-;z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k\,(a)^k\,(z-c)^{(k+\gamma)\alpha-\beta-1}}{k!\;\Gamma((k+\gamma)\alpha-\beta)} \tag{2.3}$$

which reduces to the G -function (1.5) defined by Lorenzo and Hartley [5] devoted by $G_{\alpha,\beta,\gamma}[a,c,z]$.

(iii) If we put $\gamma=1$ in (2.3).

$$K_{4(m,n)}^{(\alpha,\beta,1),(a,c),(0,0)}(-,-;z)=\sum_{k=0}^{\infty}\frac{(a)^k\,(z-c)^{(k+1)\alpha-\beta-1}}{\Gamma((k+1)\alpha-\beta)} \tag{2.4}$$

which reduces to the R -function (1.4) defined by Lorenzo and Hartley [6] and denoted by $R_{\alpha,\beta}[a,c,z]$.

(iv) If we take $c=\beta=0$ in (2.4).

$$K_{4(m,n)}^{(\alpha,0,1),(a,0),(0,0)}(-,-;z)=\sum_{k=0}^{\infty}\frac{a^k\,z^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} \tag{2.5}$$

Which reduces to the F -function defined by Lorenzo and Hartley [5] and denoted by $F_\alpha[a,z]$.

Relation between generalized K_4-function and generalized M-Series.

Putting $\beta=\alpha-\beta,\ \ \gamma=1,\ a=1$ and $c=0$ in (2.1) we obtain

$$K_{4(m,n)}^{(\alpha,\alpha-\beta,1),(1,0),(p,q)}\left(a_1,...,a_p,b_1,...,b_q;z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{(1)^k\ (1)_k\ (z)^{(k+1)\alpha-(\alpha-\beta)-1}}{k!\ \ \Gamma\left((k+1)\alpha-(\alpha-\beta)\right)}$$

$$=z^{\beta-1}\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{z^{k\alpha}}{\Gamma(k\alpha+\beta)} \tag{2.6}$$

$$=z^{\beta-1}\underset{p,q,\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left(a_1,...,a_p,b_1,...,b_q,z^{\alpha}\right)$$

also if we put $p=q=1$ in (2.6) we get

$$K_{4(m,n)}^{(\alpha,\alpha-\beta,1),(1,0),(1,1)}\left(a_1,b_1;z\right)=z^{\beta-1}\sum_{k=0}^{\infty}\frac{(a_1)_{km}\ \left(z^{\alpha}\right)^k}{(b_1)_{kn}\ \Gamma(\alpha k+\beta)}=z^{\beta-1}E_{m,n,1}^{\alpha,\beta,1}\left(z^{\alpha}\right)$$

3. Differ-integration of generalized K_4-function

In this section we will derive the relation between generalized K_4-function and the operator of differ-integral given by Oldham and Spainer [9]. The relation is presented in the next theorem as follows:

Theorem 3.1:

Let $-\infty<r<\infty,\ \mathrm{Re}(\alpha\gamma-\beta)>0,\ z>c\geq 0$ and ${}_c d_z^r$ be the operator of differintegral given by Oldham and Spainer then the relation holds:

$${}_c d_z^r K_{4(m,n)}^{(\alpha,\beta,\gamma),(a,c),(p,q)}\left(a_1,...,a_p,b_1,...,b_q;z\right)=K_{4(m,n)}^{(\alpha,\beta+r,\gamma),(a,c),(p,q)}\left(a_1,a_2,...,a_p,b_1,...,b_q;z\right) \tag{3.1}$$

Proof:

The differ-integral operator defined by Oldham and Springer [9] of function $f(z)$ is given by

$${}_c d_z^r \beta(z)=\frac{d\ f\ (z)}{d\ (z-c)^r} \tag{3.2}$$

Using (2.1) and (3.2) we have

$${}_c d_z^r\{K_{4(m,n)}^{(\alpha,\beta,\gamma),(a,c),(p,q)}(z)\}={}_c d_z^r\{\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{(\gamma)_k\ (a)^k}{k!}\frac{(z-c)^{(k+\gamma)\alpha-\beta-1}}{\Gamma\left((k+\gamma)\alpha-\beta\right)}\}$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\dots(a_p)_{km}}{(b_1)_{kn}\dots(b_q)_{kn}}\frac{(\gamma)_k\,(a)^k}{k!\quad\Gamma((k+\gamma)\alpha-\beta)}\,{}_c d_z^r\,(z-c)^{(k+\gamma)\alpha-\beta-1}$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\dots(a_p)_{km}}{(b_1)_{kn}\dots(b_q)_{kn}}\frac{(\gamma)_k\,(a)^k}{k!\;\;\Gamma((k+\gamma)\alpha-\beta)}.$$

