Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 4, 967-981
ISSN: 1927-5307

BAYESIAN PREDICTION UNDER A CLASS OF MULTIVARIATE DISTRIBUTIONS

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Abstract. In this paper the prediction problem is studied under members of a class $\Im^*$ of multivariate distributions, constructed by AL-Hussaini and Ateya $[7-8]$. More attention is paid to bivariate compound Rayleigh ($BVCR$) distribution, which is a member of this class, as illustrative example.

Keywords: Multivariate distributions, Bayesian prediction intervals.

2000 AMS Subject Classification: 62F10; 62F15; 62N01; 62N02.

1. Introduction

Suppose that a class $\Im$ of distribution functions is of the form

$$\Im = \Big\{ F : F \equiv F_{X|\Theta}(x|\theta) = 1 - exp[-\theta\delta\lambda_\eta(x)], \quad 0 \le a < x < b \le \infty, (\theta, \delta > 0, (\theta, \delta, \eta) \in \Omega) \Big\}, \tag{1}$$

where a and b are non-negative real numbers such that a may assume the value zero and b the value infinity, $\lambda_\eta(x)$ is a continuous, monotone increasing and differentiable function

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Received February 20, 2012

of x such that $\lambda_\eta(x) \to 0$ as $x \to a^+$, $\lambda_\eta(x) \to \infty$ as $x \to b^-$ and η is a parameter (could be a vector), (θ, δ, η) belongs to a parameter space Ω . This class covers some important distributions such as the Weibull, exponential, Rayleigh, compound Weibull, compound exponential (Lomax), compound Rayleigh, Pareto, power function, beta, Gompertz and compound Gompertz distributions, among others. The failure rate and survival functions corresponding to $F \in \Im$ are, respectively, $\delta\theta\lambda'_\eta(x)$ and $e^{-\theta\delta\lambda_\eta(x)}$, so that the probability density function (*pdf*) is given, for $0 \le a < x < b \le \infty$, by

$$f_{X|\Theta}(x|\theta) = \delta\theta\lambda'_\eta(x)exp[-\theta\delta\lambda_\eta(x)],\ ' \equiv \frac{d}{dx}. \tag{2}$$

The class $\Im$ was used by AL-Hussaini and Osman [9], AL-Hussaini [4], Ahmad [$1-2$], Ahmad and Fawzy [3], AL-Hussaini and Ahmad [$5-6$] and Jafar et al [12].

1.1. A Class of multivariate distributions

AL-Hussaini and Ateya [$7-8$] constructed a class of multivariate distributions by compounding members of the class $\Im$ with the gamma distribution. The resulting multivariate distributions form a class $\Im^*$, given by
$\Im^* = \left\{F^* : F^* \equiv F_{\boldsymbol{X}}(\boldsymbol{x}) = \int f_{\boldsymbol{X}}(\boldsymbol{u})d\boldsymbol{u}\right\}$,
where $\int \equiv \int_0^{x_1} ... \int_0^{x_k}$, $\boldsymbol{u} = (u_1, ..., u_k)$, $d\boldsymbol{u} = du_k...du_1$ and $f_{\boldsymbol{X}}(\boldsymbol{x})$ is the *pdf* of the random vector $\boldsymbol{X} = (X_1, ..., X_k)$, given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\left[\prod_{i=1}^{k} c_i\lambda'_{\eta_i}(x_i)\right]\left[1+\sum_{i=1}^{k} c_i\lambda_{\eta_i}(x_i)\right]^{-(\alpha+k)}, \tag{3}$$

$$c_i = \delta_i/\beta, 0 \le a < x_i < b \le \infty, i = 1, 2, ..., k.$$

It was assumed that Θ is a positive random variable following the $gamma(\alpha, \beta)$ distribution with *pdf* $g_\Theta(\theta)$ given by

$$g_\Theta(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \theta > 0, (\alpha > 0, \beta > 0). \tag{4}$$

The *pdf* $f_{\boldsymbol{X}}(\boldsymbol{x})$ in (1.3) was obtained by writing

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \int_0^\infty \left[\prod_{i=1}^{k} f_{X_i|\Theta}(x_i|\theta)\right] g_\Theta(\theta)d\theta.$$

Maximum likelihood and Bayes estimation of the parameters of members of the class $\Im^*$ were obtained by AL-Hussaini and Ateya $[7-8]$ and particularly when the underlying population distribution is bivariate compound Weibull or bivariate compound Gompertz.

