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NAIMARK-SACKER BIFURCATION OF A CERTAIN SECOND ORDER QUADRATIC FRACTIONAL DIFFERENCE EQUATION

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Abstract. We investigate the Naimark-Sacker Bifurcation of the equilibrium of some special cases of the difference equation

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}^2 + \delta x_n}{B x_n x_{n-1} + C x_{n-1}^2 + D x_n},$$

where the parameters $\beta, \gamma, \delta, B, C, D$ are nonnegative numbers which satisfy $B+C+D>0$ and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that $B x_n x_{n-1} + C x_{n-1}^2 + D x_n > 0$ for all $n \geq 0$.

Keywords: bifurcation; boundedness; difference equation; global attractivity; local stability; Naimark-Sacker

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1. Introduction and Preliminaries

In this paper we study the Naimark-Sacker bifurcation of the equilibrium of some special cases of the difference equation

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$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}^2 + \delta x_n}{B x_n x_{n-1} + C x_{n-1}^2 + D x_n}, \quad n = 0,1,2,\ldots, \tag{1.1}$$

where the parameters $\beta, \gamma, \delta, B, C, D$ are nonnegative numbers which satisfy $B+C+D>0$ and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that $Bx_nx_{n-1}+Cx_{n-1}^2+Dx_n>0$ for all $n \geq 0$.

Equation (1.1), which has been studied in [2, 3, 17], is a special case of a general second order quadratic fractional difference equation of the form

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0,1,\ldots \tag{1.2}$$

with non-negative parameters and initial conditions such that $A+B+C>0$, $a+b+c+d+e+f>0$ and $ax_n^2+bx_nx_{n-1}+cx_{n-1}^2+dx_n+ex_{n-1}+f>0, n=0,1,\ldots$. Several global asymptotic results for some special cases of Eq.(1.2) were obtained in [9, 10, 11, 19].

The change of variable $x_n = 1/u_n$ transforms Eq.(1.1) to the difference equation

$$u_{n+1} = \frac{Du_{n-1}^2 + Cu_n + Bu_{n-1}}{\delta u_{n-1}^2 + \gamma u_n + \beta u_{n-1}}, \quad n = 0,1,\ldots, \tag{1.3}$$

where we assume that $\delta+\beta+\gamma>0$ and that the non-negative initial conditions u_{-1}, u_0 are such that $\delta u_{n-1}^2+\gamma u_n+\beta u_{n-1}>0$ for all $n \geq 0$. Thus the results of this paper extends to Eq.(1.3).

The first systematic study of global dynamics of a special quadratic fractional case of Eq.(1.2) where $A=C=D=a=c=d=0$ was performed in [2, 3]. Dynamics of some related quadratic fractional difference equations was considered in the papers [9, 10, 11, 19]. Complete linear stabilty analysis of the equilibrium and the period-two solution of Eq.(1.1) was presented in [17], where it was found that 10 special cases of Eq.(1.1) exhibit locally transition from local attractor to the local repeller by passing through the critical non-hyperbolic case where both eigenvalues of characteristic equation are complex conjugate numbers on the unit circle. See Table 1-5 for the list of all special cases of Eq.(1.2), which exhibits this kind of local stability character.

In this paper we will perform the Naimark-Sacker bifurcation analysis of some special cases of Eq.(1.1), with mentioned local behavior, which are obtained when one or more coefficients of equation are set to be zero.

Now we consider bifurcation of a fixed point of map associated to Eq.(1.1) in the case where the eigenvalues are complex conjugates and of unit module. For the sake of convinience we include Naimark-Sacker bifurcation theorem, known also as Poincaré-Andronov-Hopf bifurcation theorem for maps, see [6, 8, 20]:

Theorem 3.1. *Let*

$$F : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to F(\lambda, x)$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) $F(\lambda, 0) = 0$ *for* λ *near some fixed* λ_0*;*

(ii) $DF(\lambda, 0)$ *has two non-real eigenvalues* $\mu(\lambda)$ *and* $\bar{\mu}(\lambda)$ *for* λ *near* λ_0 *with* $|\mu(\lambda_0)| = 1$*;*

(iii) $\frac{d}{d\lambda}|\mu(\lambda)| = d(\lambda_0) \neq 0$ *at* $\lambda = \lambda_0$*;*

(iv) $\mu^k(\lambda_0) \neq 1$ *for* $k = 1, 2, 3, 4$.

Then there is a smooth λ*-dependent change of coordinate bringing* f *into the form*

$$F(\lambda, x) = \mathscr{F}(\lambda, x) + O(\| x \|^5)$$

and there are smooth functions $a(\lambda)$*,* $b(\lambda)$*, and* $\omega(\lambda)$ *so that in polar coordinates the function* $\mathscr{F}(\lambda, x)$ *is given by*

$$\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} |\mu(\lambda)|r - a(\lambda)r^3 \\ \theta + \omega(\lambda) + b(\lambda)r^2 \end{pmatrix}. \tag{1.5}$$

If $a(\lambda_0) > 0$ *and* $d(\lambda_0) > 0$ $(d(\lambda_0) < 0)$*, then there is a neighborhood* U *of the origin and a* $\delta > 0$ *such that for* $|\lambda - \lambda_0| < \delta$ *and* $x_0 \in U$*, the* ω*-limit set of* x_0 *is the origin if* $\lambda < \lambda_0$ $(\lambda > \lambda_0)$ *and belongs to a closed invariant* C^1 *curve* $\Gamma(\lambda)$ *encircling the origin if* $\lambda > \lambda_0$ $(\lambda < \lambda_0)$*. Furthermore,* $\Gamma(\lambda_0) = 0$.

If $a(\lambda_0) < 0$ *and* $d(\lambda_0) > 0$ $(d(\lambda_0) < 0)$*, then there is a neighborhood* U *of the origin and a* $\delta > 0$ *such that for* $|\lambda - \lambda_0| < \delta$ *and* $x_0 \in U$*, then* α*-limit set of* x_0 *is the origin if* $\lambda > \lambda_0$ $(\lambda < \lambda_0)$ *and belongs to a closed invariant* C^1 *curve* $\Gamma(\lambda)$ *encircling the origin if* $\lambda < \lambda_0$ $(\lambda > \lambda_0)$*. Furthermore,* $\Gamma(\lambda_0) = 0$.

Consider a general map $F(\lambda, x)$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda) = \alpha(\lambda) + i\beta(\lambda)$ and $\bar{\mu}(\lambda) = \alpha(\lambda) - i\beta(\lambda)$ satisfying $\alpha(\lambda)^2 + \beta(\lambda)^2 = 1$ and $\beta(\lambda) \neq 0$.

