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THE FAMILIES OF *L*-NEIGHBORHOOD SYSTEMS, *L*-TOPOLOGIES AND *L*-QUASI-UNIFORMITIES

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Abstract. In this paper, we investigate the relations among the families of *L*-topology, *L*-neighborhood system and *L*-quasi-uniformity in complete residuated lattices. We give their examples.

Keywords: complete residuated lattices; *L*-neighborhood space; *L*-topologies; *L*-quasi-uniform spaces.

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1. Introduction

Quasi-uniformities have the following different approaches as follows the entourage approach of Lowen [2,10-14,17] , the uniform covering approach of Kotzé [13] and the unification approach of Hutton [6,9,20] based on the powersets of the form $L^{X^{L^X}}$.

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Many researcher introduced the notion of fuzzy uniformities in unit interval [0,1] ([3,4,14,15]), complete distributive lattices ([9,13,17,20]), commutative unital quantales ([8,11,12]) and complete quasi-monoidal lattices ([6,8,19]). Ramadan [18] investigated the relations among the families of L-topology, L-neighborhood system and L-quasi-uniformity in complete residuated lattices.

In this paper, we study the relations among the families of L-topology, L-neighborhood system and L-quasi-uniformity as extensions of Lowen's definitions in complete residuated lattices. We give their examples.

2. Preliminaries

Definition 2.1. [1,7] An algebra $(L,\wedge,\vee,\odot,\to,\perp,\top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L=(L,\leq,\vee,\wedge,\top,\perp)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;

(C2) $(L,\odot,\top)$ is a commutative monoid;

(C3) $x\odot y\leq z$ iff $x\leq y\to z$ for $x,y,z\in L$.

An operator $^*: L\to L$ defined by $a^*=a\to 0$ is called a *strong negation* if $a^{**}=a$.

For $\alpha\in L,\lambda\in L^A$, we denote $(\alpha\to\lambda),(\alpha\odot\lambda),\alpha_A,\top_x,\top_x^*\in L^A$ as $(\alpha\to\lambda)(x)=\alpha\to\lambda(x)$, $(\alpha\odot\lambda)(x)=\alpha\odot\lambda(x)$, $\alpha_A(x)=\alpha$,

$$\top_x(y)=\begin{cases}\top, & \text{if } y=x,\\ \perp, & \text{otherwise,}\end{cases}\quad \top_x^*(y)=\begin{cases}\perp, & \text{if } y=x,\\ \top, & \text{otherwise.}\end{cases}$$

In this paper, we assume that $(L,\vee,\wedge,\odot,\to,^*,\top,\perp)$ be a complete residuated lattice with a strong negation *.

Lemma 2.2. [1,7] *Let* $(L,\vee,\wedge,\odot,\to,^*,\top,\perp)$ *be a complete residuated lattice with a strong negation* **. For each* $x,y,z,x_i,y_i\in L$*, the following properties hold.*

(1) *If* $y\leq z$*, then* $x\odot y\leq x\odot z$*.*

(2) *If* $y\leq z$*, then* $x\to y\leq x\to z$ *and* $z\to x\leq y\to x$*.*

(3) $x\to y=\top$ *iff* $x\leq y$*.*

(4) $x \to \top = \top$ *and* $\top \to x = x$.

(5) $x \odot y \leq x \wedge y$.

(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.

(7) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ *and* $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

(9) $(x \to y) \odot x \leq y$ *and* $(x \to y) \odot (y \to z) \leq (x \to z)$.

(10) $x \to y \leq (y \to z) \to (x \to z)$ *and* $x \to y \leq (z \to x) \to (z \to y)$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ *and* $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ *and* $(x \odot y)^* = x \to y^*$.

(13) $x^* \to y^* = y \to x$ *and* $(x \to y)^* = x \odot y^*$.

(14) $y \to z \leq x \odot y \to x \odot z$.

Lemma 2.3. [4,5,16] *For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by*

$$S(\lambda,\mu) = \bigwedge_{x\in X} (\lambda(x) \to \mu(x)).$$

Then, for each $\lambda,\mu,\rho,\nu \in L^X$, and $\alpha \in L$, the following properties hold.

(1) $\lambda \leq \mu$ *iff* $S(\lambda,\mu) = \top$,

(2) *If* $\lambda \leq \mu$, *then* $S(\rho,\lambda) \leq S(\rho,\mu)$ *and* $S(\lambda,\rho) \geq S(\mu,\rho)$ *for each* $\rho \in L^X$,

(3) $S(\lambda,\mu) \odot S(\nu,\rho) \leq S(\lambda \odot \nu, \mu \odot \rho)$.

Definition 2.4. [8] A map $\tau : L^X \to L$ is called an L-topology on X if it satisfies the following conditions.

(T1) $\perp_X, \top_X \in \tau$,

(T2) if $\lambda,\rho \in \tau$, then $\lambda \odot \mu \in \tau$,

(T3) If $\lambda_i \in \tau$ for each $i \in \Gamma$, then $\bigvee_i \lambda_{i\in\Gamma} \in \tau$.

An L-topology is called enriched if

(R) if $\lambda,\rho \in \tau$, then $\alpha \odot \lambda \in \tau$ for all $\alpha \in L$.

The pair (X,τ) is called an L-topological space.

Let (X,τ_1) and (Y,τ_2) be two L-topological spaces. A mapping $\phi : X \to Y$ is said to be L-continuous iff $\phi^{\leftarrow}(\lambda) \in \tau_1$ for each $\lambda\,\tau_2$.

Definition 2.5. [8] A map $N : X \to L^{L^X}$ is called an L-neighborhood system on X if N satisfies the following conditions

(N1) $N_x(\top_X) = \top$ and $N_x(0_X) = \bot$,

(N2) $N_x(\lambda \odot \mu) \geq N_x(\lambda) \odot N_x(\mu)$ for each $\lambda, \mu \in L^X$,

(N3) If $\lambda \leq \mu$, then $N_x(\lambda) \leq N_x(\mu)$,

(N4) $N_x(\lambda) \leq \lambda(x)$ for all $\lambda \in L^X$,

(N5) $N_x(\lambda) \leq \bigvee\{N_x(\mu) \mid \mu(y) \leq N_y(\lambda),\ \forall\, y \in X\}$.