In this paper, the prediction problem is studied under members of class $\Im^*$. More attention is paid to bivariate compound Rayleigh ($BVCR$) distribution as illustrative example.

1.2. Generation of a multivariate random sample of size n from the class $\Im^*$

Knowing that $F_{X_i|\Theta}(x_i|\theta) = 1 - \exp[-\theta\delta_i\lambda_{\eta_i}(x_i)]$ and $g_\Theta(\theta) = \beta^\alpha\theta^{\alpha-1}e^{-\beta\theta}/\Gamma(\alpha)$, an observation x_{ij} is obtained by first generating θ_j from $Gamma(\alpha, \beta)$, u_i from $Uniform(0, 1)$ and then setting $x_{ij} = \lambda_{\eta_i}^{-1}\Big(-(ln\ u_i)/\theta_j\delta_i\Big), j = 1, 2, ..., n, i = 1, 2, ..., k$. This is repeated until we obtain the required multivariate random sample.

1.3. One-sample prediction

Suppose that $X_1 < X_2 < ... < X_r$ is the informative sample, representing the first r ordered lifetimes of a random sample of size n drawn from a population with probability density function (*pdf*) $f_X(x)$, cumulative distribution function (*cdf*) $F_X(x)$ and reliability function (rf) $R(x)$. In one-sample scheme the Bayesian prediction intervals (BPI) for the remaining unobserved future $(n-r)$ lifetimes are sought based on the first r observed ordered lifetimes.

For the remaining $(n-r)$ components, let $Y_s = X_{r+s}$ denote the future lifetime of the s^{th} component to fail, $1 \leq s \leq (n-r)$. The conditional density function of Y_s given that the r components had already failed is

$$g_1(y_s|\boldsymbol{\theta}) \propto [R(x_r) - R(y_s)]^{(s-1)}[R(y_s)]^{n-r-s}[R(x_r)]^{-(n-r)}f_X(y_s|\boldsymbol{\theta}), y_s > x_r, \tag{5}$$

$\boldsymbol{\theta}$ is the vector of parameters.

The predictive density function is given by

$$g_1^*(y_s|\boldsymbol{x}) = \int_{\boldsymbol{\Theta}} g_1(y_s|\boldsymbol{\theta})\pi^*(\boldsymbol{\theta}|\boldsymbol{x})d\boldsymbol{\theta}, y_s > x_r, \tag{6}$$

$\pi^*(\boldsymbol{\theta}|\boldsymbol{x})$ is the posterior density function of $\boldsymbol{\theta}$ given $\boldsymbol{x}$ and $\boldsymbol{x} = (x_1, ..., x_r)$.

A $(1-\tau)$ % BPI for y_s is an interval (L, U) such that

$$P(Y_s > L|\boldsymbol{x}) = \int_L^\infty g_1^*(y_s|\boldsymbol{x})dy_s = 1 - \frac{\tau}{2}, L > x_r, \tag{7}$$

$$P(Y_s > U|\boldsymbol{x}) = \int_U^\infty g_1^*(y_s|\boldsymbol{x})dy_s = \frac{\tau}{2}, U > x_r. \tag{8}$$

By solving equations (7) and (8), we get the interval (L, U).

1.4. Two-sample prediction

Let $X_1 < X_2 < ... < X_r$ and $Z_1 < Z_2 < ... < Z_m$ represent informative (type II censored) sample from a random sample of size n and a future ordered sample of size m, respectively. It is assumed that the two samples are independent and drawn from a population with $(pdf) f_X(x)$, $(cdf) F_X(x)$ and $(rf) R(x)$.

Our aim is to obtain the BPI for $Z_s, s = 1, 2, ..., m$. The conditional density function of Z_s, given the vector of parameters $\boldsymbol{\theta}$, is

$$g_2(z_s|\boldsymbol{\theta}) \propto [1 - R(z_s)]^{(s-1)}[R(z_s)]^{m-s}f_X(z_s|\boldsymbol{\theta}), z_k > 0, \tag{9}$$

$\boldsymbol{\theta}$ is the vector of parameters.

The predictive density function is given by

$$g_2^*(z_s|\boldsymbol{x}) = \int_{\boldsymbol{\Theta}} g_2(z_s|\boldsymbol{\theta})\pi^*(\boldsymbol{\theta}|\boldsymbol{x})d\boldsymbol{\theta}, z_s > 0, \tag{10}$$

$\pi^*(\boldsymbol{\theta}|\boldsymbol{x})$ is the posterior density function of $\boldsymbol{\theta}$ given $\boldsymbol{x}$ and $\boldsymbol{x} = (x_1, ..., x_r)$.