$$.((k+\gamma)\alpha-\beta-1)((k+\gamma)\alpha-\beta-2)\dots((k+\gamma)\alpha-\beta-r)(z-c)^{(k+\gamma)\alpha-\beta-r-1}$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\dots(a_p)_{km}}{(b_1)_{kn}\dots(b_q)_{kn}}\frac{(\gamma)_k\,(a)^k}{k!\;\;\Gamma((\alpha+k)\gamma-(\beta+r))}(z-c)^{(k+\gamma)\alpha-(\beta+r)-1}$$

$$=K_{4(m,n)}^{(\alpha,\beta+r,\gamma),(a,c),(p,q)}\left(a_1,a_2,\dots,a_p,b_1,\dots,b_q;z\right)$$

This shows that the differintegral of the generalized K_4-function is again a generalized K_4-function with indices $\beta+r$.

Particular case:

(1) ${}_c d_z^r\,G_{\alpha,\beta,\gamma}[a,c,z]=G_{\alpha,\beta+r,\gamma}[a,c,z]$

(2) ${}_c d_z^r\,R_{\alpha,\beta}[a,c,z]=R_{\alpha,\beta+r}[a,c,z]$

(3) ${}_0 d_z^r\,F_{\alpha}[a,0,z]=R_{\alpha+r}[a,0,z]$

4. Solving general fractional kinetic equation in terms of generalized K_4-function, generalized M-Series and generalized Mittag – Leffler function

The generalized Riemann – Liouville operators of fractional calculus [7,9] are defined as

$${}_a D_z^{-\nu} f(z)=\frac{1}{\Gamma(\nu)}\int_a^z (z-u)^{\nu-1} f(z)\,dz \qquad \mathrm{Re}(\nu)>0,\quad z>a \tag{4.1}$$

with

${}_a D_z^0 f(z)=f(z)$, and

$${}_a D_z^{\mu} f(z)=\frac{d^k}{d\,z^k}\left({}_a D_z^{\mu-k} f(z)\right) \qquad \mathrm{Re}(\mu)>0,\quad k-\mu>0 \tag{4.2}$$

If $p(z)=(z-a)^{p}$, we have from [10]

$$ {}_aD_z^{-\nu}\,(z-a)^{p-1}=\frac{\Gamma(p)}{\Gamma(p+\nu)}(z-a)^{p-\nu-1} \tag{4.3}$$

where $\mathrm{Re}(\nu)>0,\ \ \mathrm{Re}(p)>0,\ \ z>a$.

On integrating the standard kinetic equation

$$\frac{d}{dt}N_i(t)=-c_i\,N_i(t) \qquad c_i>0 \tag{4.4}$$

Haubold and Mathai [4] derived that

$$N_i(t)-N_i(0)=-c_i\ {}_0D_t^{-1}N_i(t) \tag{4.5}$$

Where ${}_0D_t^{-1}$ is the standard Riemann integral operator, $N_i=N_i(t)$ is number density of species i , which is a function of time t and $N_i(0)=N_0$ is the number density of that species of time $t=o$. By dropping the index i and replacing he Riemann integral ${}_0D_t^{-1}$ operator by the fractional Riemann – Liouville operator ${}_0D_t^{-\nu}$, the kinetic equation (4.5) is reduced to

$$N(t)-N_0=-c^{\nu}\ {}_0D_t^{-\nu}\,N(t) \tag{4.6}$$

Multiplying both side of(4.6) by $\left(-c^{\nu}\right)^{r}\ {}_0D_t^{-r\nu}$ and taking the sum over r from 0 to ∞ yields

$$\sum_{r=0}^{\infty}\left(-c^{\nu}\right)^{r}\ {}_0D_t^{-r\nu}\ N(t)-\sum_{r=0}^{\infty}\left(-c^{\nu}\right)^{r+1}\ {}_0D_t^{-(r+1)\nu}\ N(t)=N_0\sum_{r=0}^{\infty}\left(-c^{\nu}\right)^{r}\ {}_0D_t^{-r\nu}\ 1$$

Replacing r by $(r-1)$ in the second sum of above equation and then cancelling the equal terms on the left hand side, and applying (4.3) after putting $p=1$ on the right hand side of above equation

$$N(t)=N_0\sum_{r=0}^{\infty}\left(-c^{\nu}\right)^{r}\ {}_0D_t^{-r\nu}\ (1)$$

Or

$$N(t)=N_0\sum_{r=0}^{\infty}\frac{\left(-c^{\nu}\right)^{r}(t)^{-r\nu}}{\Gamma(1+r\nu)}=N_0\sum_{r=0}^{\infty}\frac{\left(-c^{\nu}t^{\nu}\right)^{r}}{\Gamma(1+r\nu)}$$