By putting the linear part of such a map into Jordan Canonical form, we may assume F to have the following form near the origin

$$F(\lambda,x)=\begin{pmatrix}\alpha(\lambda) & -\beta(\lambda)\\ \beta(\lambda) & \alpha(\lambda)\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}g_1(\lambda,x_1,x_2)\\ g_2(\lambda,x_1,x_2)\end{pmatrix}. \tag{1.6}$$

Then the coefficient $a(\lambda_0)$ of the cubic term in Eq.(1.4) in polar coordinates is equal to

$$a(\lambda_0)=Re\left[\frac{(1-2\mu(\lambda_0))\bar{\mu}^2(\lambda_0)}{1-\mu(\lambda_0)}\xi_{11}\xi_{20}\right]+\frac{1}{2}|\xi_{11}|^2+|\xi_{02}|^2-Re(\bar{\mu}(\lambda_0)\xi_{21}), \tag{1.7}$$

where

$$\begin{aligned}
\xi_{20}&=\frac{1}{8}\left\{(g_1)_{x_1x_1}-(g_1)_{x_2x_2}+2(g_2)_{x_1x_2}+i\left[(g_2)_{x_1x_1}-(g_2)_{x_2x_2}-2(g_1)_{x_1x_2}\right]\right\}\\
\xi_{11}&=\frac{1}{4}\left\{(g_1)_{x_1x_1}+(g_1)_{x_2x_2}+i\left[(g_2)_{x_1x_1}+(g_2)_{x_2x_2}\right]\right\},\\
\xi_{02}&=\frac{1}{8}\left\{(g_1)_{x_1x_1}-(g_1)_{x_2x_2}-2(g_2)_{x_1x_2}+i\left[(g_2)_{x_1x_1}-(g_2)_{x_2x_2}+2(g_1)_{x_1x_2}\right]\right\}\\
\xi_{21}&=\frac{1}{16}\left\{(g_1)_{x_1x_1x_1}+(g_1)_{x_1x_2x_2}+(g_2)_{x_1x_1x_2}+(g_2)_{x_2x_2x_2}\right.\\
&\quad\left.+i\left[(g_2)_{x_1x_1x_1}+(g_2)_{x_1x_2x_2}-(g_1)_{x_1x_1x_2}-(g_1)_{x_2x_2x_2}\right]\right\}.
\end{aligned}$$

TABLE 1. Equations of type (1,2)

Equation	Equilibrium point	Stability	The eigenvalues	$\mu(\delta_0)$
$x_{n+1}=\frac{\delta x_n}{Bx_nx_{n-1}+x_{n-1}^2}$	$\bar{x}=\sqrt{\frac{\delta}{B+1}}$	a repeller for $\delta>0, B>0$	$\mu(B)=\frac{1+i\sqrt{4B(B+3)+7}}{2(B+1)}$ The eigenvalues$\{\mu(\delta),\overline{\mu(\delta)}\}$	$\lvert\mu(B)\rvert=\sqrt{\frac{1}{B+1}+1}$
$x_{n+1}=\frac{\delta x_n}{Cx_{n-1}^2+x_n}$	$\bar{x}=\frac{\sqrt{4C\delta+1}-1}{2C}$	LAS for $c\delta<2$ a repeller for $c\delta>2$ a non-hyperbolic for $c\delta=2$	$\mu(\delta)=\frac{4C\delta}{C\delta+i\sqrt{C\delta\left(7C\delta+4\sqrt{4C\delta+1}+4\right)}}$ The eigenvalues $\{\mu(\delta),\overline{\mu(\delta)}\}$	$\mu(\delta_0)=\frac{1}{4}\left(1-i\sqrt{15}\right)$ $\delta_0=\frac{2}{C}$ $\lvert\mu(\delta_0)\rvert=1$

The rest of the paper is organized into one section with two subsections where Naimark-Sacker bifurcation analysis is performed for one equation of type (2,2) and one equation of type (1,3). It will turn out that indeed both considered equations undergo the Naimark-Sacker bifurcation. The bifurcation analysis of the remaining two equations of type (2,2) will be performed in the separate paper as these two equations exhibit both Naimark-Sacker bifurcation and also period-doubling bifurcation, see [14, 15, 16]. We believe that in the case when these equations undergo period-doubling bifurcation we can make our results global.

TABLE 2. Equations of type (2,1)

Equation	Equilibrium point	Stability	The eigenvalues	$\mu(\delta_0)$
$x_{n+1}=\frac{\beta x_{n-1}x_n+\delta x_n}{x_{n-1}^2}$	$\bar{x}=\frac{\sqrt{\beta^2+4\delta}+\beta}{2}$	a repeller for $\beta,\delta>0$	$\mu(\delta)=\frac{\sqrt{\beta^2+4\delta}+i\sqrt{6\beta\sqrt{\beta^2+4\delta}+6\beta^2+28\delta}+\beta}{2\left(\sqrt{\beta^2+4\delta}+\beta\right)}$ The eigenvalues $\{\mu(\delta),\overline{\mu(\delta)}\}$	$\lvert\mu(\delta)\rvert=\sqrt{\frac{2\sqrt{\beta^2+4\delta}}{\sqrt{\beta^2+4\delta}+\beta}}$
$x_{n+1}=\frac{\gamma x_{n-1}^2+\delta x_n}{x_{n-1}^2}$	$\bar{x}=\frac{\sqrt{\gamma^2+4\delta}+\gamma}{2}$	LAS for $\delta<2\gamma^2$ a repeller for $\delta>2\gamma^2$ a non-hyp. for $\delta=2\gamma^2$	$\mu(\delta)=\frac{4\delta}{i\sqrt{\delta\left(4\gamma\left(\sqrt{\gamma^2+4\delta}+\gamma\right)+7\delta\right)}+\delta}$ The eigenvalues $\{\mu(\delta),\overline{\mu(\delta)}\}$	$\mu(\delta_0)=\frac{1}{4}\left(1-i\sqrt{15}\right)$ $\delta_0=2\gamma^2$ $\lvert\mu(\delta_0)\rvert=1$

2. Naimark-Sacker Bifurcation for Maps

2.1. The case $x_{n+1}=\frac{\beta x_n x_{n-1}+\delta x_n}{Cx_{n-1}^2+x_n}$, $\delta, C>0,\ \beta\geq 0$

It is easy to see that equation

$$x_{n+1}=\frac{\beta x_n x_{n-1}+\delta x_n}{Cx_{n-1}^2+x_n},\quad n=0,1,\ldots \tag{2.1}$$

has the equilibrium point

$$\bar{x}=\frac{\sqrt{4C\delta+(\beta-1)^2}+\beta-1}{2C}. \tag{2.2}$$

In order to apply Theorem we make a change of variable $y_n=x_n-\bar{x}$. Then, transformed equation is given by

$$y_{n+1}=\frac{(\bar{x}+y_n)\left(\beta\left(\bar{x}+y_{n-1}\right)+\delta\right)}{C\left(\bar{x}+y_{n-1}\right)^2+\bar{x}+y_n}-\bar{x}. \tag{2.3}$$

By using the substitution $u_n=y_{n-1},\ v_n=y_n$ we write Eq.(2.3) in the equivalent form:

$$\begin{aligned} u_{n+1}&=v_n\\ v_{n+1}&=\frac{(2\bar{x}+v_n)\left(2\beta\bar{x}+\beta u_n+\delta\right)}{C\left(2\bar{x}+u_n\right)^2+2\bar{x}+v_n}-2\bar{x}. \end{aligned} \tag{2.4}$$

Let F be the function defined by:

$$F(u,v)=\begin{pmatrix} v\\ \frac{(\bar{x}+v)(\beta(\bar{x}+u)+\delta)}{C(\bar{x}+u)^2+\bar{x}+v}-\bar{x}\end{pmatrix}.$$