A $(1-\tau)$ % BPI for z_s is an interval (L, U) such that

$$P(Z_s > L|\,\boldsymbol{x}) = \int_L^\infty g_2^*(z_s|\,\boldsymbol{x})dz_s = 1 - \frac{\tau}{2}, \tag{11}$$

$$P(Z_s > U|\,\boldsymbol{x}) = \int_U^\infty g_2^*(z_s|\,\boldsymbol{x})dz_s = \frac{\tau}{2}. \tag{12}$$

By solving equations (11) and (12), we get the interval (L, U).

2. Baysian prediction intervals for future bivariate observations

The main goal in this section is to study the one-sample and two-sample prediction problems in case of bivariate informative observations.

While ordering a set of univariate random variables is a clear and straight-forward matter as it can be done by simply ordering the set of random variables, such ordering is not as clear if we are dealing with a set of random vectors.

Barnett [10] classified the principles used for ordering multivariate date into four principles : marginal, reduced (aggregate), partial and conditional (sequential) ordering. An interesting detailed discussion of such principles with illustrative examples are given in Barnett's paper.

In our paper, we wish to predict bivariate random vectors. The first components of the predicted random vectors are based on the ordered first components of the informative sample, as is done in the univariate case. To predict the second components, we compute the norms of each vector of the informative sample, order the norms and then predict the future norms as is done in the univariate case. The relation between the components of vectors and norms enables us to obtain the second components of the predicted vectors. In other words, we obtain the second component of a predicted vector from the knowledge of the values of the first component and the norm of the vector.

2.1. One-sample prediction

Let $(X_1, Y_1), ..., (X_r, Y_r)$ be the first r bivariate informative observations from a random sample of size n of bivariate observations. Suppose that the first components of such informative vectors are ordered, that is $X_1 < X_2 < ... < X_r$ and that their norms are given by $Z_1, Z_2, ..., Z_r$.

To obtain $BPI's$ for the remaining future vectors, denoted by $(X_1^*, Y_1^*), ..., (X_{n-r}^*, Y_{n-r}^*)$, where $X_1^* < X_2^* < ... < X_{n-r}^*$ and norms $Z_1^* < Z_2^* < ... < Z_{n-r}^*$ we apply the following steps:

(1) based on ordered $Z_1, Z_2, ..., Z_r$, denoted by $Z_{1:r}, Z_{2:r}, ..., Z_{r:r}$ compute the $BPI's$ for $Z_s^*, s = 1, 2, ..., (n-r)$, say (L_{1s}, U_{1s}),

(2) based on $X_1 < X_2 < ... < X_r$ compute the $BPI's$ for $X_s^*, s = 1, 2, ..., (n-r)$, say (L_{2s}, U_{2s}),

(3) from (1) and (2), compute the $BPI's$ for $Y_s^*, s = 1, 2, ..., (n-r)$ which are $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$. This is true, since $z_s^* = (x_s^{*\,2} + y_s^{*\,2})^{1/2}$,

(4) from (2) and (3), the $BPI's$ for $(X_s^*, Y_s^*), s = 1, 2, ..., (n-r)$ is $(L_{2s}, [L_{1s}^2 - L_{2s}^2]^{1/2}), (U_{2s}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

2.2. Two-sample prediction

In this case the first r bivariate informative observations $(X_1, Y_1), ..., (X_r, Y_r)$ from a random sample of size n is such that $X_1 < X_2 < ... < X_r$ with norms $Z_1, Z_2, ..., Z_r$. An independent future sample of size m is $(X_1^*, Y_1^*), ..., (X_m^*, Y_m^*)$, where $X_1^* < X_2^* < ... < X_m^*$ and norms $Z_1^* < Z_2^* < ... < Z_m^*$. To obtain the $BPI's$ of the future sample, we apply the following steps:

(1) based on ordered $Z_1, Z_2, ..., Z_r$, denoted by $Z_{1:r}, Z_{2:r}, ..., Z_{r:r}$ compute the $BPI's$ for $Z_s^*, s = 1, 2, ..., m$, say (L_{1s}, U_{1s}) ,

(2) based on $X_1 < X_2 < ... < X_r$ compute the $BPI's$ for $X_s^*, s = 1, 2, ..., m$, say (L_{2s}, U_{2s}),