Hence,

$$N(t)=N_0\,\underset{0,0}{\overset{\nu,1}{M}}\left(-,-;-c^{\nu}t^{\nu}\right)$$

Theorem 4.1:

If $a,\beta,\alpha,>0\ ;\ v>0$, then the solution of the general fractional kinetic equation

$$N(t)-N_0\, t^{\beta-1} E_{\alpha,\beta,n}^{a,b,m}\left(at^{\alpha}\right)=-c^{v}\ {}_0D_t^{-v}\ N(t) \tag{4.7}$$

is given by

$$N(t)=N_0\sum_{r=0}^{\infty}\left(-c^{v}\right)^{r} K_{4(m,n)}^{(\alpha,\alpha-\beta-rv,1),(a,0),(1,1)}(t) \tag{4.8}$$

Proof:

Multiplying both side of (4.7) by $\left(-c^{v}\right)^{r}\ {}_0D_t^{-rv}$ and taking the sum over r from 0 to ∞, we get

$$\sum_{r=0}^{\infty}\left(-c^{v}\right)^{r}\ {}_0D_t^{-rv}\ N(t)-\sum_{r=0}^{\infty}\left(-c^{v}\right)^{r+1}\ {}_0D_t^{-(r+1)v}\ N(t)$$

$$=N_0\sum_{r=0}^{\infty}\left(-c^{v}\right)^{r}\ {}_0D_t^{-rv}\ t^{\beta-1} E_{\alpha,\beta,n}^{a,b,m}\left(at^{\alpha}\right)$$

Replacing r by $(r-1)$ in the second sum of above equation and then cancelling the equal terms on the left hand side,

$$N(t)=N_0\sum_{r=0}^{\infty}\left(-c^{v}\right)^{r}\ {}_0D_t^{-rv}\ t^{\beta-1} E_{\alpha,\beta,n}^{a,b,m}\left(at^{\alpha}\right)$$

Now

$${}_0D_t^{-rv}\ t^{\beta-1} E_{\alpha,\beta,n}^{a,b,m}(at^{\alpha})=\sum_{k=0}^{\infty}\frac{(a)_{km}\ (a)^{k}}{(b)_{kn}\ \Gamma(\alpha k+\beta)}\ {}_0D_t^{-rv}\ t^{\alpha k+\beta-1}$$

and by applying (4.3), we get

$${}_0D_t^{-rv}\ t^{\beta-1} E_{\alpha,\beta,n}^{a,b,m}\left(at^{\alpha}\right)=\sum_{k=0}^{\infty}\frac{(a)_{km}\ (a)^{k}}{(b)_{kn}\ \Gamma(\alpha k+\beta)}\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta+rv)}t^{rv+\alpha k+\beta-1}$$

$$=\sum_{k=0}^{\infty}\frac{(a)_{km}\ (a)^{k}\ (1)_k\ t^{(k+1)\alpha-(\alpha-\beta-rv)-1}}{(b)_{kn}\ k!\ \Gamma\left(\alpha(k+1)-(\alpha-\beta-rv)\right)}$$

Then the solution become

$$N(t)=N_0\sum_{r=0}^{\infty}\left(-c^{v}\right)^{r} K_{4(m,n)}^{(\alpha,\alpha-\beta-rv,1),(a,0),(1,1)}(t)$$

Theorem4.2:

If $\alpha,\beta,a>0$ and $v>0$, then the solution of the general fractional kinetic equation

$$N(t) - N_0 t^{\beta-1} \underset{p,q,\mathrm{m,n}}{\overset{\alpha,\beta}{M}} \left(at^{\alpha}\right) = -c^{v} {}_0D_t^{-v} N(t) \tag{4.9}$$

is given by

$$N(t) = N_0 \sum_{r=0}^{\infty} \left(-c^{v}\right)^{r} K_{4(m,n)}^{(\alpha,\alpha-\beta-rv,1),(a,0),(p,q)}(t) \tag{4.10}$$

Proof:

Multiplying both side of (4.9) by $\left(-c^{v}\right)^{r} {}_0D_t^{-rv}$ and taking the sum over r from 0 to ∞, we get

$$\sum_{r=0}^{\infty} \left(-c^{v}\right)^{r} {}_0D_t^{-rv} N(t) - \sum_{r=0}^{\infty} (-c)^{r+1} {}_0D_t^{-(r+1)v} N(t)$$

$$= N_0 \sum_{r=0}^{\infty} \left(-c^{v}\right)^{r} {}_0D_t^{-rv} t^{\beta-1} \underset{p,q,\mathrm{m,n}}{\overset{\alpha,\beta}{M}} \left(at^{\alpha}\right)$$

