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L-NEIGHBORHOOD SYSTEMS, *L*-TOPOLOGIES AND *L*-UNIFORMITIES

A.A. RAMADAN[1], E.H. ELKORDY[1], YONG CHAN KIM[2,*]

[1]Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

[2]Department of Mathematics, Gangneung-Wonju National University,
Gangneung, Gangwondo 210-702, Korea

Abstract. In this paper, we study the relations among *L*-topology, *L*-neighborhood system and *L*-uniformity in complete residuated lattices. We give their examples.

Keywords: complete residuated lattices; *L*-neighborhood space; *L*-topologies; *L*-uniform spaces.

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1. Introduction

Quasi-uniformities have the following different approaches as follows the entourage approach of Lowen [2,10-14,17] , the uniform covering approach of Kotzé [13] and the unification approach of Hutton [6,9,19] based on the powersets of the form $L^{X^{L^X}}$.

Many researcher introduced the notion of fuzzy uniformities in unit interval [0,1] ([3,4,14,15]), complete distributive lattices ([9,13,17,19]), commutative unital quantales ([8,11,12]) and complete quasi-monoidal lattices ([6,8,18]).

*Corresponding author

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In this paper, we study the relations among L-topology, L-neighborhood system and L-uniformity as extensions of Lowen's definitions in complete residuated lattices. We give their examples.

2. Preliminaries

Definition 2.1. [1,7] An algebra $(L,\wedge,\vee,\odot,\to,\bot,\top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L=(L,\leq,\vee,\wedge,\top,\bot)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(C2) $(L,\odot,\top)$ is a commutative monoid;

(C3) $x\odot y\leq z$ iff $x\leq y\to z$ for $x,y,z\in L$.

An operator $^*: L\to L$ defined by $a^*=a\to 0$ is called a *strong negation* if $a^{**}=a$.

For $\alpha\in L,\lambda\in L^A$, we denote $(\alpha\to\lambda),(\alpha\odot\lambda),\alpha_A,\top_x,\top_x^*\in L^A$ as $(\alpha\to\lambda)(x)=\alpha\to\lambda(x)$, $(\alpha\odot\lambda)(x)=\alpha\odot\lambda(x)$, $\alpha_A(x)=\alpha$,

$$\top_x(y)=\begin{cases}\top, & \text{if } y=x,\\ \bot, & \text{otherwise},\end{cases}\quad \top_x^*(y)=\begin{cases}\bot, & \text{if } y=x,\\ \top, & \text{otherwise}.\end{cases}$$

In this paper, we assume that $(L,\vee,\wedge,\odot,\to,^*,\top,\bot)$ be a complete residuated lattice with a strong negation *.

Lemma 2.2. [1,7] Let $(L,\vee,\wedge,\odot,\to,^*,\top,\bot)$ be a complete residuated lattice with a strong negation *. For each $x,y,z,x_i,y_i\in L$, the following properties hold.

(1) If $y\leq z$, then $x\odot y\leq x\odot z$.

(2) If $y\leq z$, then $x\to y\leq x\to z$ and $z\to x\leq y\to x$.

(3) $x\to y=\top$ iff $x\leq y$.

(4) $x\to\top=\top$ and $\top\to x=x$.

(5) $x\odot y\leq x\wedge y$.

(6) $x\odot(\bigvee_{i\in\Gamma}y_i)=\bigvee_{i\in\Gamma}(x\odot y_i)$ and $(\bigvee_{i\in\Gamma}x_i)\odot y=\bigvee_{i\in\Gamma}(x_i\odot y)$.

(7) $x\to(\bigwedge_{i\in\Gamma}y_i)=\bigwedge_{i\in\Gamma}(x\to y_i)$ and $(\bigvee_{i\in\Gamma}x_i)\to y=\bigwedge_{i\in\Gamma}(x_i\to y)$.

(8) $\bigvee_{i\in\Gamma}x_i\to\bigvee_{i\in\Gamma}y_i\geq\bigwedge_{i\in\Gamma}(x_i\to y_i)$ and $\bigwedge_{i\in\Gamma}x_i\to\bigwedge_{i\in\Gamma}y_i\geq\bigwedge_{i\in\Gamma}(x_i\to y_i)$.

(9) $(x \to y) \odot x \leq y$ and $(x \to y) \odot (y \to z) \leq (x \to z)$.

(10) $x \to y \leq (y \to z) \to (x \to z)$ and $x \to y \leq (z \to x) \to (z \to y)$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.

(13) $x^* \to y^* = y \to x$ and $(x \to y)^* = x \odot y^*$.

(14) $y \to z \leq x \odot y \to x \odot z$.