(3) from (1) and (2), compute the $BPI's$ for $Y_s^*, s = 1, 2, ..., m$ which are $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

(4) from (2) and (3), the $BPI's$ for $(X_s^*, Y_s^*), s = 1, 2, ..., m$
is $(L_{2s}, [L_{1s}^2 - L_{2s}^2]^{1/2}), (U_{2s}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

3. One-sample prediction in case of (BVCR) distribution

If, in (3), $k = 2, \lambda_\eta(x) = x^2, \lambda_\eta(y) = y^2, \delta_1 = \delta_2 = 1$ so that $c_1 = c_2 = 1/\beta = c$, then (X, Y) has a bivariate compound Rayleigh $(BVCR)\, pdf$, given by

$$f_{X,Y}(x, y) = 4\,\alpha\,(\alpha + 1)\,c^2\,x\,y\,[1 + c(x^2 + y^2)]^{-(\alpha+2)}, x > 0, y > 0. \tag{13}$$

The marginal $pdf's$ of the random variables X and Y are given, respectively, by

$$f_X(x) = 2\,\alpha\,c\,x\,[1 + c\,x^2]^{-(\alpha+1)}, x > 0, \tag{14}$$

$$f_Y(y) = 2\,\alpha\,c\,y\,[1 + c\,y^2]^{-(\alpha+1)}, y > 0. \tag{15}$$

In this section we apply the steps given in Subsection 2.1.

Step 1

The norm Z of the vector (X, Y) is given by $Z = (X^2 + Y^2)^{1/2}$. In APPENDIX A the pdf and hence cdf and rf are derived. Such functions are given by

$$f_Z(z) = 2\,\alpha\,(\alpha + 1)\,c^2\,z^3\,[1 + cz^2]^{-(\alpha+2)}, z > 0, \tag{16}$$

$$F_Z(z) = 1 - \alpha\,c\,z^2\,[1 + cz^2]^{-(\alpha+1)} - [1 + cz^2]^{-\alpha}, z > 0, \tag{17}$$

$$R(z) = \alpha\,c\,z^2\,[1 + cz^2]^{-(\alpha+1)} + [1 + cz^2]^{-\alpha}, z > 0. \tag{18}$$

From (16) and (18), the conditional density of Z_s^* given (c, α) is obtained (see APPENDIX B), as

$$\begin{aligned} g_1(z_s^* | \, c, \alpha) \propto \sum^{*} B_{i,j,l,s}\, c^{k_3}\, \alpha^{k_4}\, (\alpha + 1)\, z_s^{*(2(k_1 - j) + 3)}\, (1 + c\, z_s^{*2})^{-\alpha\, k_1 - k_1 + j - \alpha - 2} \\ .z_{r:r}^{2(k_2 - l)}\, (1 + c\, z_{r:r}^2)^{-\alpha\, k_2 - k_2 + l}, \end{aligned} \tag{19}$$

where

$$\sum^{*}=\sum_{i=0}^{s-1}\sum_{j=0}^{k_1}\sum_{l=0}^{k_2},\ B_{i,j,l,s}=(-1)^i\binom{s-1}{i}\binom{k_1}{j}\binom{k_2}{l},$$

$$k_1=n-r+i-s, k_2=s-i-(n-r)-1, k_3=1-j-l, k_4=-j-l.$$

Suppose that the prior belief of the experimenter is given by the *pdf* $\pi(c,\alpha)=\pi_1(c|\alpha)\,\pi_2(\alpha), c|\alpha\sim Gamma(c_1,\alpha)$ and $\alpha\sim Gamma(c_2,c_3)$. So that

$$\pi(c,\alpha)\propto\alpha^{c_1+c_2-1}\,c^{c_1-1}\,e^{-\alpha(c+c_3)}. \tag{20}$$

The likelihood function of (c,α) given $Z_{1:r},...,Z_{r:r}$ is given by

$$\begin{aligned}L(c,\alpha|z_{1:r},...,z_{r:r})&\propto[R(z_{r:r})]^{n-r}\prod_{i_1=1}^{r}f(z_i)\\ &=2^r\,\alpha^r\,c^{2r}(\alpha+1)^r\left(\prod_{i_1}^{r}z_{i_1}\right)^3\left(\prod_{i_1}^{r}(1+cz_{i_1}^2)\right)^{-(\alpha+2)}\sum_{l_1}^{n-r}\binom{n-r}{l_1}\alpha^{n-r-l_1}c^{n-r-l_1}\\ &\quad z_{r:r}^{2(n-r-l_1)}(1+c\,z_{r:r}^2)^{-\alpha(n-r)-(n-r)+l_1}.\end{aligned}\tag{21}$$