An L-neighborhood system is called stratified if

(R) $N_x(\alpha \odot \lambda) \geq \alpha \odot N_x(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

The pair (X, N) is called an L-neighborhood space.

Let (X, N) and (Y, M) be two L-neighborhood spaces. A mapping $\phi : X \to Y$ is said to be L-continuous at $x \in X$ iff $M_{\phi(x)}(\lambda) \leq N_x(\phi^{\leftarrow}(\lambda))$ for each $\lambda \in L^Y, \phi$ is L-continuous if it is L-continuous at every $x \in X$.

We define L-quasi-uniformity in a sense of Lowen [14].

Definition 2.6. [18] A map $U \subset L^{X\times X}$ is called an L-quasi-uniformity on X iff the following conditions are fulfilled

(QU1) $\top_{X\times X} \in U$,

(QU2) If $v \leq u$ and $v \in U$, then $u \in U$,

(QU3) For every $u, v \in U$, $u \odot v \in U$,

(QU4) If $u \in U$ then $\top_\triangle \leq u$ where

$$\top_\triangle(x,y) = \begin{cases} \top, & \text{if x = y} \\ \bot, & \text{if } x \neq y, \end{cases}$$

(QU5) For each $u \in U$, there exists $v \in U$ such that $v \circ v \leq u$ where

$$v \circ v(x,y) = \bigvee_{z \in X} v(x,z) \odot v(z,y),\ \forall\, x, y \in X.$$

An L-quasi-uniformity U on X is said to be stratified if

(S) if $u \in U$, then $\alpha \odot u \in U$.

An L-quasi-uniformity U on X is said to be L-uniformity if

(U) if $u \in U$, then $u^{-1} \in U$ where $u^{-1}(x,y) = u(y,x)$.

The pair (X,U) is called an L-uniform space.

Let (X,U) and (Y,V) be L-quasi-uniform spaces, and $\phi : X \to Y$ be a mapping. Then ϕ is said to be L-quasi-uniformly continuous if $(\phi \times \phi)^{\leftarrow}(v) \in U$, for every $v \in V$.

Theorem 2.7. [18] *Let (X,τ) be an L-topological space. Define a map $N^{\tau} : X \to L^{L^X}$ by*

$$N_x^{\tau}(\lambda) = \bigvee \{\rho(x) \mid \rho \leq \lambda,\ \rho \in \tau\}.$$

Then (X,N^{τ}) is an L-neighborhood space.

Theorem 2.8. [18] *Let (X,N) be an L-neighborhood space. Define $\tau_N \subset L^X$ as follows*

$$\tau_N = \{\lambda \in L^X \mid \lambda(x) = N_x(\lambda), \forall x \in X\}.$$

Then,

(1) τ_N is an L-topology on X such that $N = N^{\tau_N}$.

(2) If (X,τ) is an L-topological space, then $\tau = \tau_{N^{\tau}}$.

Theorem 2.9. [18] *$\phi : (X,\tau_X) \to (Y,\tau_Y)$ is L-continuous iff $\phi : (X,N^{\tau_X}) \to (Y,N^{\tau_Y})$ is L-continuous.*

Theorem 2.10. [18] *Let (X,U) be an L-quasi uniform space. Define two maps $rN^U, lN^U : X \to L^{L^X}$ by*

$$rN_x^U(\lambda) = \bigvee_{u \in U} S(u[x],\lambda),\ \forall\ \lambda \in L^X,\ x \in X,$$

$$lN_x^U(\lambda) = \bigvee_{u \in U} S(u[[x]],\lambda),\ \forall\ \lambda \in L^X,\ x \in X,$$

where $u[x](y) = u(y,x)$ and $u[[x]](y) = u(x,y)$.

Then (X,rN^U) and (X,lN^U) are stratified L-neighborhood spaces.

Theorem 2.11. [18] *Let (X,U) be an L-quasi uniform space, (X,rN^U) and (X,lN^U) L-neighborhood spaces. Define $\tau_U^r, \tau_U^l \subset L^X$ as follows*

$$\tau_U^r = \{\lambda \in L^X \mid \lambda(x) = rN_x^U(\lambda), \forall x \in X\},$$

$$\tau_U^l = \{\lambda \in L^X \mid \lambda(x) = lN_x^U(\lambda), \forall x \in X\}.$$

Then (1) τ_U^r *is an enriched L-topology on X such that* $rN^U = N^{\tau_U^r}$.

(2) τ_U^l *is an enriched L-topology on X such that* $lN^U = N^{\tau_U^l}$.

3. L-neighborhood systems, L-topologies and L-quasi-uniformities

Let $\lambda \in L^X$ and $\mu \in L^Y$, we define the product $\lambda \times \mu \in L^{X\times Y}$ as follows:

$$(\lambda \times \mu)(x,z) = \lambda(x) \odot \mu(z).$$

Theorem 3.1. *If* N_1 *and* N_2 *are L-neighborhood systems on X and Y respectively satisfying* $N_1(\lambda_1) \odot N_2(\lambda_2) = \perp$ *for each* $\lambda_1 \times \lambda_2 = \perp_{X\times Y}$*, then their product* $N_1 \times N_2$ *is an L-neighborhood system on* $X \times Y$ *defined by for all* $\mu \in L^{X\times Y}$,

$$(N_1 \times N_2)(\mu) = \bigvee\{N_1(\lambda_1) \odot N_2(\lambda_2) \mid \lambda_1 \times \lambda_2 \le \mu\}.$$

Proof. (LN1)

$$\begin{aligned}&(N_1 \times N_2)(1_{X\times Y}) = \bigvee\{N_1(1_X) \odot N_2(1_Y) \mid 1_X \times 1_Y \le 1_{X\times Y}\}\\&= \bigvee\{\top \odot \top \mid \top \odot \top \le \top\} = \top.\end{aligned}$$

Also, $(N_1 \times N_2)(0_{X\times Y}) = \perp$.