Definition 2.3.[1,4,5,16] Let X be a set. A function $R : X \times X \to L$ is called an *L-prtial order* if it satisfies the following conditions:

(E1) reflexive if $R(x,x) = \top$ for all $x \in X$,

(E2) transitive if $R(x,y) \odot R(y,z) \leq R(x,z)$, for all $x,y,z \in X$,

(E3) if $R(x,y) = R(y,x) = \top$, then $x = y$.

Lemma 2.4. [4,5,16] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(\lambda,\mu) = \bigwedge_{x\in X} (\lambda(x) \to \mu(x)).$$

Then, for each $\lambda,\mu,\rho,\nu \in L^X$, and $\alpha \in L$, the following properties hold.

(1) S is an L-partial order on L^X.

(2) $\lambda \leq \mu$ iff $S(\lambda,\mu) = \top$,

(3) If $\lambda \leq \mu$, then $S(\rho,\lambda) \leq S(\rho,\mu)$ and $S(\lambda,\rho) \geq S(\mu,\rho)$ for each $\rho \in L^X$,

(4) $S(\lambda,\mu) \odot S(\nu,\rho) \leq S(\lambda \odot \nu, \mu \odot \rho)$.

Let $\phi : X \to Y$ be an ordinary mapping. Define $\phi^{\to} : L^X \to L^Y$ and $\phi^{\leftarrow} : L^Y \to L^X$ by

$$\phi^{\to}(\lambda)(y) = \bigvee_{\phi(x)=y} \lambda(x) \ for \ \lambda \in L^X,\ y \in Y,$$

$$\phi^{\leftarrow}(\mu)(x) = \mu(\phi(x)) = \mu \circ \phi(x) \ for \ \mu \in L^Y,$$

respectively.

Lemma 2.5. [5,16] Let $\phi : X \to Y$ be an ordinary mapping. Define $\phi^{\to} : L^X \to L^Y$ and $\phi^{\leftarrow} : L^Y \to L^X$ by

$$\phi^{\to}(\lambda)(y) = \bigvee_{\phi(x)=y} \lambda(x), \;\; \forall \lambda \in L^X,\, y \in Y,$$

$$\phi^{\leftarrow}(\mu)(x) = \mu(\phi(x)) = \mu \circ \phi(x), \;\; \forall \mu \in L^Y.$$

Then for $\lambda, \mu \in L^X$ and $\rho, \nu \in L^Y$,

$$S(\lambda, \mu) \le S(\phi^{\to}(\lambda), \phi^{\to}(\mu)),$$

$$S(\rho, \nu) \le S(\phi^{\leftarrow}(\rho), \phi^{\leftarrow}(\nu)),$$

and the equalities hold if ϕ is bijective.

Definition 2.6. [8] A map $\tau : L^X \to L$ is called an L-topology on X if it satisfies the following conditions.

(T1) $\perp_X, \top_X \in \tau$,

(T2) if $\lambda, \rho \in \tau$, then $\lambda \odot \mu \in \tau$,

(T3) If $\lambda_i \in \tau$ for each $i \in \Gamma$, then $\bigvee_i \lambda_{i \in \Gamma} \in \tau$.

An L-topology is called enriched if

(R) if $\lambda, \rho \in \tau$, then $\alpha \odot \lambda \in \tau$ for all $\alpha \in L$.

The pair (X, τ) is called an L-topological space.

Let (X, τ_1) and (Y, τ_2) be two L-topological spaces. A mapping $\phi : X \to Y$ is said to be L-continuous iff $\phi^{\leftarrow}(\lambda) \in \tau_1$ for each $\lambda \tau_2$.

Definition 2.7. [8] A map $N : X \to L^{L^X}$ is called an L-neighborhood system on X if N satisfies the following conditions

(N1) $N_x(\top_X) = \top$ and $N_x(0_X) = \perp$,

(N2) $N_x(\lambda \odot \mu) \ge N_x(\lambda) \odot N_x(\mu)$ for each $\lambda, \mu \in L^X$,

(N3) If $\lambda \le \mu$, then $N_x(\lambda) \le N_x(\mu)$,

(N4) $N_x(\lambda) \le \lambda(x)$ for all $\lambda \in L^X$,

(N5) $N_x(\lambda) \le \bigvee \{N_x(\mu) \mid \mu(y) \le N_y(\lambda),\, \forall\, y \in X\}$.

An L-neighborhood system is called stratified if

(R) $N_x(\alpha \odot \lambda) \geq \alpha \odot N_x(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

The pair (X,N) is called an L-neighborhood space.

Let (X,N) and (Y,M) be two L-neighborhood spaces. A mapping $\phi : X \to Y$ is said to be L-continuous at $x \in X$ iff $M_{\phi(x)}(\lambda) \leq N_x(\phi^{\leftarrow}(\lambda))$ for each $\lambda \in L^Y, \phi$ is L-continuous if it is L-continuous at every $x \in X$.