Since the posterior density $\pi^*(c,\alpha|z_{1:r},...,z_{r:r})\propto\pi(c,\alpha)\,L(c,\alpha|z_{1:r},...,z_{r:r})$, it follows, from (19) – (21) that

$$\begin{aligned}&g_1(z_s^*|\,c,\alpha)\pi^*(c,\alpha|\,z_{1:r},...,z_{r:r})=A\sum^{**}B^*_{i,j,l,s,l_1}\,c^{n+r+c_1-j-l-l_1}\\ &\alpha^{n+c_1+c_2-j-l-l_1-1}(\alpha+1)^{r+1}\left(\prod_{i_1}^{r}z_{i_1}\right)^3\left(\prod_{i_1}^{r}(1+cz_{i_1}^2)\right)^{-(\alpha+2)}z_s^{*(2(k_1-j)+3)}\\ &(1+c\,z_s^{*2})^{-\alpha\,k_1-k_1+j-\alpha-2}\,z_{r:r}^{2(s-i-l_1-l-1)}\,(1+c\,z_{r:r}^2)^{-\alpha\,(s-i-1)-s+i+l_1+l+1}\\ &\exp[-\alpha\,c-\alpha\,c_3],\end{aligned}\tag{22}$$

where A is a normalizing constant and

$$\sum^{**}=\sum^{*}\sum_{l_1=0}^{n-r},\ B^*_{i,j,l,s,l_1}=B_{i,j,l,s}\binom{n-r}{l_1}.$$

It then follows, from (6) and (22) that the predictive density function of Z_s^* is given by

$$g_1^*(z_s^*|\,z_{1:r},...,z_{r:r})=\int_0^\infty\int_0^\infty g_1(z_s^*|\,c,\alpha)\pi^*(c,\alpha|\,z_{1:r},...,z_{r:r})dc\,d\alpha. \tag{23}$$

To obtain $(1-\tau)$ % BPI for Z_s^*, say (L_{1s}, U_{1s}),we solve the following two nonlinear equations, numerically,

$$P(Z_s^* > L_{1s}|\, z_{1:r}, ..., z_{r:r}) = \int_{L_{1s}}^{\infty} g_1^*(z_s^*|\, z_{1:r}, ..., z_{r:r}) dz_s^* = 1 - \frac{\tau}{2}, L_{1s} > z_{r:r}, \tag{24}$$

$$P(Z_s^* > U_{1s}|\, z_{1:r}, ..., z_{r:r}) = \int_{U_{1s}}^{\infty} g_1^*(z_s^*|\, z_{1:r}, ..., z_{r:r}) dz_s^* = \frac{\tau}{2}, U_{1s} > z_{r:r}. \tag{25}$$

Step 2

By using the *pdf* (14) and its *cdf*, the predictive density function of X_s^* can be written as follows

$$g_1^*(x_s^*|\, x_1, ..., x_r) = \int_0^{\infty}\int_0^{\infty} g_1(x_s^*|\, c, \alpha)\pi^*(c, \alpha|\, x_1, ..., x_r) dc\, d\alpha, \tag{26}$$

where

$$g_1(x_s^*|\, c, \alpha)\pi^*(c, \alpha|\, x_1, ..., x_r) = A_1 \sum_{i=0}^{s-1} B_{i,s}\, c^{c_1+r} \alpha^{c_1+c_2+r} \left(\prod_{i_1}^{r} x_{i_1}\right)$$
$$\left(\prod_{i_1}^{r}(1 + c\, x_{i_1}^2)\right)^{-(\alpha+1)} x_s^*\, (1 + c\, x_s^{*2})^{(-\alpha\,(n-r+i-s+1)-1)}\, (1 + c\, x_r^2)^{-\alpha\,(s-i-1)} \tag{27}$$
$$\exp[-\alpha\, c - \alpha\, c_3],$$

where A_1 is a normalizing constant and $B_{i,s} = (-1)^i \binom{s-1}{i}$.