TABLE 3. Equations of type (2,2)

Equation	Equilibrium point	Stability	Eigenvalues	$\mu(\delta_0)$
$x_{n+1}=\frac{\beta x_{n-1}x_n+\delta x_n}{Bx_nx_{n-1}+x_{n-1}^2}$	$\bar{x}=\frac{\sqrt{4B\delta+\beta^2+4\delta}+\beta}{2(B+1)}$	LAS for $B\beta^2>\delta$ a repeller for $B\beta^2<\delta$ non-hyp. $B\beta^2=\delta$	$\mu(\delta)=\frac{i\sqrt{\Delta}+\beta\bar{x}+\delta}{2(B+1)^2\bar{x}^2}$ $\Delta=4(B\bar{x}+\bar{x})^2(\beta\bar{x}+(B+2)\delta)$ $-(\beta\bar{x}+\delta)^2>0$ The eigenvalues $\{\mu(\delta),\overline{\mu(\delta)}\}$	$\mu(\delta_0)=\frac{1+i\sqrt{4B(B+2)+3}}{2(B+1)}$ $\delta_0=B\beta^2$ $\lvert\mu(\delta_0)\rvert=1$
$x_{n+1}=\frac{\beta x_{n-1}x_n+\delta x_n}{Cx_{n-1}^2+x_n}$	$\bar{x}=\frac{\sqrt{4C\delta+(\beta-1)^2}+\beta-1}{2C}$	a LAS for $c\delta<2(1+\beta)$ a repeller $c\delta>2(1+\beta)$ a non-hyp $c\delta=2(1+\beta)$	$\mu_{\pm}(\delta)=\frac{C\bar{x}\beta+C\delta\pm\sqrt{\Delta}}{2(C\bar{x}+1)^2}$ $\Delta=(C\bar{x}\beta+C\delta)^2$ $-4(C\bar{x}+1)^2(\beta(C\bar{x}-1)+2C\delta)$ The eigenvalues $\{\mu_+(\delta),\mu_-(\delta)\}$	$\mu(\delta_0)=\frac{1+\beta+i\sqrt{\Gamma}}{2(\beta+2)}$ $\Gamma=(\beta+3)(3\beta+5)$ $\delta_0=\frac{2(1+\beta)}{C}$ $\lvert\mu(\delta_0)\rvert=1$
$x_{n+1}=\frac{\gamma x_{n-1}^2+\delta x_n}{Bx_{n-1}x_n+x_{n-1}^2}$	$\bar{x}=\frac{\sqrt{4B\delta+\gamma^2+4\delta}+\gamma}{2(B+1)}$	LAS for $\frac{3(B-1)\gamma^2}{(B+2)^2}<4\delta$ or $\delta<2(2B+1)\gamma^2$ saddle point for $B>1\wedge\delta<\frac{3(B-1)\gamma^2}{4(B+2)^2}$ non-hyperbolic eq. for $(4b+2)\gamma^2=\delta$ or $4(B+2)^2\delta=3(B-1)\gamma^2$ repeller for $\delta>2(2B+1)\gamma^2$	$\mu_{\pm}(\delta)=\frac{-B\gamma\bar{x}+\delta\pm\sqrt{\Delta}}{2(B+1)^2\bar{x}^2}$ $\Delta=4(B+1)^2\bar{x}^2(B\gamma\bar{x}-(B+2)\delta)$ $+(B\gamma\bar{x}-\delta)^2$ The eigenvalues $\{\mu_+(\delta),\mu_-(\delta)\}$	$\mu(\delta_0)=\frac{1+i\sqrt{\Gamma}}{4(1+B)}$ $\Gamma=(4B+3)(4B+5)$ $\delta_0=(4B+2)\gamma^2$ $\lvert\mu(\delta_0)\rvert=1$
$x_{n+1}=\frac{\gamma x_{n-1}^2+\delta x_n}{Cx_{n-1}^2+x_n}$	$\bar{x}=\frac{\sqrt{4C\delta+(1-\gamma)^2}+\gamma-1}{2C}$	LAS $\delta<\frac{(\gamma+2)(2\gamma+1)}{C}$ and $\frac{(3-\gamma)(3\gamma-1)}{16C}<\delta$ a repeller for $\delta>\frac{(\gamma+2)(2\gamma+1)}{C}$ a saddle for $\frac{(3-\gamma)(3\gamma-1)}{16C}>\delta$ a non-hyperbolic eq. for $\frac{(3-\gamma)(3\gamma-1)}{16C}=\delta$ or $\delta=\frac{(\gamma+2)(2\gamma+1)}{C}$	$\mu_{\pm}(\delta)=\frac{C\delta-\gamma\pm\sqrt{\Delta}}{2\left(C^2\bar{x}^2+2C\bar{x}+1\right)}$ $\Delta=(\gamma-C\delta)^2$ $-4\left(C^2\bar{x}^2+2C\bar{x}+1\right)(2C\delta-2\gamma)$ $\Delta<0$ for $c\delta>\gamma$ The eigenvalues $\{\mu_+(\delta),\mu_-(\delta)\}$	$\mu(\delta_0)=\frac{1}{4}\left(1+i\sqrt{15}\right)$ $\delta_0=\frac{(\gamma+2)(2\gamma+1)}{C}$ $\lvert\mu(\delta_0)\rvert=1$

Then F has the unique fixed point $(0,0)$ and maps $(-\bar{x},\infty)^2$ into $(-\bar{x},\infty)^2$. The Jacobian matrix of F is given by

$$Jac_F(u,v)=\begin{pmatrix} 0 & 1 \\ \frac{(y+\bar{x})\left(\beta\left(C(x+\bar{x})^2+y+\bar{x}\right)-2C(x+\bar{x})(\delta+\beta(x+\bar{x}))\right)}{\left(C(x+\bar{x})^2+y+\bar{x}\right)^2} & \frac{C(x+\bar{x})^2(\delta+\beta(x+\bar{x}))}{\left(Cx^2+2C\bar{x}x+C\bar{x}^2+y+\bar{x}\right)^2} \end{pmatrix}.$$

Equation	Equilibrium point	Stability	The eigenvalues	$\mu(\delta_0)$
$x_{n+1}=\frac{\beta x_n x_{n-1}+\gamma x_{n-1}^2+\delta x_n}{x_{n-1}^2}$	$\bar{x}=\frac{\sqrt{(\beta+\gamma)^2+4\delta}+\beta+\gamma}{2}$	LAS for $\delta<\gamma(\beta+2\gamma)$	$\mu(\delta)=\frac{\beta\bar{x}+\delta+i\sqrt{\Delta}}{2\bar{x}^2}$	$\mu(\delta_0)=\frac{i\sqrt{\Gamma}+\beta+\gamma}{2(\beta+2\gamma)}$
			$\Delta=4\bar{x}^2(\beta\bar{x}+2\delta)-(\beta\bar{x}+\delta)^2$	$\delta_0=\gamma(\beta+2\gamma)$
		a repeller $\delta>\gamma(\beta+2\gamma)$		$\Gamma=(\beta+3\gamma)(3\beta+5\gamma)$
			The eigenvalues $\{\mu(\delta),\overline{\mu(\delta)}\}$	$\lvert\mu(\delta_0)\rvert=1$
		a non-hyp. $\delta=\gamma(\beta+2\gamma)$		