(LN2) For $\lambda_1, \mu_1 \in L^X$ and $\lambda_2, \mu_2 \in L^Y$, we can prove that

$$\begin{aligned}&(N_1 \times N_2)(\lambda) \odot (N_1 \times N_2)(\mu)\\&= \bigvee\{N_1(\lambda_1) \odot N_2(\lambda_2) \mid \lambda_1 \times \lambda_2 \le \lambda\} \odot \bigvee\{N_1(\mu_1) \odot N_2(\mu_2) \mid \mu_1 \times \mu_2 \le \mu\}\\&\le \bigvee\{N_1(\lambda_1) \odot N_1(\mu_1) \odot N_2(\lambda_2) \odot N_2(\mu_2) \mid (\lambda_1 \times \lambda_2) \odot (\mu_1 \times \mu_2) \le \lambda \odot \mu\}\\&\le \bigvee\{N_1(\lambda_1 \odot \mu_1) \odot N_2(\lambda_2 \odot \mu_2) \mid (\lambda_1 \odot \mu_1) \times (\lambda_2 \odot \mu_2) \le \lambda \odot \mu\}\\&\le \bigvee\{N_1(\rho) \odot N_2(\nu) \mid \rho \times \nu \le \lambda \odot \mu\} = (N_1 \times N_2)(\lambda \odot \mu).\end{aligned}$$

(LN3) and (LN4) are clearly true.

(LN5) Let $\lambda \in L^{X\times Y}$, $\lambda_1 \in L^X$ and $\lambda_2 \in L^Y$, then we have

$$
\begin{aligned}
&(N_1 \times N_2)(\lambda) = \bigvee\{N_1(\lambda_1) \odot N_2(\lambda_2) \mid \lambda_1 \times \lambda_2 \leq \lambda\} \\
&\leq \bigvee\{\bigvee\{N_x(\mu_1) \mid \mu_1 \leq N_y(\lambda_1)\} \odot \bigvee\{N_z(\mu_2) \mid \mu_2 \leq N_w(\lambda_2)\} \mid \lambda_1 \times \lambda_2 \leq \lambda\} \\
&\leq \bigvee\{\bigvee\{N_x(\mu_1) \odot N_z(\mu_2) \mid \mu_1 \odot \mu_2 \leq N_y(\lambda_1) \odot N_w(\lambda_2)\}, \lambda_1 \times \lambda_2 \leq \lambda\} \\
&= \bigvee\{\bigvee\{N_x(\mu_1) \odot N_z(\mu_2) \mid \mu_1(y) \times \mu_2(w) \leq N_y(\lambda_1) \odot N_w(\lambda_2)\},\ \lambda_1 \times \lambda_2 \leq \lambda\} \\
&\leq \bigvee\{N_x \odot N_z(\mu_1 \times \mu_2) \mid \mu_1 \times \mu_2(y,w) \leq \bigvee\{N_y \odot N_w(\lambda_1 \times \lambda_2)\}\} \\
&\leq \bigvee\{N_1 \odot N_2(\mu) \mid \mu \leq N_1 \odot N_2(\lambda)\}.
\end{aligned}
$$

Theorem 3.2. *Let (X,N) be an L-neighborhood space. Then $\phi : (X,N) \to (X \times X, N \times N)$ defined by $\phi(x) = (x,x)$ is L-continuous.*

Proof. If $x \in X$ and $\lambda \in L^{X \times X}$, then we have

$$
\begin{aligned}
&(N \times N)_{\phi(x)}(\lambda) = \bigvee\{N_x(\lambda_1) \odot N_x(\lambda_2) \mid (\lambda_1 \times \lambda_2)(x) \leq \lambda(\phi(x))\} \\
&\leq \bigvee\{N_x(\lambda_1 \odot \lambda_2) \mid \lambda_1(x) \times \lambda_2(x) \leq \lambda(\phi(x)) = \phi^{\leftarrow}(\lambda)(x)\} \\
&\leq N_x(\phi^{\leftarrow}(\lambda)).
\end{aligned}
$$

Theorem 3.3. *Let $f : (X,N) \to (Y,M^1)$ and $g : (X,N) \to (Z,M^2)$ be L-continuous. Define a mapping $\phi : X \to Y \times Z$ by $\phi(x) = (f(x), g(x))$. Then $\phi : (X,N) \to (Y \times Z, M^1 \times M^2)$ is L-continuous.*

Proof. Let $\lambda \in L^{Y \times Z}, \lambda_1 \in L^Y$ and $\lambda_2 \in L^Z$, then for all $x \in X$ we have

$$
\begin{aligned}
&\phi^{\leftarrow}(\lambda_1 \times \lambda_2)(x) = \lambda_1 \times \lambda_2(\phi(x)) \\
&= \lambda_1 \times \lambda_2(f(x), g(x)) = (\lambda_1(f(x)), \lambda_2(g(x))) \\
&= (f^{\leftarrow}(\lambda_1)(x), g^{\leftarrow}(\lambda_2)(x)) = (f^{\leftarrow}(\lambda_1) \times g^{\leftarrow}(\lambda_2))(x) \\
&= (f^{\leftarrow}(\lambda_1) \odot g^{\leftarrow}(\lambda_2))(x).
\end{aligned}
$$