To obtain $(1-\tau)$ % BPI for X_s^*, say (L_{2s}, U_{2s}),we solve the following two nonlinear equations, numerically,

$$P(X_s^* > L_{2s}|\, x_1, ..., x_r) = \int_{L_{2s}}^{\infty} g_1^*(x_s^*|\, x_1, ..., x_r) dx_s^* = 1 - \frac{\tau}{2}, L_{2s} > x_r, \tag{28}$$

$$P(X_s^* > U_{2s}|\, x_1, ..., x_r) = \int_{U_{2s}}^{\infty} g_1^*(x_s^*|\, x_1, ..., x_r) dx_s^* = \frac{\tau}{2}, U_{2s} > x_r. \tag{29}$$

Step 3

From steps 2 and 3, a $(1-\tau)$ % BPI for Y_s^* is $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

4. Two-sample prediction in case of (BVCR) distribution

In this case we apply the steps in Subsection 2.2 as follows

Step 1

Substituting from (16) and (18) in (9) and then using (20) and (21) we can write

$$\begin{aligned} g_2(z_s^*|\, c,\alpha)\pi^*(c,\alpha|\, z_{1:r},...,z_{r:r}) &= A\sum^{**} B^*_{i,j,s,m}\, c^{n+r+c_1-l_1+k-j+1} \\ &\alpha^{n+c_1+c_2+k-j-l_1}(\alpha+1)^{r+1}\left(\prod_{i_1}^{r} z_{i_1}\right)^3\left(\prod_{i_1}^{r}(1+cz_{i_1}^2)\right)^{-(\alpha+2)} z_s^{*(2(k-j)+3)} \\ &(1+c\,z_s^{*2})^{-\alpha\,k-k+j-\alpha-2}\, z_{r:r}^{2(n-r-l_1)}\,(1+c\,z_{r:r}^2)^{-\alpha\,(n-r)-(n-r)+l_1} \\ &\exp[-\alpha\,c-\alpha\,c_3], \end{aligned} \tag{30}$$

where

$$\sum^{**}=\sum_{i=0}^{s-1}\sum_{j=0}^{k}\sum_{l_1=0}^{n-r},\, B^*_{i,j,s,m}=(-1)^i\binom{s-1}{i}\binom{k}{j}\binom{n-r}{l_1}, k=m-s+i,$$

and A is a normalizing constant.

It then follows that the predictive density function of Z_s^* is given by

$$g_2^*(z_s^*|\, z_{1:r},...,z_{r:r})=\int_0^\infty\int_0^\infty g_1(z_s^*|\, c,\alpha)\pi^*(c,\alpha|\, z_{1:r},...,z_{r:r})dc\,d\alpha. \tag{31}$$

To obtain $(1-\tau)$ % BPI for Z_s^*, say (L_{1s},U_{1s}),we solve the following two nonlinear equations, numerically,

$$P(Z_s^*>L_{1s}|\, z_{1:r},...,z_{r:r})=\int_{L_{1s}}^\infty g_2^*(z_s^*|\, z_{1:r},...,z_{r:r})dz_s^*=1-\frac{\tau}{2}, L_{1s}>0, \tag{32}$$

$$P(Z_s^*>U_{1s}|\, z_{1:r},...,z_{r:r})=\int_{U_{1s}}^\infty g_2^*(z_s^*|\, z_{1:r},...,z_{r:r})dz_s^*=\frac{\tau}{2}, U_{1s}>0. \tag{33}$$

Step 2

Using the *pdf* (14), its *cdf* and the same prior as in (20) the predictive density function of X_s^* is given by

$$g_2^*(x_s^*|\, x_1,...,x_r) = \int_0^\infty \int_0^\infty g_2(x_s^*|\, c,\alpha)\pi^*(c,\alpha|\, x_1,...,x_r) dc\, d\alpha, \tag{34}$$

where

$$g_2(x_s^*|\, c,\alpha)\pi^*(c,\alpha|\, x_1,...,x_r) = A_1 \sum_{i=0}^{s-1} B_{i,s}\, c^{r+c_1} \alpha^{c_1+c_2+r} \left(\prod_{i_1}^{r} x_{i_1}\right)$$
$$\left(\prod_{i_1}^{r}(1+c\, x_{i_1}^2)\right)^{-(\alpha+1)} x_s^* (1+c\, x_s^{*2})^{(-\alpha\,(m+i-s+1)-1)} (1+c\, x_r^2)^{-\alpha\,(n-r)} \tag{35}$$
$$\exp[-\alpha\, c - \alpha\, c_3],$$