TABLE 4. Equations of type (1,3)

Equation	Equilibrium point	Stability	The eigenvalues	$\mu(\delta_0)$
$x_{n+1}=\frac{x_n}{Bx_n x_{n-1}+Cx_{n-1}^2+Dx_n}$	$\bar{x}=\frac{\sqrt{4B+4C+D^2}-D}{2(B+C)}$	LAS for	$\mu(B)=\frac{C+i\sqrt{\Delta}}{2(\bar{x}(B+C)+D)^2}$	$\mu(B_0)=\frac{2C\left(D^2+\sqrt{\Gamma}\right)}{\left(\lvert D^2-2C\rvert+D^2\right)^2}$
		$B>\frac{C^2-2CD^2}{D^2}$	$\Delta=4(B+2C)(\bar{x}(B+C)+D)^2-C^2$	$\Gamma=-2D^2\lvert D^2-2C\rvert$
			$\Delta>0$	$-\left(D^2-2C\right)^2$
		a repeller for		
		$B<\frac{C^2-2CD^2}{D^2}$		$B_0=\frac{C^2-2CD^2}{D^2}$
			The eigenvalues $\{\mu(\delta),\overline{\mu(\delta)}\}$	if $C-2D^2>0$ then
		a non-hyperbolic		$\mu(B_0)=\frac{D^2+i\sqrt{4C^2-D^4}}{2C}$
		$B=\frac{C^2-2CD^2}{D^2}$		$\lvert\mu(B_0)\rvert=1$

At $(0,0)$, $Jac_F(u,v)$ has the form

$$J_0=JacF(0,0)=\begin{pmatrix} 0 & 1 \\ \frac{-C\bar{x}\beta+\beta-2C\delta}{(C\bar{x}+1)^2} & \frac{C(\delta+\beta\bar{x})}{(C\bar{x}+1)^2} \end{pmatrix}. \tag{2.5}$$

The eigenvalues of (2.5) are $\mu_\pm(\delta)$ where

$$\mu_\pm(\delta)=\frac{\pm\sqrt{(C\beta\bar{x}+C\delta)^2-4\left(C\bar{x}+1\right)^2\left(\beta\left(C\bar{x}-1\right)+2C\delta\right)}+C\beta\bar{x}+C\delta}{2\left(C\bar{x}+1\right)^2}.$$

Then we have that

$$F\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ \frac{-C\bar{x}\beta+\beta-2C\delta}{(C\bar{x}+1)^2} & \frac{C(\delta+\beta\bar{x})}{(C\bar{x}+1)^2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}+\begin{pmatrix} f_1(\delta,u,v) \\ f_2(\delta,u,v) \end{pmatrix}, \tag{2.6}$$

and

$$f_1(\delta,u,v)=0$$

$$f_2(\delta,u,v)=\frac{(\bar{x}+v)\left(\beta\left(\bar{x}+u\right)+\delta\right)}{C\left(\bar{x}+u\right)^2+\bar{x}+v}+\frac{u\left(\beta\left(C\bar{x}-1\right)+2C\delta\right)}{\left(C\bar{x}+1\right)^2}-\frac{Cv\left(\beta\bar{x}+\delta\right)}{\left(C\bar{x}+1\right)^2}-\bar{x}.$$

Let

$$\delta_0=\frac{2(1+\beta)}{C}. \tag{2.7}$$

For $\delta = \delta_0$ we obtain

$$\bar{x} = \frac{1+\beta}{C} \text{ and } J_0 = \begin{pmatrix} 0 & 1 \\ -1 & \frac{\beta+1}{\beta+2} \end{pmatrix}.$$

The eigenvalues of J_0 are $\mu(\delta_0)$ and $\overline{\mu(\delta_0)}$ where

$$\mu(\delta_0) = \frac{\beta + i\sqrt{(\beta+3)(3\beta+5)} + 1}{2(\beta+2)}.$$

The eigenvectors corresponding to $\mu(\delta)$ and $\overline{\mu(\delta)}$ are $v(\delta_0)$ and $\overline{v(\delta_0)}$ where

$$v(\delta_0) = \left(\frac{\beta - i\sqrt{(\beta+3)(3\beta+5)} + 1}{2(\beta+2)}, 1 \right).$$

One can prove that

$$|\mu(\delta_0)| = 1,$$

$$\mu^2(\delta_0) = -\frac{\beta(\beta+6)+7}{2(\beta+2)^2} + \frac{i(\beta+1)\sqrt{(\beta+3)(3\beta+5)}}{2(\beta+2)^2}$$

$$\mu^3(\delta_0) = -\frac{(\beta+1)(2\beta(\beta+5)+11)}{2(\beta+2)^3} - \frac{i\sqrt{(\beta+3)(3\beta+5)}(2\beta+3)}{2(\beta+2)^3}$$

$$\mu^4(\delta_0) = \frac{17-(\beta-2)\beta(\beta(\beta+6)+10)}{2(\beta+2)^4} - \frac{i(\beta+1)\sqrt{(\beta+3)(3\beta+5)}(\beta(\beta+6)+7)}{2(\beta+2)^4},$$

from which follows that $\mu^k(\delta_0) \neq 1$ for $k = 1,2,3,4$ and $\beta \geq 0$. Substituting $\delta = \delta_0$ and $\bar{x}$ into (2.6) we get

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{\beta+1}{\beta+2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix},$$

and

$$h_1(u,v) = f_1(\delta_0,u,v) = 0$$

$$h_2(u,v) = f_2(\delta_0,u,v) = -\frac{C\left(-Cu^3(\beta+2) + u^2(\beta+1)(Cv-\beta-2) + uv\beta(\beta+1) + v^2(\beta+1)\right)}{(\beta+2)\left(C(u(Cu+2)+v) + 2Cu\beta + \beta^2 + 3\beta + 2\right)}.$$

Hence, for $\delta = \delta_0$ system (2.4) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{\beta+1}{\beta+2} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} h_1(u_n,v_n) \\ h_2(u_n,v_n) \end{pmatrix}. \quad (2.8)$$

Let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}$$

where

$$P = \begin{pmatrix} \frac{\beta+1}{2(\beta+2)} & \frac{\sqrt{(\beta+3)(3\beta+5)}}{2(\beta+2)} \\ 1 & 0 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{2(\beta+2)}{\sqrt{(\beta+3)(3\beta+5)}} & -\frac{\beta+1}{\sqrt{(\beta+3)(3\beta+5)}} \end{pmatrix}.$$

Then system (2.8) is equivalent to its normal form

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\beta+1}{2\beta+4} & -\frac{\sqrt{(\beta+3)(3\beta+5)}}{2(\beta+2)} \\ \frac{\sqrt{(\beta+3)(3\beta+5)}}{2(\beta+2)} & \frac{\beta+1}{2\beta+4} \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + G \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix},$$

where

$$H \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix}.$$

Let

$$G \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix} = P^{-1} H \left(P \begin{pmatrix} u \\ v \end{pmatrix} \right).$$