Thus,

$$\begin{aligned}
&(M^1 \times M^2)_{\phi(x)}(\lambda) = \bigvee\{M^1_{f(x)}(\lambda_1) \odot M^2_{g(x)}(\lambda_2) \mid \lambda_1 \times \lambda_2 \leq \lambda\} \\
&\leq \bigvee\{N_x(f^{\leftarrow}(\lambda_1)) \odot N_x(g^{\leftarrow}(\lambda_2)) \mid \lambda_1 \times \lambda_2(\phi(x)) \leq \lambda(\phi(x))\} \\
&\leq \bigvee\{N_x(f^{\leftarrow}(\lambda_1)) \odot N_x(g^{\leftarrow}(\lambda_2)) \mid \phi^{\leftarrow}(\lambda_1 \times \lambda_2) \leq \phi^{\leftarrow}(\lambda)\} \\
&\leq \bigvee\{N_x(f^{\leftarrow}(\lambda_1) \odot g^{\leftarrow}(\lambda_2)) \mid f^{\leftarrow}(\lambda_1) \times g^{\leftarrow}(\lambda_2) \leq \phi^{\leftarrow}(\lambda)\} \\
&= \bigvee\{N_x(f^{\leftarrow}(\lambda_1) \odot g^{\leftarrow}(\lambda_2)) \mid f^{\leftarrow}(\lambda_1) \odot g^{\leftarrow}(\lambda_2) \leq \phi^{\leftarrow}(\lambda)\} \\
&\leq N_x(\phi^{\leftarrow}(\lambda)).
\end{aligned}$$

Theorem 3.4. *Let N_1 and N_2 be L-neighborhood systems on X satisfying $N_1(\mu_1) \odot N_2(\mu_2) = \perp$ for all $\mu_1 \odot \mu_2 = \perp_X$. Then $N_1 \oplus N_2$ is an L-neighborhood system defined for all $\mu \in L^X$ by*

$$(N_1 \oplus N_2)(\mu) = \bigvee\{N_1(\mu_1) \odot N_2(\mu_2) \mid \mu_1 \odot \mu_2 \leq \mu\}.$$

Proof. It is similar to the proof of Theorem 3.1.

Theorem 3.5. *Let τ_1 and τ_2 be L-topologies on X. We define*

$$\tau_1 \oplus \tau_2 = \{\bigvee_{j\in\Gamma} \lambda_j \mid \lambda_j = \lambda_{j1} \odot \lambda_{j2},\ \lambda_{j1} \in \tau_1, \lambda_{j2} \in \tau_2\}.$$

Then the following properties hold.

(1) $\tau_1 \oplus \tau_2$ is the coarsest L-topology on X which is finer than τ_1 and τ_2.

(2) $N^{\tau_1} \oplus N^{\tau_2} = N^{\tau_1 \oplus \tau_2}$.

(3) If N_1 and N_2 are L-neighborhood systems on X, then $\tau_{N_1} \oplus \tau_{N_2} = \tau_{N_1 \oplus N_2}$.

Proof. (1) (T1) $\top_X \in \tau_1 \oplus \tau_2$ because $\top_X \odot \top_X = \top_X$ for $\top_X \in \tau_i, i = 1,2$. Similarly, $\perp_X \in \tau_1 \oplus \tau_2$.

(T2) For every $\lambda, \rho \in \tau_1 \oplus \tau_2$, then there exist $\lambda_{ki} \in \tau_i, \rho_{ji} \in \tau_i$ with

$$\lambda = \bigvee_{k\in K} (\lambda_{k1} \odot \lambda_{k2}),\ \ \rho = \bigvee_{j\in J} (\rho_{j1} \odot \rho_{k2})$$

Then

$$\begin{aligned}
\lambda \odot \rho &= \textstyle\bigvee_{k\in K}(\lambda_{k1} \odot \lambda_{k2}) \odot \bigvee_{j\in J}(\rho_{j1} \odot \rho_{k2}) \\
&= \textstyle\bigvee_{k\in K} \bigvee_{j\in J}((\lambda_{k1} \odot \rho_{j1}) \odot (\lambda_{k2} \odot \rho_{k2})) \in \tau_1 \oplus \tau_2.
\end{aligned}$$

(T3) For every $\lambda_k \in \tau_1 \oplus \tau_2$, then there exist $\lambda_{ki_1} \in \tau_1, \lambda_{ki_2} \in \tau_1$ with $\lambda_k = \bigvee_{i\in I_k}(\lambda_{ki_2} \odot \lambda_{ki_2})$. Then

$$\lambda = \bigvee_{k\in K} \lambda_k = \bigvee_{k\in K} \bigvee_{i\in I_k} (\lambda_{ki_2} \odot \lambda_{ki_2}) \in \tau_1 \oplus \tau_2$$

If $\lambda_1 \in \tau_1$, then $\lambda_1 = \lambda_1 \odot \top_X$ such that $\lambda_1 \in \tau_1, \top_X \in \tau_2$. Hence $\lambda_1 \in \tau_1 \oplus \tau_2$;i.e. $\tau_1 \subset \tau_1 \oplus \tau_2$. Similarly, $\tau_1 \subset \tau_1 \oplus \tau_2$. If $\tau_i \subset \tau$ and τ is an L-topology, for $\lambda \in \tau_1 \oplus \tau_2$, there exist $\lambda_{ki} \in \tau_i \in \tau_i$ with

$$\lambda = \bigvee_{k\in K} (\lambda_{k1} \odot \lambda_{k2}).$$

Since $\lambda_{ki} \in \tau$, then $\lambda_{k1} \odot \lambda_{k2} \in \tau$. Hence $\lambda \in \tau$. So, $\tau_1 \oplus \tau_2 \subset \tau$.

(2) Since $\lambda_1 \odot \lambda_2 = \perp_X$, then $N_x^{\tau_1}(\lambda_1) \odot N_x^{\tau_2}(\lambda_2) \le \lambda_1(x) \odot \lambda_2(x) = \perp$. Hence $N_x^{\tau_1} \oplus N_x^{\tau_2}$ exists. Since $N_x^{\tau_i}(N_-^{\tau_i}(\lambda_i)) = N_x^{\tau_i}(\lambda_i)$ and $\tau_{N^{\tau_i}} = \tau_i$ from Theorem 2.8(2) for $i = 1,2$, then $N_-^{\tau_i}(\lambda_i) \in \tau_i$ for $i = 1,2$.