where A_1 is a normalizing constant and

$$B_{i,s} = (-1)^i \binom{s-1}{i}.$$

To obtain $(1-\tau)$ % BPI for X_s^*, say (L_{2s}, U_{2s}),we solve the following two nonlinear equations, numerically,

$$P(X_s^* > L_{2s}|\, x_1,...,x_r) = \int_{L_{2s}}^\infty g_2^*(x_s^*|\, x_1,...,x_r) dx_s^* = 1 - \frac{\tau}{2}, L_{2s} > 0, \tag{36}$$

$$P(X_s^* > U_{2s}|\, x_1,...,x_r) = \int_{U2s}^\infty g_2^*(x_s^*|\, x_1,...,x_r) dx_s^* = \frac{\tau}{2}, U_{2s} > 0. \tag{37}$$

Step 3

From steps 2 and 3, a $(1-\tau)$ % BPI for Y_s^* is $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

5. Numerical example

In this section we follow the steps

(1) given the set of prior parameters, generate the parameters (c,α),

(2) using the generated population parameters, generate a bivariate random sample of size n, say $(X_1,Y_1),...,(X_n,Y_n)$ as shown in subsection 1.2

(3) follow steps in Subsections 2.1 and 2.2.

In Tables (1) and (2) 95% $BPI's$ are computed in case of the one- and two-sample predictions, respectively, with the same parameters c, α, hyperparameters c_1, c_2, c_3 and using informative samples of different sizes, r.

Table(1):One-Sample prediction: 95 % $BPI's$ for Z_s^*, Y_s^* and $X_s^*, s = 1, 2, 3$.

r	$c_1 = 1.0, c_2 = 1.5, c_3 = 2.0$ $c = 1.3, \alpha = 0.76$	z_1^*	z_2^*	z_3^*
10	Coverage Percentage	97.43	98.65	98.97
	BPI	(3.9064,5.6565)	(4.4398,6.6373)	(4.8985,7.8809)
	BPI Length	1.7501	2.1975	2.9824
20	Coverage Percentage	96.33	97.42	97.99
	BPI	(3.8761,5.4953)	(4.4523,6.4451)	(4.8723,7.1942)
	BPI Length	1.6192	1.9928	2.3219
45	Coverage Percentage	95.80	96.12	96.87
	BPI	(3.7670,4.8779)	(4.3687,6.1819)	(4.7585,6.8615)
	BPI Length	1.1109	1.8132	2.1030
r		x_1^*	x_2^*	x_3^*
10	Coverage Percentage	96.11	98.41	98.84
	BPI	(2.4110,3.0393)	(2.7269,3.7051)	(3.1654,4.4564)
	BPI Length	0.6283	0.9782	1.2910
20	Coverage Percentage	95.88	96.23	97.16
	BPI	(2.3720,2.9688)	(2.5971,3.4690)	(3.0912,4.1933)
	BPI Length	0.5968	0.8719	1.1021
45	Coverage Percentage	95.41	95.92	96.10
	BPI	(2.2891,2.7694)	(2.4870,3.2379)	(2.9714,3.9531)
	BPI Length	0.4803	0.7509	0.9817
r		y_1^*	y_2^*	y_3^*
10	Coverage Percentage	97.40	98.04	98.67
	BPI	(3.0736,4.7706)	(3.5036,5.5069)	(3.7389,6.4999)
	BPI Length	1.6970	2.0033	2.7610
20	Coverage Percentage	96.89	97.08	97.68
	BPI	(3.0655,4.6243)	(3.6164,5.4319)	(3.7661,5.8457)
	BPI Length	1.5588	1.8154	2.0796
45	Coverage Percentage	95.88	96.50	97.12
	BPI	(2.9917,4.0155)	(3.5917,5.2661)	(3.7167,5.6083)
	BPI Length	1.0238	1.6744	1.8916

Table(2):Two-Sample prediction: 95 % $BPI's$ for Z_s^*, Y_s^* and $X_s^*, s = 1, 2, 3.$