By straightforward calculation we obtain that

$$\begin{aligned} g_1(u,v) &= \frac{\frac{\Delta_1}{\Delta_2} - 2u(\beta+1)}{2(\beta+2)}, \\ g_2(u,v) &= -\frac{\beta+1}{\sqrt{(\beta+3)(3\beta+5)}} g_1(u,v), \end{aligned}$$

where

$$\Delta_1 = \left(C\left(u\beta + u + v\sqrt{(\beta+3)(3\beta+5)}\right) + 2(\beta+1)(\beta+2)\right)$$
$$\left(Cu^2(\beta+1)^2 + 2u(\beta+1)\left(Cv\sqrt{(\beta+3)(3\beta+5)} + 2\beta(\beta+4) + 8\right) + Cv^2(\beta+3)(3\beta+5)\right)$$

$$\begin{aligned} \Delta_2 = {} & C^2\left(u^2(\beta+1)^2 + 2uv(\beta+1)\sqrt{(\beta+3)(3\beta+5)} + v^2(\beta+3)(3\beta+5)\right) \\ & + 4C(\beta+2)\left(u(\beta(\beta+3)+3) + v\sqrt{(\beta+3)(3\beta+5)}(\beta+1)\right) + 4(\beta+1)(\beta+2)^3 \end{aligned}$$

and furthermore

$$\xi_{20}(0,0)=\frac{C\sqrt{(\beta+3)(3\beta+5)}(-\beta-3)+iC\left(\beta^2+2\beta-1\right)}{4(\beta+2)^2\sqrt{(\beta+3)(3\beta+5)}},$$

$$\xi_{11}(0,0)=\frac{C(\beta+1)(\beta+4)\left(-i\beta+\sqrt{(\beta+3)(3\beta+5)}-i\right)}{4(\beta+2)^3\sqrt{(\beta+3)(3\beta+5)}},$$

$$\xi_{02}(0,0)=\frac{C\sqrt{(\beta+3)(3\beta+5)}\left(-\beta^2-4\beta-5\right)+iC\left(\beta^3+7\beta^2+17\beta+13\right)}{4(\beta+2)^3\sqrt{(\beta+3)(3\beta+5)}},$$

$$\xi_{21}(0,0)=\frac{C^2\beta\sqrt{(\beta+3)(3\beta+5)}\left(\beta^2+3\beta\right)+iC^2\beta\left(3\beta^3+24\beta^2+57\beta+40\right)}{8(\beta+1)(\beta+2)^4\sqrt{(\beta+3)(3\beta+5)}}.$$

Since

$$\begin{aligned}
Re\left(\overline{\mu(\delta_0)}\xi_{21}(\delta_0)\right)&=\frac{C^2\beta(\beta(\beta+5)+5)}{4(\beta+1)(\beta+2)^4}\\
Re\left(\frac{(1-2\mu(\lambda_0))\bar{\mu}^2(\lambda_0)}{1-\mu(\lambda_0)}\xi_{11}(\delta_0)\xi_{20}(\delta_0)\right)&=\frac{C^2(\beta+1)(\beta+4)(3\beta(\beta+5)+19)}{8(\beta+2)^4(\beta+3)(3\beta+5)}\\
\xi_{11}(\delta_0)\overline{\xi_{11}(\delta_0)}&=\frac{C^2(\beta+1)^2(\beta+4)^2}{4(\beta+2)^4(\beta+3)(3\beta+5)}\\
\xi_{02}(\delta_0)\overline{\xi_{02}(\delta_0)}&=\frac{C^2(\beta(\beta(\beta+7)+18)+17)}{4(\beta+2)^3(\beta+3)(3\beta+5)}
\end{aligned}\tag{2.9}$$

then by using (2.9) and after lengthy calculation we obtain that

$$a(\delta_0)=\frac{C^2(\beta(\beta(2\beta+17)+43)+32)}{8(\beta+1)(\beta+2)^4(\beta+3)}>0.$$

One can see that

$$|\mu(\delta)|^2=\mu(\delta)\overline{\mu(\delta)}=\frac{\beta\left(C\bar{x}-1\right)+2C\delta}{\left(C\bar{x}+1\right)^2}=\frac{2\left(\beta\left(\sqrt{4C\delta+(\beta-1)^2}+\beta-3\right)+4C\delta\right)}{\left(\sqrt{4C\delta+(\beta-1)^2}+\beta+1\right)^2},$$

from which we get

$$\frac{d}{d\delta}|\mu(\delta)|=\frac{\sqrt{2}C(\beta+2)}{\left(\sqrt{4C\delta+(\beta-1)^2}+\beta+1\right)\sqrt{(4C\delta+(\beta-1)^2)\left(\beta\left(\sqrt{4C\delta+(\beta-1)^2}+\beta-3\right)+4C\delta\right)}},$$

which by substituting (2.7) simplifies to

$$\frac{d|\mu(\delta)|}{d\delta}|_{\delta=\delta_0}=\frac{C}{2\left(\beta^2+5\beta+6\right)}>0.$$

Thus we have proved the following result:

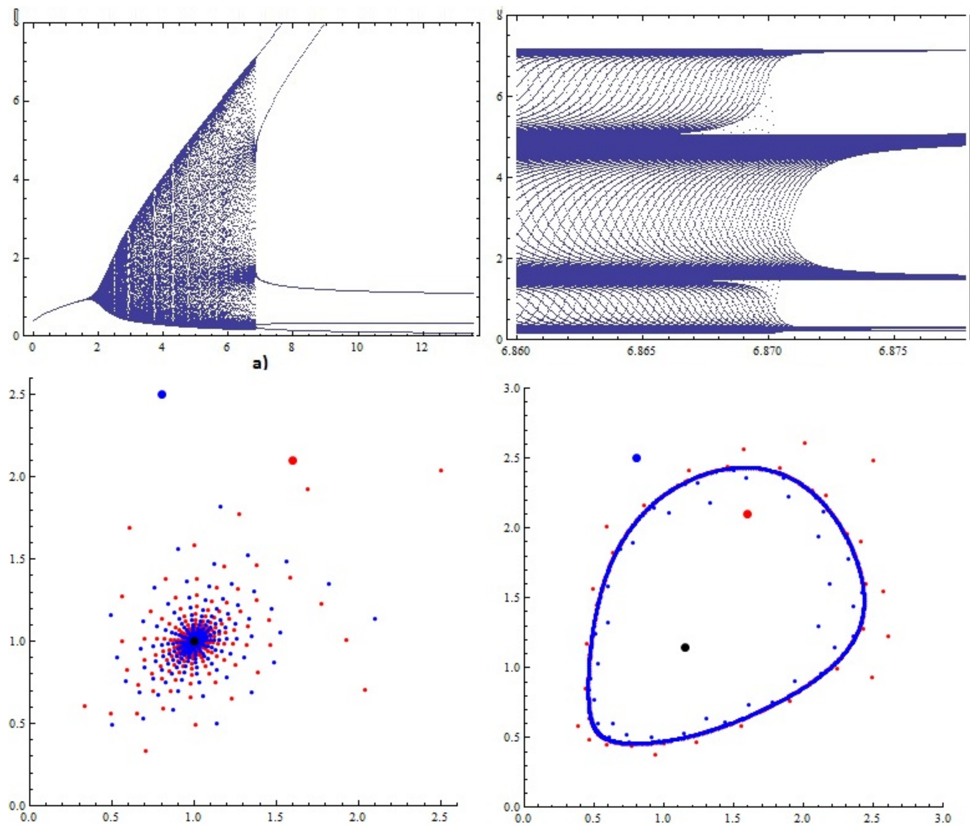

FIGURE 1. Figures a) and b): Bifurcation diagram in $(\delta - x)$ plane for $\beta = 2.2$ and $C = 3.1$; Figure c): Phase portraits when $\delta = 1.9$, $x_{-1} = 1.6$, $x_0 = 2.1$ (blue) $x_{-1} = 0.8$, $x_0 = 2.5$ (red) Figure d): Phase portraits when $\delta = 2.7$, $x_{-1} = 1.6$, $x_0 = 2.1$ (blue) $x_{-1} = 0.8$, $x_0 = 2.5$ (red).