Suppose there exist $x \in X, \lambda \in L^X$ such that

$$N_x^{\tau_1\oplus\tau_2}(\lambda) \not\geq (N_x^{\tau_1} \oplus N_x^{\tau_2})(\lambda).$$

By the definition of $N_x^{\tau_1} \oplus N_x^{\tau_2}$, there exist λ_i with $\lambda_1 \odot \lambda_2 \le \lambda$ such that

$$N_x^{\tau_1\oplus\tau_2}(\lambda) \not\geq N_x^{\tau_1}(\lambda_1) \odot N_x^{\tau_2})(\lambda_2).$$

On the other hand,

$$\begin{aligned} &N_x^{\tau_1\oplus\tau_2}(\lambda) = \bigvee\{\rho(x) \mid \rho \le \lambda,\ \rho \in \tau_1 \oplus \tau_2\} \\ &\ge \bigvee\{N_x^{\tau_1}(\lambda_1) \odot N_x^{\tau_2}(\lambda_2) \mid N_x^{\tau_1}(\lambda_1) \odot N_x^{\tau_2}(\lambda_2) \le \lambda,\ N_x^{\tau_i}(\lambda_i) \in \tau_i\} \end{aligned}$$

It is a contradiction. Hence $N_x^{\tau_1\oplus\tau_2} \ge N_x^{\tau_1} \oplus N_x^{\tau_2}$.

Suppose there exist $x \in X, \rho \in L^X$ such that

$$N_x^{\tau_1\oplus\tau_2}(\rho) \not\leq (N_x^{\tau_1} \oplus N_x^{\tau_2})(\rho).$$

Then there exist $\rho_j = \rho_{j1} \odot \rho_{j2}$ with $\rho_{ji} \in \tau_i$ such that

$$\rho_j(x) \not\leq (N_x^{\tau_1} \oplus N_x^{\tau_2})(\rho).$$

On the other hand, since $N_x^{\tau_i}(\rho_{ji}) = \rho_{ji}(x), i = 1,2$, we have

$$\begin{aligned}(N_x^{\tau_1} \oplus N_x^{\tau_2})(\rho) &\geq (N_x^{\tau_1} \oplus N_x^{\tau_2})(\rho_j) \\ &\geq N_x^{\tau_1}(\rho_{j1}) \odot N_x^{\tau_2}(\rho_{j2}) = \rho_{j1}(x) \odot \rho_{j2}(x) = \rho_j(x)\end{aligned}$$

It is a contradiction. Hence $N_x^{\tau_1 \oplus \tau_2} \leq N_x^{\tau_1} \oplus N_x^{\tau_2}$. Thus, $N_x^{\tau_1 \oplus \tau_2} \leq N_x^{\tau_1} \oplus N_x^{\tau_2}$.

(3) Let $\lambda \in \tau_{N_1} \oplus \tau_{N_2}$. Then $\lambda = \bigvee_{j \in J} \lambda_j$ with $\lambda_j = \lambda_{j1} \odot \lambda_{j2}$ and $\lambda_{ji} \in \tau_{N_i}, i = 1,2$. Thus,

$$\begin{aligned}\lambda_j(x) &= \lambda_{j1}(x) \odot \lambda_{j2}(x) = (N_1)_x(\lambda_{j1}) \odot (N_2)_x(\lambda_{j2}) \\ &\leq (N_1 \oplus N_2)_x(\lambda_j) \leq \lambda_j(x).\end{aligned}$$

Then $\lambda_j = (N_1 \oplus N_2)(\lambda_j)$ for each $j \in J$.

$$\lambda = \bigvee_{j \in J} \lambda_j = \bigvee_{j \in J} N_x^{N_1 \oplus N_2}(\lambda_j) \leq N_x^{N_1 \oplus N_2}(\bigvee_{j \in J} \lambda_j) \leq \bigvee_{j \in J} \lambda_j = \lambda.$$

Hence $\lambda = N_x^{N_1 \oplus N_2}(\lambda)$. So, $\lambda \in \tau_{N_1 \oplus N_2}$.

Let $\rho \in \tau_{N_1 \oplus N_2}$. Then $\rho(x) = (N_1 \oplus N_2)_x(\rho) = ((N_1)_x \oplus (N_2)_x)(\rho) = \bigvee\{(N_1)_x(\rho_1) \odot (N_2)_x(\rho_2) \mid \rho_1 \odot \rho_2 \leq \rho\}$. Since $(N_i)_x((N_i)_-(\rho_i)) = (N_i)_-(\rho_i), i = 1,2$, then $(N_i)_-(\rho_i) \in \tau_{N_i}$. So, $\rho \in \tau_{N_1} \oplus \tau_{N_2}$.

Theorem 3.6. *Let U_1 and U_2 be L-quasi-uniformities on X. We define*

$$U_1 \oplus U_2 = \{u \in L^{X \times X} \mid u_1 \odot u_2 \leq u,\ u_1 \in U_1, u_2 \in U_2\}.$$

Then we have the following properties.

(1) $U_1 \oplus U_2$ is the coarsest L-quasi-uniformity on X which is finer than U_1 and U_2.

(2) $rN^{U_1} \oplus rN^{U_2} = rN^{U_1 \oplus U_2}$ and $lN^{U_1} \oplus lN^{U_2} = lN^{U_1 \oplus U_2}$.

(3) $\tau^r_{U_1} \oplus \tau^r_{U_2} = \tau^r_{U_1 \oplus U_2}$ where

$$\tau^r_{U_1} \oplus \tau^r_{U_2} = \{\lambda = \lambda_1 \odot \lambda_2 \mid \lambda_i \in \tau^r_{U_i}, i = 1,2\}.$$

(4) $\tau^l_{U_1} \oplus \tau^l_{U_2} = \tau^l_{U_1 \oplus U_2}$ where

$$\tau^l_{U_1} \oplus \tau^l_{U_2} = \{\lambda = \lambda_1 \odot \lambda_2 \mid \lambda_i \in \tau^l_{U_i}, i = 1,2\}.$$

Proof. (1) (QU1) $\top_{X \times X} \in U_1 \oplus U_2$ because $\top_{X \times X} \odot \top_{X \times X} = \top_{X \times X}$ for $\top_{X \times X} \in U_i, i = 1,2$.