We define L-uniformity in a sense of Lowen.

Definition 2.8. [24] A map $U \subset L^{X\times X}$ is called an L-quasi-uniformity on X iff the following conditions are fulfilled

(QU1) $\top_{X\times X} \in U$,

(QU2) If $v \leq u$ and $v \in U$, then $u \in U$,

(QU3) For every $u, v \in U$, $u \odot v \in U$,

(QU4) If $u \in U$ then $\top_\triangle \leq u$ where

$$\top_\triangle(x,y) = \begin{cases} \top, & \text{if x = y} \\ \bot, & \text{if } x \neq y, \end{cases}$$

(QU5) For each $u \in U$, there exists $v \in U$ such that $v \circ v \leq u$ where

$$v \circ v(x,y) = \bigvee_{z\in X} v(x,z) \odot v(z,y),\ \forall\, x,y \in X.$$

An L-quasi-uniformity U on X is said to be stratified if

(S)If $u \in U$, then $\alpha \odot u \in U$.

An L-quasi-uniformity U on X is said to be L-uniformity if

(US)If $u \in U$, then $u^{-1} \in U$ where $u^{-1}(x,y) = u(y,x)$.

The pair (X,U) is called an L-uniform space.

Let (X,U) and (Y,V) be L-uniform spaces, and $\phi : X \to Y$ ba a mapping. Then ϕ is said to be L-uniformly continuous if $(\phi \times \phi)^{\leftarrow}(v) \in U$, for every $v \in V$.

3. L-neighborhood systems, L-topologies and L-uniformities

Theorem 3.1. Let (X,τ) be an L-topological space. Define a map $N^{\tau}: X \to L^{L^X}$ by

$$N^{\tau}_x(\lambda) = \bigvee\{\rho(x) \mid \rho \le \lambda,\ \rho \in \tau\}.$$

Then the following properties hold.

(1) (X,N^{τ}) is an L-neighborhood space.

(2) If τ is enriched, then N^{τ} is stratified and

$$N^{\tau}_x(\lambda) = \bigvee_{\rho\in\tau}(\rho(x)\odot S(\rho,\lambda).$$

Proof. (1) (N1) Since $\top_X, \perp \in \tau$, $N^{\tau}_x(\top_X) = \top$ and $N^{\tau}_x(\perp) = \perp$.

(N2)

$$\begin{aligned}
&N^{\tau}_x(\lambda)\odot N^{\tau}_x(\rho)\\
&= \left(\bigvee\{\lambda_1(x) \mid \lambda_1 \le \lambda,\ \lambda_1 \in \tau\}\right)\odot\left(\bigvee\{\rho_1(x) \mid \rho_1 \le \rho,\ \rho_1 \in \tau \ge s\}\right)\\
&\le \bigvee\{(\lambda_1\odot\rho_1)(x) \mid \lambda_1\odot\rho_1 \le \lambda\odot\rho,\ \lambda_1\odot\rho_1 \in \tau\}\\
&\le N^{\tau}_x(\lambda\odot\rho).
\end{aligned}$$

(N3-5) follow from the definition of N^{τ}.

(N6) Put $N^{\tau}_{-}(\lambda,r) = \bigvee\{\rho \mid \rho \le \lambda,\ \rho \in \tau\}$ with $N^{\tau}_{-}(x) = N^{\tau}_x$. Then $N^{\tau}_{-}(\lambda) \in \tau$. By (N3) and the definition of N^{τ},

$$N^{\tau}_x(N^{\tau}_{-}(\lambda)) = N^{\tau}_x(\lambda).$$

For $r > r_1$,

$$\begin{aligned}
N^{\tau}_x(\lambda) &= N^{\tau}_x(N^{\tau}_{-}(\lambda))\\
&\le \bigvee\{N^{\tau}_x(\rho) \mid \rho(y) \le N^{\tau}_y(\lambda)\}.
\end{aligned}$$

Thus (X,N^{τ}) is an L-neighborhood space.

(2)

$$\begin{aligned}
\alpha\odot N^{\tau}_x(\lambda) &= \alpha\odot\bigvee\{\rho \mid \rho \le \lambda,\ \rho\in\tau\}\\
&\le \bigvee\{\alpha\odot\rho \mid \alpha\odot\rho \le \alpha\odot\lambda,\ \alpha\odot\rho \in \tau\} \le N^{\tau}_x(\alpha\odot\lambda).
\end{aligned}$$

Put $\gamma(x) = \bigvee_{\rho\in\tau}(\rho(x)\odot S(\rho,\lambda))$. Let ρ with $\rho \le \lambda$ and $\rho \in \tau$. Then $\rho(x)\odot S(\rho,\lambda) = \rho(x)\odot\top = \rho(x)$. Thus $\rho(x) \le \gamma(x)$. Therefore $N^{\tau}_x(\lambda) \le \gamma(x)$.