Theorem 2.1. *Let*

$$\delta_0 = \frac{2(1+\beta)}{C} \text{ and } \bar{x} = \frac{\sqrt{4C\delta + (\beta-1)^2} + \beta - 1}{2C}.$$

Assume that $C, \delta > 0$ and $\beta \geq 0$. Then there is a neighborhood U of the equilibrium point $\bar{x}$ and a $\rho > 0$ such that for $|\delta - \delta_0| < \rho$ and $x_0, x_{-1} \in U$, the ω-limit set of solution of Eq.(2.1), with initial condition x_0, x_{-1} is the equilibrium point $\bar{x}$ if $\delta < \delta_0$ and belongs to a closed invariant C^1 curve $\Gamma(\delta)$ encircling $\bar{x}$ if $\delta > \delta_0$. Furthermore, $\Gamma(\delta_0) = 0$.

The visual illustration of Theorem 2.1. is given in Figure 1.

2.2. The case $x_{n+1}=\frac{x_n}{Bx_nx_{n-1}+Cx_{n-1}^2+Dx_n}$, $B,C,D>0$

Equation

$$x_{n+1}=\frac{x_n}{Bx_nx_{n-1}+Cx_{n-1}^2+Dx_n},\quad n=0,1,\dots \tag{2.10}$$

has the equilibrium point $\bar{x}=\frac{2}{\sqrt{4(B+C)+D^2}+D}$.

In order to apply Theorem we make a change of variable $y_n=x_n-\bar{x}$. Then, new equation is given by

$$y_{n+1}=\frac{\bar{x}+v_n}{B(\bar{x}+v_n)(\bar{x}+v_{n-1})+C(\bar{x}+v_{n-1})^2+D(\bar{x}+v_n)}-\bar{x}. \tag{2.11}$$

By using the substitution $u_n=y_{n-1},\ v_n=y_n$ we write Eq.(2.11) in the equivalent form:

$$\begin{aligned} u_{n+1}&=v_n\\ v_{n+1}&=\frac{\bar{x}+v_n}{B(\bar{x}+u_n)(\bar{x}+v_n)+C(\bar{x}+u_n)^2+D(\bar{x}+v_n)}-\bar{x}. \end{aligned} \tag{2.12}$$

Define the map F as:

$$F(u,v)=\begin{pmatrix} v\\ \frac{\bar{x}+v}{B(\bar{x}+u)(\bar{x}+v)+C(\bar{x}+u)^2+D(\bar{x}+v)}-\bar{x}\end{pmatrix}.$$

Then F has the unique fixed point $(0,0)$ and maps $(-\bar{x},\infty)^2$ into $(-\bar{x},\infty)^2$. The Jacobian matrix of F is given by

$$Jac_F(u,v)=\begin{pmatrix} 0 & 1\\ -\frac{(v+\bar{x})(2C(u+\bar{x})+B(v+\bar{x}))}{(D(v+\bar{x})+(u+\bar{x})(C(u+\bar{x})+B(v+\bar{x})))^2} & \frac{C(u+\bar{x})^2}{(D(v+\bar{x})+(u+\bar{x})(C(u+\bar{x})+B(v+\bar{x})))^2}\end{pmatrix}.$$

At $(0,0)$, $Jac_F(u,v)$ has the form

$$J_0=JacF(0,0)=\begin{pmatrix} 0 & 1\\ -\frac{B+2C}{(D+(B+C)\bar{x})^2} & \frac{C}{(D+(B+C)\bar{x})^2}\end{pmatrix}. \tag{2.13}$$

The eigenvalues of (2.13) are $\mu(B)$ and $\overline{\mu(B)}$ where

$$\mu(B)=\frac{C+i\sqrt{4(B+2C)(\bar{x}(B+C)+D)^2-C^2}}{2(\bar{x}(B+C)+D)^2},$$

since

$$C^2-4(B+2C)(\bar{x}(B+C)+D)^2<0.$$

Now we have

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{B+2C}{(D+(B+C)\bar{x})^2} & \frac{C}{(D+(B+C)\bar{x})^2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(\delta,u,v) \\ f_2(\delta,u,v) \end{pmatrix}, \tag{2.14}$$

and

$$f_1(\delta,u,v) = 0$$

$$f_2(\delta,u,v) = \frac{\bar{x}+v}{(\bar{x}+u)\left(B\left(\bar{x}+v\right)+C\left(\bar{x}+u\right)\right)+D\left(\bar{x}+v\right)} + \frac{u(B+2C)}{\left(\bar{x}(B+C)+D\right)^2} - \frac{Cv}{\left(\bar{x}(B+C)+D\right)^2} - \bar{x}.$$

Let

$$B_0 = \frac{C\left(C-2D^2\right)}{D^2} \text{ and } C-2D^2 > 0. \tag{2.15}$$

For $B = B_0$ we obtain

$$\bar{x} = \frac{D}{C} \text{ and } J_0 = \begin{pmatrix} 0 & 1 \\ -1 & \frac{D^2}{C} \end{pmatrix}.$$

The eigenvalues of J_0 are $\mu(B_0)$ and $\overline{\mu(B_0)}$ where

$$\mu(B_0) = \frac{D^2 + i\sqrt{4C^2-D^4}}{2C}.$$

The eigenvectors corresponding to $\mu(B)$ and $\overline{\mu(B)}$ are $v(B_0)$ and $\overline{v(B_0)}$ where

$$v(B_0) = \left(\frac{D^2 - i\sqrt{4C^2-D^4}}{2C}, 1\right).$$

One can prove that

$$\begin{aligned}
|\mu(B_0)| &= 1, \\
\mu^2(B_0) &= \frac{D^4}{2C^2} + \frac{iD^2\sqrt{4C^2-D^4}}{2C^2} - 1, \\
\mu^3(B_0) &= \frac{D^6 - 3C^2D^2}{2C^3} - \frac{i\left(C^2-D^4\right)\sqrt{4C^2-D^4}}{2C^3}, \\
\mu^4(B_0) &= \frac{D^8}{2C^4} - \frac{2D^4}{C^2} + \frac{iD^2\sqrt{4C^2-D^4}\left(D^4-2C^2\right)}{2C^4} + 1
\end{aligned}$$

from which follows that $\mu^k(B_0) \neq 1$ for $k = 1,2,3,4$.