(QU2) If $v \leq u$ and $v \in U_1 \oplus U_2$, then there exist $v_i \in U_i, i = 1,2$, with $v_1 \odot v_2 \leq v \leq u$. Thus $u \in U_1 \oplus U_2$.

(QU3) For every $u, v \in U_1 \oplus U_2$, then there exist $u_i, v_i \in U_i, i = 1,2$, with $u_1 \odot u_2 \leq u$ and $v_1 \odot v_2 \leq v$. Thus $u_1 \odot u_2 \odot v_1 \odot v_2 \leq u \odot v$. Hence $u \odot v \in U_1 \oplus U_2$.

(QU4) If $u \in U_1 \oplus U_2$, then there exist $u_i \in U_i, i = 1,2$, with $u_1 \odot u_2 \leq u$. Since $u_i \in U_i, i = 1,2$, by (QU4), $\top_\triangle \leq u_i, i = 1,2$. Hence $\top_\triangle \leq u$.

(QU5) For each $u \in U_1 \oplus U_2$, there exist $u_1 \in U_1$ and $u_2 \in U_2$ such that $u_1 \odot u_2 \leq u$. For each $u_i \in U_i, i = 1,2$, there exists $v_i \in U_i$ such that $v_i \circ v_i \leq u_i$.

$$
\begin{aligned}
&((v_1 \odot v_2) \circ (v_1 \odot v_2))(x,y) \\
&= \bigvee_{z \in X} ((v_1 \odot v_2)(x,z) \odot (v_1 \odot v_2)(z,y)) \\
&= \bigvee_{z \in X} ((v_1(x,z) \odot v_1(z,y)) \odot (v_2(x,z) \odot v_2(z,y))) \\
&\leq \bigvee_{z \in X} ((v_1(x,z) \odot v_1(z,y)) \odot \bigvee_{w \in X} (v_2(x,w) \odot v_2(w,y))) \\
&= (v_1 \circ v_1)(x,y) \odot (v_2 \circ v_2)(x,y) \\
&= u_1(x,y) \odot u_2(x,y) \leq u(x,y).
\end{aligned}
$$

Thus, there exists $(v_1 \odot v_2) \in U_1 \oplus U_2$ such that $(v_1 \odot v_2) \circ (v_1 \odot v_2) \leq u$.

If $u_1 \in U_1$, then $u_1 \odot \top_{X \times X} = u_1$ such that $u_1 \in U_1, \top_{X \times X} \in U_2$. Hence $u_1 \in U_1 \oplus U_2$;i.e. $U_1 \subset U_1 \oplus U_2$. Similarly, $U_1 \subset U_1 \oplus U_2$. If $U_i \subset V$ and V is an L-quasi-uniformity, for $u \in U_1 \oplus U_2$, there exists $u_i \in U_i$ such that $u_1 \odot u_2 \leq u$. Since $u_i \in V$, then $u_1 \odot u_2 \in V$. Hence $u \in V$. So, $U_1 \oplus U_2 \subset V$.

(2)

$$
\begin{aligned}
&(rN_x^{U_1} \oplus rN_x^{U_2})(\rho) = \bigvee_{\lambda \odot \mu \leq \rho} rN_x^{U_1}(\lambda) \odot rN_x^{U_2}(\mu) \\
&= \bigvee_{\lambda \odot \mu \leq \rho} \{ \bigvee_{u_1 \in U_1} S(u_1[x], \lambda) \} \odot \bigvee_{u_2 \in U_2} S(u_2[x], \mu) \} \} \\
&= \bigvee_{\lambda \odot \mu \leq \rho} \{ \bigvee_{u_1 \in U_1, u_2 \in U_2} (S(u_1[x], \lambda) \odot S(u_2[x], \mu)) \} \} \\
&\leq \bigvee_{\lambda \odot \mu \leq \rho} \{ \bigvee_{u_1 \in U_1, u_2 \in U_2} S(u_1[x] \odot u_2[x], \lambda \odot \mu) \} \} \ \text{(by Lemma 2.3 (3))} \\
&\leq \bigvee_{\lambda \odot \mu \leq \rho} \{ \bigvee_{u_1 \odot u_2 \in U_1 \oplus U_2} S((u_1 \odot u_2)[x], \lambda \odot \mu) \} \} \\
&\leq \bigvee_{\lambda \odot \mu \leq \rho} \{ \bigvee_{u \in U_1 \oplus U_2} S(u[x], \lambda \odot \mu) \} \\
&\leq \bigvee_{\lambda \odot \mu \leq \rho} rN^{U_1 \oplus U_2}(\lambda \odot \mu) \\
&= rN_x^{U_1 \oplus U_2}(\rho).
\end{aligned}
$$

Suppose there exist $x \in X, \rho \in L^X$ such that

$$rN_x^{U_1 \oplus U_2}(\rho) \not\leq (rN_x^{U_1} \oplus rN_x^{U_2})(\rho).$$

Then there exist $u \in U_1 \oplus U_2$ such that

$$S(u[x], \rho) \not\leq (rN_x^{U_1} \oplus rN_x^{U_2})(\rho).$$

Then there exist $u_i \in U_i$ with $u_1 \odot u_2 \leq u$ such that

$$S((u_1 \odot u_2)[x], \rho) = S(u_1[x], u_2[x] \to \rho) \not\leq (rN_x^{U_1} \oplus rN_x^{U_2})(\rho).$$

On the other hand, since $(u_2[x] \to \rho) \odot u_2[x] \leq \rho$, we have

$$\begin{aligned} & rN_x^{U_1 \oplus U_2}(\rho) \geq rN_x^{U_1}(u_2[x] \to \rho) \odot rN_x^{U_1}(u_2[x]) \\ & \geq S(u_1[x], u_2[x] \to \rho) \odot S(u_2[x], u_2[x]) = S(u_1[x], u_2[x] \to \rho) \odot \top. \end{aligned}$$

It is a contradiction. Hence $rN_x^{U_1 \oplus U_2} \leq rN_x^{U_1} \oplus rN_x^{U_2}$. Thus, $rN_x^{U_1 \oplus U_2} = rN_x^{U_1} \oplus rN_x^{U_2}$.