Let $\rho(x)\odot S(\rho,\lambda)$ with $\rho\in\tau$. Since τ is enriched, $\rho\odot S(\rho,\lambda)\in\tau$ and $\rho(x)\odot S(\rho,\lambda)\leq \rho(x)\odot(\rho(x)\to\lambda(x))\leq\lambda(x)$. Then $\gamma(x)\leq N^{\tau}_x(\lambda)$.

Theorem 3.2. Let (X,N) be an L-neighborhood space. Define $\tau_N\subset L^X$ as follows

$$\tau_N=\{\lambda\in L^X\mid\lambda(x)=N_x(\lambda),\forall x\in X\}.$$

Then,

(1) τ_N is an L-topology on X,

(2) If N is stratified, then τ_N is an enriched L-topology.

(3) $N=N^{\tau_N}$.

(4) If (X,τ) is an L-topological space, then $\tau=\tau_{N^\tau}$.

Proof. (1) (T1) Since $N_x(\top_X)=\top$ and $N_x(\bot_X)=\bot$, we have $\top_X,\bot_X\in\tau_N$.

(T2) Let $\lambda,\rho\in\tau_N$. Since $N_x(\lambda\odot\rho)\geq N_x(\lambda)\odot N_x(\rho)=(\lambda\odot\rho)(x)$ and (N4), then $\lambda\odot\rho\in\tau_N$.

(T3) Let $\lambda_i\in\tau_N$ for all $i\in\Gamma$. Since $N_x(\bigvee_{i\in\Gamma}\lambda_i)\geq\bigvee_{i\in\Gamma}N_x(\lambda_i)=\bigvee_{i\in\Gamma}\lambda_i$ and (N4), then $\bigvee_{i\in\Gamma}\lambda_i\in\tau_N$.

(2) (R) Let $\lambda\in\tau_N$. Since $N_x(\alpha\odot\lambda)\geq\alpha\odot N_x(\lambda)=\alpha\odot\lambda(x)$ and (N4), then $\alpha\odot\lambda\in\tau_N$.

(3) Since $N_x(\lambda)\leq N_x(N_-(\lambda))\leq N_x(\lambda)$ from (N3) and (N5), $N_x(\lambda)\leq N_x(N_-(\lambda))$ for all $x\in X$. Since $N_-(\lambda)\in\tau$, by the definition of N^{τ_N}, $N_x(\lambda)\leq N^{\tau_N}_x(\lambda)$.

Since $N^{\tau_N}_x(\lambda)=\bigvee\{\rho_i(x)\mid\rho_i\leq\lambda,\ \rho_i\in\tau_N\}$ and $\rho_i(x)=N_x(\rho_i)$, then

$$\bigvee_i\rho_i(x)=\bigvee_i N_x(\rho_i)\leq N_x(N^{\tau_N}_-(\lambda))=N_x(\bigvee_i\rho_i)\leq\bigvee_i\rho_i(x).$$

Hence $N_x(N^{\tau_N}_-(\lambda))=N^{\tau_N}_x(\lambda)$. Since $N^{\tau_N}_-(\lambda)\leq\lambda$, by (N3),

$$N^{\tau_N}_x(\lambda)=N_x(N^{\tau_N}_-(\lambda))\leq N_x(\lambda).$$

Thus $N^{\tau_N}_x=N_x$ for all $x\in X$.

(4) Let $\lambda\in\tau_{N^\tau}$. Then $\lambda=N^{\tau}_-(\lambda)\in\tau$.

Let $\rho\in\tau$. Then $\rho(x)=N^{\tau}_x(\rho)$ for all $x\in X$. Then $\rho\in\tau_{N^\tau}$.

Theorem 3.3. $\phi:(X,\tau_X)\to(Y,\tau_Y)$ is L-continuous iff $\phi:(X,N^{\tau_X})\to(Y,N^{\tau_Y})$ is L-continuous.

Proof. ($\Rightarrow$) Since $\phi^{\leftarrow}(\rho) \in \tau_X$ for each $\rho \in \tau_Y$, we have

$$
\begin{aligned}
& N^{\tau_Y}_{\phi(x)}(\lambda) = \bigvee\{\rho(\phi(x)) \mid \rho \le \lambda, \rho \in \tau_Y\} \\
& = \bigvee\{\phi^{\leftarrow}(\rho)(x) \mid \phi^{\leftarrow}(\rho) \le \phi^{\leftarrow}(\lambda), \phi^{\leftarrow}(\rho) \in \tau_X\} \\
& \le N^{\tau_Y}_{\phi(x)}(\phi^{\leftarrow}(\lambda)).
\end{aligned}
$$

($\Leftarrow$) Let $\lambda \in \tau_Y$. Since $\tau_Y = \tau_{N^{\tau_Y}}$ from Theorem 3.2(4), $\lambda(\phi(x)) = N^{\tau_Y}_{\phi(x)}(\lambda) \le N^{\tau_Y}_x(\phi^{\leftarrow}(\lambda))$. Hence $\phi^{\leftarrow}(\lambda) \in \tau_Y$.