Substituting $B = B_0$ and $\bar{x}$ into (2.14) we obtain

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{D^2}{C} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix},$$

and

$$h_1(u,v) = f_1(B_0,u,v) = 0,$$

$$h_2(u,v) = f_2(B_0,u,v) = \\ = \frac{C^3u^2v + C^2Du\left(Du^2 - 2Duv - Dv^2 + u\right) - CD^3\left(u^2(Dv+1) - 2Duv^2 + v^2\right) + D^5v^2}{C\left(C^2uv + D^2(Cu(u-2v)+1) + CD(u+v) + D^3(-v)\right)}.$$

Hence, for $B = B_0$ system (2.12) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{D^2}{C} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} h_1(u_n,v_n) \\ h_2(u_n,v_n) \end{pmatrix}. \tag{2.16}$$

Let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}$$

where

$$P = \begin{pmatrix} \frac{D^2}{2C} & \frac{\sqrt{4C^2-D^4}}{2C} \\ 1 & 0 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{2C}{\sqrt{4C^2-D^4}} & -\frac{D^2}{\sqrt{4C^2-D^4}} \end{pmatrix}.$$

Then system (2.16) is equivalent to its normal form

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{D^2}{2C} & -\frac{\sqrt{4C^2-D^4}}{2C} \\ \frac{\sqrt{4C^2-D^4}}{2C} & \frac{D^2}{2C} \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + P^{-1}H\left(P\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}\right),$$

where

$$H\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix}.$$

Let

$$G\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix} = P^{-1}H\left(P\begin{pmatrix} u \\ v \end{pmatrix}\right).$$

By a straightforward calculation we obtain that

$$g_1(u,v) = \frac{\Lambda(u,v)}{\Upsilon(u,v)}$$

$$g_2(u,v) = -\frac{D^2\Lambda(u,v)}{\Upsilon(u,v)\sqrt{4C^2-D^4}},$$

where

$$\begin{aligned}\Lambda(u,v) = 8C^4uv^2 + 8C^3Dv^2(1-2Du) - 2C^2D^3\left(Du^3 - Duv^2 + 4u^2 + 4v^2\right)\\ - D^2v\sqrt{4C^2-D^4}\left(-4C^2v^2 - 4CDu + D^3\left(Du^2 + Dv^2 + 4u\right)\right)\\ + 2CD^5\left(2Du^3 + 2Duv^2 + 5u^2 - v^2\right) + D^7\left(-\left(Du^3 + Duv^2 + 2u^2 - 2v^2\right)\right),\end{aligned}$$

$$\begin{aligned}\Upsilon(u,v) = 2CD\left(2C^2\left(Du^2 + 2Dv^2 + 2u\right) - 2CD\left(2D^2u^2 + Du - 2\right) + D^5\left(u^2 - v^2\right)\right)\\ + 4Cv\sqrt{4C^2-D^4}\left(C^2u + C\left(D - 2D^2u\right) + D^4u\right).\end{aligned}$$

By *Mathematica* aided calculation we obtain that

$$\begin{aligned}\xi_{20}(0,0) &= -\frac{\left(C-D^2\right)\left(C\sqrt{4C^2-D^4} + D^2\sqrt{4C^2-D^4} + iCD^2 - iD^4\right)}{4CD\sqrt{4C^2-D^4}}\\ \xi_{11}(0,0) &= \frac{\left(C-D^2\right)^2\left(\sqrt{4C^2-D^4} - iD^2\right)}{2CD\sqrt{4C^2-D^4}}\\ \xi_{02}(0,0) &= -\frac{\left(C-D^2\right)\left(-D^4\sqrt{4C^2-D^4} + CD^2\left(\sqrt{4C^2-D^4} - iD^2\right) + C^2\left(\sqrt{4C^2-D^4} - 3iD^2\right) + iD^6\right)}{4C^2D\sqrt{4C^2-D^4}}\\ \xi_{21}(0,0) &= \frac{\left(C-2D^2\right)\left(6iC^3 - 2D^4\sqrt{4C^2-D^4} + 3CD^2\sqrt{4C^2-D^4} - 5iCD^4 + 2iD^6\right)}{8CD^2\sqrt{4C^2-D^4}}.\end{aligned}$$

Now we have

$$\begin{aligned} Re\left(\overline{\mu(B_0)}\xi_{21}(B_0)\right) &= \frac{\left(C-2D^2\right)\left(3C^2-D^4\right)}{8CD^2}\\ Re\left(\frac{(1-2\mu(B_0))\bar{\mu}^2(B_0)}{1-\mu(\lambda_0)}\xi_{11}(\delta_0)\xi_{20}(\delta_0)\right) &= \frac{\left(C-D^2\right)^3\left(3C^2+4CD^2-D^4\right)}{16C^3D^2-4CD^6}\\ \xi_{11}(B_0)\overline{\xi_{11}(B_0)} &= \frac{\left(C-D^2\right)^4}{4C^2D^2-D^6}\\ \xi_{02}(B_0)\overline{\xi_{02}(B_0)} &= \frac{\left(C-D^2\right)^2\left(C^3+2C^2D^2+CD^4-D^6\right)}{16C^3D^2-4CD^6}.\end{aligned} \tag{2.17}$$

By using (2.17) after lengthy calculation we obtain that

$$a(B_0) = \frac{C^3 - 10C^2D^2 + 9CD^4 - 2D^6}{8C\left(D^2-2C\right)} > 0.$$

One can see that

$$|\mu(B)|^2 = \mu(B)\overline{\mu(B)} = \frac{B+2C}{(\bar{x}(B+C)+D)^2} = \frac{B+2C}{\left(\frac{2(B+C)}{\sqrt{4(B+C)+D^2}+D}+D\right)^2},$$

from which we obtain

$$\frac{d}{dB}|\mu(B)| = \frac{D^2\left(\sqrt{4(B+C)+D^2}+D\right)-2C\sqrt{4(B+C)+D^2}+2BD}{\sqrt{(B+2C)\left(4(B+C)+D^2\right)}\left(\sqrt{4(B+C)+D^2}+D\right)\left(D\left(\sqrt{4(B+C)+D^2}+D\right)+2B+2C\right)},$$

which in view of (2.15) yields

$$\frac{d|\mu(B)|}{dB}|_{B=B_0} = \frac{D^4}{2C^2\left(D^2-2C\right)} < 0.$$

Thus we have proved the following.