Using the same technique of theorem (4.1) , $N(t)$ become

$$N(t) = N_0 \sum_{r=0}^{\infty} \left(-c^{v}\right)^{r} {}_0D_t^{-rv} t^{\beta-1} \underset{p,q,\mathrm{m,n}}{\overset{\alpha,\beta}{M}} \left(at^{\alpha}\right)$$

Now

$${}_0D_t^{-rv} t^{\beta-1} \underset{p,q,\mathrm{m,n}}{\overset{\alpha,\beta}{M}} \left(at^{\alpha}\right) = \sum_{k=0}^{\infty} \frac{(a_1)_{km} \cdots (a_p)_{km} (a)^k \;\; {}_0D_t^{-rv} \left(t^{\alpha k+\beta-1}\right)}{(b_1)_{kn} \cdots (b_q)_{kn} \;\; \Gamma(\alpha k+\beta)}$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_{km} \cdots (a_p)_{km}}{(b_1)_{kn} \cdots (b_q)_{kn}} \frac{(a)^k}{\Gamma(\alpha k+\beta)} \frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta+rv)} t^{rv+\alpha k+\beta-1}$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_{km} \cdots (a_p)_{km}}{(b_1)_{kn} \cdots (b_q)_{kn}} \frac{(1)_k \; (a)^k}{k\,!} \frac{t^{(k+1)\alpha-(\alpha-\beta-rv)-1}}{\Gamma\left((k+1)\alpha-(\alpha-\beta-rv)\right)}$$

$$= K_{4(m,n)}^{(\alpha,\alpha-\beta-rv,1),(a,0),(p,q)}(t)$$

So that the solution become

$$N(t) = N_0 \sum_{r=0}^{\infty} \left(-c^{v}\right)^{r} K_{4(m,n)}^{(\alpha,\alpha-\beta-rv,1),(a,0),(p,q)}(t).$$

Theorem 4.3:

If $a,c,\gamma,\alpha,\beta,a>0;\ v>0$ and $\mathrm{Re}(\gamma\alpha-\beta)>0$, then the equation

$$N(t) - N_0 K_{4(m,n)}^{(\alpha,\beta,\gamma);(a,b),(p,q)}(t) = -c^{v} {}_0D_t^{-v} N(t) \tag{4.11}$$

has the solution

$$N(t)=N_0\sum_{r=0}^{\infty}\left(-c^{v}\right)^{r}K_{4(m,n)}^{(\alpha,\beta-rv,\gamma);(a,b),(p,q)}(t) \tag{4.12}$$

Proof:

Repeating the process applied in the theorem (4.1), (4.2), we get

$$\sum_{r=0}^{\infty}\left(-c^{v}\right)^{r}{}_0D_t^{-rv}N(t)-\sum_{r=0}^{\infty}\left(-c^{v}\right)^{r+1}{}_0D_t^{-(r+1)v}N(t)$$

$$=N_0\sum_{r=0}^{\infty}\left(-c^{v}\right)^{r}{}_0D_t^{-rv}K_{4(m,n)}^{(\alpha,\beta,\gamma),(a,c),(p,q)}(t)$$

Hence $N(t)$ become

$$N(t)=N_0\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left(-c^{v}\right)^{r}(a_1)_{km}\ldots(a_p)_{km}}{(b_1)_{kn}\ldots(b_q)_{kn}\,k!}\frac{(\gamma)_k\,(a)^k\;{}_0D_t^{-rv}\;(t-b)^{(k+\gamma)\alpha-\beta-1}}{\Gamma\left((k+\gamma)\alpha-\beta\right)}$$

and by applying (4.3), the solution become

$$N(t)=N_0\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left(-c^{v}\right)^{r}(a_1)_{km}\ldots(a_p)_{km}}{(b_1)_{kn}\ldots(b_q)_{kn}\,k!}\frac{(\gamma)_k\,(a)^k}{\Gamma\left((k+\gamma)\alpha-\beta\right)}\frac{\Gamma\left((k+\gamma)\alpha-\beta\right)(t-b)^{(k+\gamma)\alpha-\beta+rv-1}}{\Gamma\left((k+\gamma)\alpha-\beta+rv\right)}$$

Or

$$N(t)=N_0\sum_{r=0}^{\infty}\left(-c^{v}\right)^{r}K_{4(m,n)}^{(\alpha,\beta-rv,\gamma),(a,c),(p,q)}(t).$$

Conflict of Interests

The author declares that there is no conflict of interests.

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