Theorem 3.4. Let (X,U) be an L-quasi uniform space. Define two maps $rN^U, lN^U : X \to L^{L^X}$ by

$$rN^U_x(\lambda) = \bigvee_{u \in U} S(u[x], \lambda), \ \forall\, \lambda \in L^X,\ x \in X,$$

$$lN^U_x(\lambda) = \bigvee_{u \in U} S(u[[x]], \lambda), \ \forall\, \lambda \in L^X,\ x \in X,$$

where $u[x](y) = u(y,x)$ and $u[[x]](y) = u(x,y)$.

Then

(1) (X, rN^U) is a stratified L-neighborhood space.

(2) (X, lN^U) is a stratified L-neighborhood space.

(3) $rN^U_x(\lambda) = \bigvee\ \{\ \rho(x) \mid u[\rho] \le \lambda \mid u \in U\ \} = \bigvee\{\rho(x) \odot S(u[\rho], \lambda) \mid u \in U\}$ where

$$u[\rho](x) = \bigvee_{y \in X} u(x,y) \odot \rho(y),$$

(4) $lN^U_x(\lambda) = \bigvee\ \{\ \rho(x) \mid u[[\rho]] \le \lambda \mid u \in U\ \} = \bigvee\{\rho(x) \odot S(u[[\rho]], \lambda) \mid u \in U\}$ where

$$u[[\rho]](x) = \bigvee_{y \in X} u(y,x) \odot \rho(y),$$

Proof. (1) (N1) For $u \in U$, by (QU4), $\top_\triangle \le u$. Then

$$
\begin{aligned}
rN^U_x(\bot_X) &= \textstyle\bigvee_{u \in U} S(u[x], \bot_X) \\
&\le \textstyle\bigvee_{u \in U} (u(x,x) \to \bot) = \bot.
\end{aligned}
$$

Hence $rN_x^U(\bot_X)=\bot$. Also, $rN_x^U(\top_X)=\top$, because

$$rN_x^U(\top_X)\geq \bigwedge_{y\in X}(\top_\triangle(x,y)\to \top_X(y))=\top.$$

(N2) By Lemma 2.4 (4) , we have

$$\begin{aligned}
&rN_x^U(\lambda)\odot rN_x^U(\mu)=\big(\bigvee_{u\in U}S(u[x],\lambda)\big)\odot\big(\bigvee_{v\in U}S(v[x],\mu)\big)\\
&=\bigvee_{u\odot v\in U}S(u[x],\lambda)\odot S(v[x],\mu)\leq \bigvee_{u\odot v\in U}S((u\odot v)[x],\lambda\odot\mu)\\
&\leq \bigvee_{w\in U}S(w[x],\lambda\odot\mu)=rN_x^U(\lambda\odot\mu).
\end{aligned}$$

(N3) By Lemma 2.4 (3), we have

$$\begin{aligned}
rN_x^U(\lambda)\ &=\bigvee_{u\in U}S(u[x],\lambda)\\
&\leq \bigvee_{u\in U}S(u[x],\mu)=rN_x^U(\mu).
\end{aligned}$$

(N4) For $u\in U$, by (QU4), $\top_\triangle\leq u$. We have

$$\begin{aligned}
rN_x^U(\lambda)\ &=\bigvee_{u\in U}\bigwedge_{y\in X}(u(y,x)\to\lambda(y))\\
&\leq \bigvee_{u\in U}(u(x,x)\to\lambda(x))\leq\lambda(x).
\end{aligned}$$

(N5)

$$\begin{aligned}
&rN_x^U(\lambda)=\bigvee_{u\in U}S(u[x],\lambda)\\
&=\bigvee_{u\in U}\bigwedge_{y\in X}(u(y,x)\to\lambda(y))\\
&\leq \bigvee_{v\in U}\bigwedge_{y\in X}((v\circ v)(y,x)\to\lambda(y))\\
&=\bigvee_{v\in U}\bigwedge_{y\in X}((\bigvee_{z\in X}v(z,x)\odot v(y,z))\to\lambda(y))\\
&=\bigvee_{v\in U}\bigwedge_{y\in X}\bigwedge_{z\in X}((v(z,x)\odot v(y,z))\to\lambda(y))\\
&\text{(by Lemma 2.2 (12))}\\
&=\bigvee_{v\in U}\bigwedge_{y\in X}\bigwedge_{z\in X}(v(z,x)\to(v(y,z)\to\lambda(y))\\
&=\bigvee_{v\in U}\bigwedge_{z\in X}(v(z,x)\to\bigwedge_{y\in X}(v(y,z)\to\lambda(y)).
\end{aligned}$$