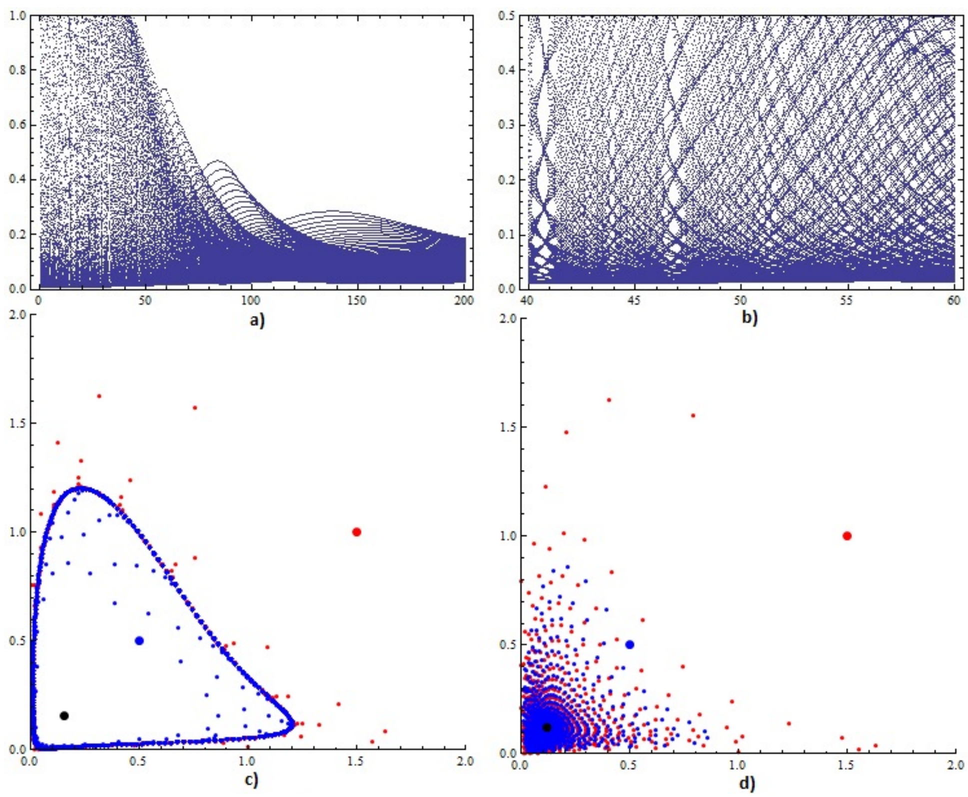

FIGURE 2. Figures a) and b): Bifurcation diagram in $(B-x)$ plane for $C = 3.6$ and $D = 0.5$; Figure c): Phase portraits when $B = 33.6$, $x_{-1} = 0.5$, $x_0 = 0.5$ (blue) $x_{-1} = 1.5$, $x_0 = 1.0$ (red) Figure d): Phase portraits when $B = 60.6$, $x_{-1} = 0.5$, $x_0 = 0.5$ (blue) $x_{-1} = 1.5$, $x_0 = 1.0$ (red).

Theorem 2.2. *Let*

$$B_0 = \frac{C(C-2D^2)}{D^2},\ C-2D^2 > 0 \text{ and } \bar{x} = \frac{D}{C}.$$

Then there is a neighborhood U of the equilibrium point $\bar{x}$ and a $\rho > 0$ such that for $|B - B_0| < \rho$ and $x_0, x_{-1} \in U$, the ω-limit set of a solution of Eq.(2.10), with initial condition x_0, x_{-1} is the equilibrium point $\bar{x}$ if $B > B_0$ and belongs to a closed invariant C^1 curve $\Gamma(B)$ encircling $\bar{x}$ if $B < B_0$. Furthermore, $\Gamma(B_0) = 0$.

Visual illustration of Theorems is given in Figure 2. Based on our simulations we pose the following.

Conjecture 2.1. *In both considered equations the equilibrium is globally asymptotically stable whenever is locally stable.*

3. Conclusion

In this paper we have found normal forms and performed bifurcation analysis for two special cases of Eq.(1.1) for which the local stability analysis indicates the possibility of Naimark-Sacker bifurcation. Indeed, we found the regions of parameters where Naimark-Sacker bifurcation occurs showing in such a way the presence of a locally stable periodic solution of unknown period. We conjectured that the equilibrium is globally asymptotically stable in the complement of the parametric region where Naimark-Sacker bifurcation occurs.

Conflict of Interests

The authors declare that they have no competing interests.

References

[1] L. J. Allen, An Introduction to Mathematical Biology, Pearson, 2006.

[2] A. M. Amleh, E. Camouzis, and G. Ladas, On the Dynamics of a Rational Difference Equation, Part I, Int. J. Difference Equ. 3 (2008), 1–35.

[3] A. M. Amleh, E. Camouzis, and G. Ladas, On the Dynamics of a Rational Difference Equation, Part II, Int. J. Difference Equ. 3 (2008), 195–225.

[4] E. Camouzis and G. Ladas, Dynamics of third-order rational difference equations with open problems and conjectures. Advances in Discrete Mathematics and Applications, 5. Chapman & Hall/CRC, Boca Raton, FL, 2008.

[5] S. N. Elaydi, An introduction to difference equations. Third edition. Undergraduate Texts in Mathematics. Springer, New York, 2005.

[6] S. N. Elaydi, Discrete chaos. With applications in science and engineering. Second edition, Chapman & Hall/CRC, Boca Raton, FL, 2008 .

[7] E. G. Enciso and E. D. Sontag, Global attractivity, I/O monotone small-gain theorems, and biological delay systems. Discrete Contin. Dyn. Syst. 14 (2006), 549–578.

[8] J. K. Hale and H. Kocak,Dynamics and Bifurcation, Texts in Applied Mathematics, 3. Springer-Verlag, New York, 1991.

[9] E. A Janowski and M. R. S. Kulenović, Attractivity and Global Stability for Linearizable Difference Equations, Comput. Math. Appl. 57(2009), 1592–1607.

[10] C. M. Kent and H. Sedaghat, Global attractivity in a quadratic-linear rational difference equation with delay. J. Difference Equ. Appl. 15 (2009), 913–925.

[11] C. M. Kent and H. Sedaghat, Global attractivity in a rational delay difference equation with quadratic terms, J. Difference Equ. Appl., 17 (2011), 457–466.

[12] M. R. S. Kulenović and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall/CRC, Boca Raton, London, 2001.

[13] M. R. S. Kulenović and O. Merino, Discrete Dynamical Systems and Difference Equations with Mathematica, Chapman and Hall/CRC, Boca Raton, London, 2002.

[14] M. R. S. Kulenović and O. Merino, Competitive-exclusion versus competitive-coexistence for systems in the plane, Discrete Contin. Dyn. Syst. Ser. B 6 (2006), 1141–1156.

[15] M. R. S. Kulenović and O. Merino, Global Bifurcations for Competitive Systems in the Plane, Discrete Contin. Dyn. Syst. Ser. B 12 (2009), 133–149.

[16] M. R. S. Kulenović and O. Merino, Invariant Manifolds for Competitive Discrete Systems in the Plane, Int. J. Bifurcations Chaos, 20 (2010), 2471–2486.

[17] M. R. S. Kulenović, E. Pilav and E. Silić, Local Dynamics and Global Attractivity of a Certain Second Order Quadratic Fractional Difference Equation, Adv. Difference Equ. 2014, 32 pp.

[18] H. Sedaghat, Nonlinear difference equations. Theory with applications to social science models, Mathematical Modelling: Theory and Applications, 15. Kluwer Academic Publishers, Dordrecht, 2003.

[19] H. Sedaghat, Global behaviours of rational difference equations of orders two and three with quadratic terms. J. Difference Equ. Appl. 15 (2009), 215–224.

[20] S. Wiggins, Introduction to applied nonlinear dynamical systems and chaos. Second edition. Texts in Applied Mathematics, 2. Springer-Verlag, New York, 2003.