Similarly, $lN_x^{U_1 \oplus U_2} = lN_x^{U_1} \oplus lN_x^{U_2}$.

(3) Let $\lambda \in \tau^r_{U_1} \oplus \tau^r_{U_2}$ such that $\lambda = \bigvee_{j \in J} \lambda_j$. Then there exist $\lambda_{ji} \in \tau^r_{U_i}, i = 1,2$ such that

$$\begin{aligned} & \lambda_j(x) = \lambda_{j1}(x) \odot \lambda_{j2}(x) = rN_x^{U_1}(\lambda_{j1}) \odot rN_x^{U_2}(\lambda_{j2}) \\ & = rN_x^{U_1} \oplus rN_x^{U_2}(\lambda) \leq rN_x^{U_1 \oplus U_2}(\lambda_j) \leq \lambda_j(x). \end{aligned}$$

Then $\lambda_j = rN_x^{U_1 \oplus U_2}(\lambda_j)$ for each $j \in J$.

$$\lambda = \bigvee_{j \in J} \lambda_j = \bigvee_{j \in J} rN_x^{U_1 \oplus U_2}(\lambda_j) \leq rN_x^{U_1 \oplus U_2}(\bigvee_{j \in J} \lambda_j) \leq \bigvee_{j \in J} \lambda_j = \lambda.$$

Hence $\lambda = rN_x^{U_1 \oplus U_2}(\lambda)$. So, $\lambda \in \tau^r_{U_1 \oplus U_2}$.

Let $\rho \in \tau^r_{U_1 \oplus U_2}$. Then $\rho(x) = rN_x^{U_1 \oplus U_2}(\rho) = (rN_x^{U_1} \oplus rN_x^{U_2})(\rho) = \bigvee \{ rN_x^{U_1}(\rho_1) \odot rN_x^{U_2}(\rho_2) \mid \rho_1 \odot \rho_2 \leq \rho \}$. Since $rN_x^{U_i}(rN_-^{U_i}(\rho_i)) = rN_-^{U_i}(\rho_i), i = 1,2$, then $rN_-^{U_i}(\rho_i) \in \tau^r_{U_i}$. So, $\rho \in \tau^r_{U_1} \oplus \tau^r_{U_2}$.

(4) It is similarly proved as (3).

Example 3.7. *Let* $(L = [0,1], \odot, \to)$ *be a complete residuated lattice defined by*

$$x \odot y = x \wedge y, \;\; x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise}. \end{cases}$$

Let $X = \{x, y, z\}$ *be a set and* $u_1, u_2, w \in L^{X\times X}$ *such that*

$$u_2 = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.6 & 1 & 0.6 \\ 0.3 & 0.3 & 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 & 0.6 & 0.8 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.5 & 1 \end{pmatrix}$$

$$w = u_1 \wedge u_2 = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.4 & 1 & 0.4 \\ 0.3 & 0.3 & 1 \end{pmatrix}.$$

Define $U_i = \{u \in L^{X\times X} \mid u \geq u_i\}$ *for each* $i = 1, 2$.

(1) Since $u_i \circ u_i = u_i$, U_i *is an L- quasi-uniformity on X for each* $i = 1, 2$. *By Theorem 3.6(1),* $U_1 \oplus U_2 = \{u \in L^{X\times X} \mid u \geq w\}$ *is an L- quasi-uniformity on* X.

(2) Since $rN_x^{U_i}(\lambda) = \bigvee_{u\in U_i} S(u_i[x], \lambda)$, *we have*

$$rN_x^{U_1}(\lambda) = \bigvee_{u\in U} S(u[x], \lambda) = \lambda(x) \wedge (0.6 \to \lambda(y)) \wedge (0.3 \to \lambda(z)),$$
$$rN_y^{U_1}(\lambda) = \bigvee_{u\in U} S(u[y], \lambda) = (0.5 \to \lambda(x)) \wedge \lambda(y) \wedge (0.3 \to \lambda(z)),$$
$$rN_z^{U_1}(\lambda) = \bigvee_{u\in U} S(u[z], \lambda) = (0.5 \to \lambda(x)) \wedge (0.6 \to \lambda(y)) \wedge \lambda(z).$$

$$rN_x^{U_2}(\lambda) = \bigvee_{u\in U} S(u[x], \lambda) = \lambda(x) \wedge (0.4 \to \lambda(y)) \wedge (0.5 \to \lambda(z)),$$
$$rN_y^{U_2}(\lambda) = \bigvee_{u\in U} S(u[y], \lambda) = (0.6 \to \lambda(x)) \wedge \lambda(y) \wedge (0.5 \to \lambda(z)),$$
$$rN_z^{U_2}(\lambda) = \bigvee_{u\in U} S(u[z], \lambda) = (0.8 \to \lambda(x)) \wedge (0.4 \to \lambda(y)) \wedge \lambda(z).$$

By Theorem 3.6(2), we have $rN^{U_1} \oplus rN^{U_2} = rN^{U_1\oplus U_2}$ *from:*

$$rN_x^{U_1\oplus U_2}(\lambda) = S(w[x], \lambda) = \lambda(x) \wedge (0.4 \to \lambda(y)) \wedge (0.3 \to \lambda(z)),$$
$$rN_y^{U_1\oplus U_2}(\lambda) = S(w[y], \lambda) = (0.5 \to \lambda(x)) \wedge \lambda(y) \wedge (0.3 \to \lambda(z)),$$
$$rN_z^{U_1\oplus U_2}(\lambda) = S(w[z], \lambda) = (0.5 \to \lambda(x)) \wedge (0.4 \to \lambda(y)) \wedge \lambda(z).$$