Let $\rho(z)=\bigwedge_{y\in X}(v(y,z)\to\lambda(y))$. Then $\rho(z)\leq rN_x^U(\lambda)$ for all $z\in X$. Thus,

$$\begin{aligned}
rN_x^U(\lambda)\ &\leq \bigvee_{v\in U}\{\bigwedge_{z\in X}(v(z,x)\to\rho(z))\mid \rho(z)\leq N_z^U(\lambda)\}\\
&\leq \bigvee_v\big\{rN_x^U(\rho)\mid \rho(z)\leq N_z^U(\lambda)\big\}.
\end{aligned}$$

Thus, (X,rN^U) is an L-neighborhood space.

Since $\alpha\odot u[x](y)\odot S(u[x],\lambda)\leq\alpha\odot u[x](y)\odot(u[x](y)\to\lambda(y))\leq\alpha\odot\lambda(y)$, we have

$$\alpha\odot S(u[x],\lambda)\leq S(u[x],\alpha\odot\lambda).$$

Thus, rN^U is stratified from:

$$\begin{aligned}\alpha\odot rN^U_x(\lambda) &= \alpha\odot\bigvee_{u\in U}S(u[x],\lambda)=\bigvee_{u\in U}(\alpha\odot S(u[x],\lambda))\\ &\le\bigvee_{u\in U}(S(u[x],\alpha\odot\lambda))=rN^U_x(\alpha\odot\lambda).\end{aligned}$$

(2) It is similarly proved as (1).

(3) Put $\gamma=\bigvee\{\rho(x)\mid u[\rho]\le\lambda|u\in U\}$. We show that $rN^U_-=\gamma$ from the following statements.

Let $\rho=\bigwedge_{x\in X}(u(x,y)\to\lambda(x))$. Then

$$\begin{aligned}&u[\rho](z)=\bigvee_{y\in X}(u(z,y)\odot\rho(y))\\ &=\bigvee_{y\in X}(u(z,y)\odot(\bigwedge_{x\in X}(u(x,y)\to\lambda(x))))\\ &\le\bigvee_{y\in X}(u(z,y)\odot(u(z,y)\to\lambda(z)))\le\lambda(z).\end{aligned}$$

Hence $rN^U_-\le\gamma$.

Let $u[\rho](z)=\bigvee_{y\in X}(u(z,y)\odot\rho(y))\le\lambda(z)$. Then

$$\rho(y)\le\bigwedge_{z\in X}(u(z,y)\to\lambda(z)).$$

Hence $rN^U_-\ge\gamma$.

Put $\delta=\bigvee\{\rho(x)\odot S(u[\rho],\lambda)\mid u\in U\}$. We show that $\delta=\gamma$ from the following statements.

Let $\rho\in L^X$ with $u[\rho]\le\lambda$ and $u\in U$. Then $S(u[\rho],\lambda)=\top$. Hence $\rho(x)\odot S(u[\rho],\lambda)=\rho(x)\le\delta(x)$. So, $\gamma(x)\le\delta(x)$.

Let $\rho\odot S(u[\rho],\lambda)$ with $u\in U$. Since

$$\begin{aligned}&u[\rho\odot S(u[\rho],\lambda)](x)=\bigvee_{y\in X}(u(x,y)\odot\rho(y)\odot S(u[\rho],\lambda))\\ &=u[\rho](x)\odot S(u[\rho],\lambda)\le\lambda(x).\end{aligned}$$

we have $u[\rho\odot S(u[\rho],\lambda)]\le\lambda$. Then $\rho(x)\odot S(u[\rho],\lambda)\le\gamma(x)$. Thus, $\delta=\gamma$.

Theorem 3.5. Let (X,U) be an L-uniform space, (X,rN^U) and (X,lN^U) L-neighborhood spaces. Define $\tau^r_U,\tau^l_U\subset L^X$ as follows

$$\tau^r_U=\{\lambda\in L^X\mid\lambda(x)=rN^U_x(\lambda),\forall x\in X\},$$

$$\tau^l_U=\{\lambda\in L^X\mid\lambda(x)=lN^U_x(\lambda),\forall x\in X\}.$$

Then,

(1) τ_U^r is an enriched L-topology on X.

(2) τ_U^l is an enriched L-topology on X.

(3) $rN^U = N^{\tau_U^r}$.

(4) $lN^U = N^{\tau_U^l}$.