r	$c_1 = 1.0, c_2 = 1.5, c_3 = 2.0$ $c = 1.3, \alpha = 0.76$	z_1^*	z_2^*	z_3^*
10	Coverage Percentage	96.98	97.78	98.65
	BPI	(1.4319,2.1823)	(2.2627,3.4651)	(3.3804,5.2912)
	BPI Length	0.7501	1.2014	1.9108
20	Coverage Percentage	95.79	96.45	97.03
	BPI	(1.4053,2.0401)	(2.2816,3.1608)	(3.2239,4.5159)
	BPI Length	0.6348	0.8792	1.2920
45	Coverage Percentage	94.98	95.14	96.39
	BPI	(1.7919,1.9721)	(2.2502,3.0318)	(3.1705,4.1634)
	BPI Length	0.1801	0.7816	0.9925
r		x_1^*	x_2^*	x_3^*
10	Coverage Percentage	97.53	97.99	98.36
	BPI	(0.8941,1.2541)	(1.3730,1.9512)	(2.1106,2.9016)
	BPI Length	0.3601	0.5782	0.7910
20	Coverage Percentage	96.55	96.98	97.13
	BPI	(0.8714,1.2152)	(1.2537,1.6696)	(2.0943,2.7255)
	BPI Length	0.3438	0.4159	0.63111
45	Coverage Percentage	95.81	96.30	97.03
	BPI	(0.8680,0.6083)	(1.2301,1.6013)	(2.0805,2.5665)
	BPI Length	0.2403	0.3709	0.5861
r		y_1^*	y_2^*	y_3^*
10	Coverage Percentage	98.63	98.70	99.49
	BPI	(1.1184,1.7859)	(1.7985,3.2524)	(2.6405,4.4264)
	BPI Length	0.6676	1.4539	1.7840
20	Coverage Percentage	97.97	98.13	99.01
	BPI	(1.1025,1.6387)	(1.9062,2.6839)	(2.4510,3.6011)
	BPI Length	0.5362	0.7776	1.1496
45	Coverage Percentage	96.78	96.90	97.62
	BPI	(1.0681,1.5677)	(1.8842,2.5744)	(2.3924,3.2783)
	BPI Length	0.4816	0.6902	0.8859

6. Concluding remarks

In Tables (1) and (2) we take different sizes for the informative sample, 10, 20 and 45 and predict the first three future observations .

In these tables, we observe that

(1) The length of the $BPI's$ and the number of samples which cover these intervals increase by increasing s and decrease by increasing the informative sample size.

(2) The results become better as the informative sample size r gets larger.

(3) In all cases, the simulated percentage coverages are at least 95%.

(4) There is no particular reason for choosing the hyperparameters (c_1, c_2, c_3) as $(1, 1.5, 2)$.

(5) If the hyperparameters are unknown, they can be estimated by using the empirical Bayes method [see Maritz and Lwin[13]] or the hierarchical method [see Bernardo and Smith[11]].

APPENDIX A

Proof of equations (16)-(18)

From the joint density function of the random variables X and Y which is given by (13) and using the transforms $X = Z\cos\Theta$ and $Y = Z\sin\Theta$ we get the joint density function of the random variables Z and Θ in the form

$$f_{Z,\Theta}(z,\theta) = 4\,\alpha\,(\alpha+1)\,c^2\,z^3\,\sin\theta\,\cos\theta\,[1+c\,z^2]^{-(\alpha+2)}, z>0, 0\le\theta\le\pi/2. \quad (A.1)$$

Integrating $(A.1)$ with respect to θ, we get the density function of Z as in (16).

The (*cdf*) of the random variable Z is given by

$$F_Z(z) = 2\,\alpha\,(\alpha+1)\,c^2\int_0^z u^3\,[1+c\,u^2]^{-(\alpha+2)}du. \quad (A.2)$$

The *cdf* (17) is obtained by integrating by parts the integral in $(A.2)$. The rf is then obtained as in (18), since $R(z) = 1 - F_Z(z)$.

APPENDIX B

Proof of equation (19)

From $(5), (16)$ and (18) we have

$$\begin{aligned} g_1(z_s^*|c,\alpha) &\propto [R(z_{r:r}) - R(z_s^*)]^{(s-1)}[R(z_s^*)]^{n-r-s}[R(z_{r:r}^*)]^{-(n-r)} f_Z(z_s^*) \\ &= \sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}[R(z_s^*)]^{n-r-s+i}\,[R(z_{r:r})]^{s-i-(n-r)-1}\,f_Z(z_s^*), \quad (B.1) \end{aligned}$$

where the reliability function $R(z)$, given by (18) yields

$$[R(z)]^k = \sum_{i=0}^{k} \binom{k}{i} c^{k-i} \alpha^{k-i} z^{2(k-i)} (1 + c z^2)^{-\alpha k - k + i}. \qquad (B.2)$$

Using $(B.2)$ and (16) in $(B.1)$ we get (19)

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