(3) Since $lN_x^U(\lambda) = \bigvee_{u\in U} S(u[[x]], \lambda)$, *we have*

$$lN_x^{U_1}(\lambda) = \bigvee_{u\in U} S(u[[x]], \lambda) = \lambda(x) \wedge (0.5 \to \lambda(y)) \wedge (0.5 \to \lambda(z)),$$
$$lN_y^{U_1}(\lambda) = \bigvee_{u\in U} S(u[[y]], \lambda) = (0.6 \to \lambda(x)) \wedge \lambda(y) \wedge (0.6 \to \lambda(z)),$$
$$lN_z^{U_1}(\lambda) = \bigvee_{u\in U} S(u[[z]], \lambda) = (0.3 \to \lambda(x)) \wedge (0.3 \to \lambda(y)) \wedge \lambda(z).$$

$$lN_x^{U_2}(\lambda) = \bigvee_{u\in U} S(u[[x]],\lambda) = \lambda(x) \wedge (0.6 \to \lambda(y)) \wedge (0.8 \to \lambda(z)),$$
$$lN_y^{U_2}(\lambda) = \bigvee_{u\in U} S(u[[y]],\lambda) = (0.6 \to \lambda(x)) \wedge \lambda(y) \wedge (0.6 \to \lambda(z)),$$
$$lN_z^{U_2}(\lambda) = \bigvee_{u\in U} S(u[[z]],\lambda) = (0.5 \to \lambda(x)) \wedge (0.5 \to \lambda(y)) \wedge \lambda(z).$$

By Theorem 3.6(2), we have $lN^{U_1} \oplus lN^{U_2} = lN^{U_1 \oplus U_2}$ *from:*

$$lN_x^{U_1\oplus U_2}(\lambda) = \bigvee_{u\in U} S(u[[x]],\lambda) = \lambda(x) \wedge (0.5 \to \lambda(y)) \wedge (0.5 \to \lambda(z)),$$
$$lN_y^{U_1\oplus U_2}(\lambda) = \bigvee_{u\in U} S(u[[y]],\lambda) = (0.4 \to \lambda(x)) \wedge \lambda(y) \wedge (0.4 \to \lambda(z)),$$
$$lN_z^{U_1\oplus U_2}(\lambda) = \bigvee_{u\in U} S(u[[z]],\lambda) = (0.3 \to \lambda(x)) \wedge (0.3 \to \lambda(y)) \wedge \lambda(z).$$

(4) Since $\tau_U^r = \{\lambda \in L^X \mid \lambda(x) = rN_x^U(\lambda), \forall x \in X\}$ *from Theorem 3.5, we have*

$$\lambda \in \tau_{U_1}^r \ \text{iff} \ \begin{cases} \lambda = \alpha_X, \\ \lambda(x) \le 0.6 \to \lambda(y), \lambda(x) \le 0.3 \to \lambda(z), \\ \lambda(y) \le 0.5 \to \lambda(x), \lambda(y) \le 0.3 \to \lambda(z), \\ \lambda(z) \le 0.5 \to \lambda(x), \lambda(z) \le 0.6 \to \lambda(z), \end{cases}$$

$$\lambda \in \tau_{U_1}^l \ \text{iff} \ \begin{cases} \lambda = \alpha_X, \\ \lambda(x) \le 0.5 \to \lambda(y), \lambda(x) \le 0.5 \to \lambda(z), \\ \lambda(y) \le 0.6 \to \lambda(x), \lambda(y) \le 0.6 \to \lambda(z), \\ \lambda(z) \le 0.3 \to \lambda(x), \lambda(z) \le 0.3 \to \lambda(z), \end{cases}$$

$$\lambda \in \tau_{U_2}^r \ \text{iff} \ \begin{cases} \lambda = \alpha_X, \\ \lambda(x) \le 0.4 \to \lambda(y), \lambda(x) \le 0.5 \to \lambda(z), \\ \lambda(y) \le 0.6 \to \lambda(x), \lambda(y) \le 0.5 \to \lambda(z), \\ \lambda(z) \le 0.8 \to \lambda(x), \lambda(z) \le 0.4 \to \lambda(z), \end{cases}$$

$$\lambda \in \tau_{U_2}^l \ \text{iff} \ \begin{cases} \lambda = \alpha_X, \\ \lambda(x) \le 0.6 \to \lambda(y), \lambda(x) \le 0.8 \to \lambda(z), \\ \lambda(y) \le 0.4 \to \lambda(x), \lambda(y) \le 0.4 \to \lambda(z), \\ \lambda(z) \le 0.5 \to \lambda(x), \lambda(z) \le 0.5 \to \lambda(z), \end{cases}$$

By Theorem 3.6(3), we have $\tau^r_{U_1} \oplus \tau^r_{U_2} = \tau^r_{U_1 \oplus U_2}$ *from:*

$$\lambda \in \tau^r_{U_1 \oplus U_2} \ \ iff \ \begin{cases} \lambda = \alpha_X, \\ \lambda(x) \leq 0.4 \to \lambda(y), \lambda(x) \leq 0.3 \to \lambda(z), \\ \lambda(y) \leq 0.5 \to \lambda(x), \lambda(y) \leq 0.3 \to \lambda(z), \\ \lambda(z) \leq 0.5 \to \lambda(x), \lambda(z) \leq 0.4 \to \lambda(z), \end{cases}$$

By Theorem 3.6(3), we have $\tau^l_{U_1} \oplus \tau^l_{U_2} = \tau^l_{U_1 \oplus U_2}$ *from:*

$$\lambda \in \tau^l_{U_1 \oplus U_2} \ \ iff \ \begin{cases} \lambda = \alpha_X, \\ \lambda(x) \leq 0.5 \to \lambda(y), \lambda(x) \leq 0.5 \to \lambda(z), \\ \lambda(y) \leq 0.4 \to \lambda(x), \lambda(y) \leq 0.4 \to \lambda(z), \\ \lambda(z) \leq 0.3 \to \lambda(x), \lambda(z) \leq 0.3 \to \lambda(z). \end{cases}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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