Proof. (1) (T1) Since $N_x^U(\top_X) = \top$ and $N_x^U(\bot_X) = \bot$, we have $\top_X, \bot_X \in \tau_U$.

(T2) Let $\lambda, \rho \in \tau_U$. Since $N_x^U(\lambda \odot \rho) \geq N_x^U(\lambda) \odot N_x^U(\rho) = (\lambda \odot \rho)(x)$ and (N4), then $\lambda \odot \rho \in \tau_U$.

(T3) Let $\lambda_i \in \tau_U$ for all $i \in \Gamma$. Since $N_x^U(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigvee_{i\in\Gamma} N_x^U(\lambda_i) = \bigvee_{i\in\Gamma} \lambda_i$ and (N4), then $\bigvee_{i\in\Gamma} \lambda_i \in \tau_U$.

(R) Let $\lambda \in \tau_U$. Since $N_x^U(\alpha \odot \lambda) \geq \alpha \odot N_x^U(\lambda) = \alpha \odot \lambda(x)$ and (N4), then $\alpha \odot \lambda \in \tau_U$.

(2) It is similarly proved as (1).

(3) Since $rN_x^U(\lambda) \leq rN_x^U(rN_-^U(\lambda)) \leq rN_x^U(\lambda)$ from (N3) and (N5), $rN_x^U(\lambda) = rN_x^U(rN_-^U(\lambda))$ for all $x \in X$. Since $rN_-^U(\lambda) \in \tau_U^r$, by the definition of $N^{\tau_U^r}$, $rN_x^U(\lambda) \leq N_x^{\tau_U^r}(\lambda)$.

Since $N^{\tau_U^r} = \bigvee\{\rho_i(x) \mid \rho_i \leq \lambda,\ \rho_i \in \tau_U^r\}$ and $\rho_i(x) = rN_x^U(\rho_i)$, then

$$\bigvee_i \rho_i(x) = \bigvee_i rN_x^U(\rho_i) \leq rN_x^U(N^{\tau_U^r}(\lambda)) = rN_x^U(\bigvee_i \rho_i) \leq \bigvee_i \rho_i(x).$$

Hence $rN_x^U(N^{\tau_U^r}(\lambda)) = N^{\tau_U^r}(\lambda)$. Since $N^{\tau_U^r}(\lambda) \leq \lambda$, by (N3), $N^{\tau_U^r}(\lambda) = rN_x^U(N^{\tau_U^r}(\lambda)) \leq rN_x^U(\lambda)$. So, $rN^U = N^{\tau_U^r}$.

(4) It is similarly proved as (3).

Theorem 3.6. If $\phi : (X,U) \to (Y,V)$ is L-quasi-uniformly continuous, then

(1) $\phi : (X, rN^U) \to (Y, rN^V)$ is L-continuous.

(2) $\phi : (X, lN^U) \to (Y, lN^V)$ is L-continuous.

(3) a map $\phi : (X, \tau_U^r) \to (Y, \tau_V^r)$ is L- continuous.

(4) a map $\phi : (X, \tau_U^l) \to (Y, \tau_V^l)$ is L- continuous.

Proof. (1) First we show that $\phi^{\leftarrow}(v[\phi(x)]) = (\phi \times \phi)^{\leftarrow}(v)[x]$ from

$$\begin{aligned} \phi^{\leftarrow}(v[\phi(x)])(z) &= v[\phi(x)](\phi(z)) = v(\phi(z), \phi(x)) \\ &= (\phi \times \phi)^{\leftarrow}(v)(z,x) = (\phi \times \phi)^{\leftarrow}(v)[x](z). \end{aligned}$$

Thus, by Lemma 2.5, we have

$$\begin{aligned} S(v[\phi(x)],\lambda) &\leq S(\phi^{\leftarrow}(v[\phi(x)]),\phi^{\leftarrow}(\lambda)) \\ &= S((\phi\times\phi)^{\leftarrow}(v)[x],\phi^{\leftarrow}(\lambda)). \end{aligned}$$

$$\begin{aligned} & rN^V_{\phi(x)}(\lambda) = \bigvee_{v\in V(v)} S(v[\phi(x)],\lambda) \leq \bigvee_{v\in V} S((\phi\times\phi)^{\leftarrow}(v)[x],\phi^{\leftarrow}(\lambda)) \\ & \leq \bigvee_{(\phi\times\phi)^{\leftarrow}(v)\in U} S((\phi\times\phi)^{\leftarrow}(v)[x],\phi^{\leftarrow}(\lambda)) \leq rN^U_x(\phi^{\leftarrow}(\lambda)). \end{aligned}$$

(2) It is similarly proved as (1).

(3) Let $\lambda\in\tau^r_V(\lambda)$. Then $\lambda = rN^V_-(\lambda)$. Then $\phi^{\leftarrow}(\lambda) = \phi^{\leftarrow}(rN^V_-(\lambda))$. Since $\phi^{\leftarrow}(rN^V_-(\lambda)) \leq rN^U_-(\phi^{\leftarrow}(\lambda))$, then $\phi^{\leftarrow}(\lambda) = \phi^{\leftarrow}(rN^V_-(\lambda)) \leq rN^U_-(\phi^{\leftarrow}(\lambda))$. By (N3), $\phi^{\leftarrow}(\lambda) = rN^U_-(\phi^{\leftarrow}(\lambda))$. Hence $\phi^{\leftarrow}(\lambda)\in\tau^r_U(\phi^{\leftarrow}(\lambda))$.

(4) It is similarly proved as (3).

Example 3.7. Let $(L=[0,1],\odot,\to)$ be a complete residuated lattice defined by

$$x\odot y = x\wedge y,\;\; x\to y = \begin{cases} 1, & \text{if } x\leq y, \\ y, & \text{otherwise.} \end{cases}$$

Let $X=\{x,y,z\}$ be a set and $w\in L^{X\times X}$ such that

$$w = \begin{pmatrix} 1 & 0.6 & 0.8 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.5 & 1 \end{pmatrix}.$$

Define $U=\{u\in L^{X\times X}\mid u\geq w\}$.

(1) Since $w\circ w = w$, U is an L- quasi-uniformity on X.

(2) Since $rN^U_x(\lambda) = \bigvee_{u\in U} S(u[x],\lambda)$, we have

$$\begin{aligned} rN^U_x(\lambda) &= \bigvee_{u\in U} S(u[x],\lambda) = \lambda(x)\wedge(0.4\to\lambda(y))\wedge(0.5\to\lambda(z)), \\ rN^U_y(\lambda) &= \bigvee_{u\in U} S(u[y],\lambda) = (0.6\to\lambda(x))\wedge\lambda(y)\wedge(0.5\to\lambda(z)), \\ rN^U_z(\lambda) &= \bigvee_{u\in U} S(u[z],\lambda) = (0.8\to\lambda(x))\wedge(0.4\to\lambda(y))\wedge\lambda(z). \end{aligned}$$

(3) Since $lN^U_x(\lambda) = \bigvee_{u\in U} S(u[[x]],\lambda)$, we have

$$lN_x^U(\lambda) = \bigvee_{u\in U} S(u[[x]],\lambda) = \lambda(x) \wedge (0.6 \to \lambda(y)) \wedge (0.8 \to \lambda(z)),$$
$$lN_y^U(\lambda) = \bigvee_{u\in U} S(u[[y]],\lambda) = (0.6 \to \lambda(x)) \wedge \lambda(y) \wedge (0.6 \to \lambda(z)),$$
$$lN_z^U(\lambda) = \bigvee_{u\in U} S(u[[z]],\lambda) = (0.5 \to \lambda(x)) \wedge (0.5 \to \lambda(y)) \wedge \lambda(z).$$

(3) Since $\tau_U^r = \{\lambda \in L^X \mid \lambda(x) = rN_x^U(\lambda), \forall x \in X\}$ from Theorem 3.5, we have

$$\lambda \in \tau_U^r \text{ iff } \begin{cases} \lambda = \alpha_X, \\ \lambda(x) \le 0.4 \to \lambda(y), \lambda(x) \le 0.5 \to \lambda(z), \\ \lambda(y) \le 0.6 \to \lambda(x), \lambda(y) \le 0.5 \to \lambda(z), \\ \lambda(z) \le 0.8 \to \lambda(x), \lambda(z) \le 0.4 \to \lambda(z), \end{cases}$$

$$\lambda \in \tau_U^l \text{ iff } \begin{cases} \lambda = \alpha_X, \\ \lambda(x) \le 0.6 \to \lambda(y), \lambda(x) \le 0.8 \to \lambda(z), \\ \lambda(y) \le 0.4 \to \lambda(x), \lambda(y) \le 0.4 \to \lambda(z), \\ \lambda(z) \le 0.5 \to \lambda(x), \lambda(z) \le 0.5 \to \lambda(z), \end{cases}$$

For $\lambda = (0.6, 0.5, 0.6), \lambda \in \tau^r_{rN^U}, \lambda \notin \tau^l_{lN^U}$,

$$\lambda = (0.1, 0.9, 0.5), \lambda \notin \tau^r_{rN^U}, \lambda \notin \tau^l_{lN^U},$$

$$\lambda = (0.5, 0.5, 0.6), \lambda \notin \tau^r_{rN^U}, \lambda \in \tau^l_{lN^U}.$$

Conflict of Interests

The author declares that there is no conflict of interests.

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