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RESOLUTION OF THE IDENTITY OF THE OPERATOR ASSOCIATED WITH A SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract: Consider the system of second order differential equations

$$Ly(x) + \lambda^2 R(x)y(x) = 0$$

where $x \in (a,b)$, a,b finite or infinite; λ, a complex parameter and $y(x) = (y_1(x), y_2(x))^T$,

$$L = \begin{pmatrix} D^2 + p(x) & r(x) \\ r(x) & D^2 + q(x) \end{pmatrix}, D^2 = \frac{d^2}{dx^2}, R(x) = \begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix},$$

$p(x), q(x), r(x), s(x), t(x)$ are all assumed to be real-valued functions summable on (a,b).

In this paper we determine the resolution of the identity of the operator L_A generated by the matrix differential operator L under the general boundary conditions where $s(x), t(x)$ are assumed to be greater than zero for $x \in (a,b)$, a,b being finite or infinite.

Keywords: Parseval theorem; Bessel's inequality, mean convergence theorem; functions of bounded variations.

2010 AMS Subject Classification: 37K40.

1. INTRODUCTION

Consider the system of second order differential equations

$$Ly(x) + \lambda^2 R(x)y(x) = 0 \quad (1)$$

where

$$L \equiv \begin{pmatrix} D^2 + p(x) & r(x) \\ r(x) & D^2 + q(x) \end{pmatrix}, D^2 \equiv \frac{d^2}{dx^2},\ y(x) = (y_1(x), y_2(x))^T, R(x) = \begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix}, (2)$$

$p(x), q(x), r(x), s(x), t(x)$ are all assumed to be real-valued functions summable on (a,b), a,b finite or infinite and λ is a complex parameter.

The boundary conditions at a,b satisfied by a solution $U(x,\lambda) = (U_1(x,\lambda), U_2(x,\lambda))^T$ of the equation (1) are

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$$[U(x,\lambda),\phi_i](a) = 0, [U(x,\lambda),\phi_j](b) = 0 \tag{3}$$

i= 1, 2; j= 3,4, where $\phi_l = \phi_l(x,\lambda), l = 1,2,3,4,$ called boundary condition vectors, are the solutions of (1) which together with their first derivatives take some prescribed values at $x = a, x = b$ and [. , .](α) is the value at $x = \alpha$ of the bilinear-concomitant [., .]. (See Sengupta [10]). The boundary condition vectors ϕ_1, ϕ_2 at $x = a$ and ϕ_3, ϕ_4 at $x = b$ are linearly independent of each other and moreover if

$$[\phi_1, \phi_2](a) = [\phi_3, \phi_4](b) = 0. \tag{4}$$

then the boundary value problem (1)—(3) leads to a self-adjoint eigenvalue problem over the interval (a, b) (see Chakravarty[3]).

For the system (1) with $s(x) = t(x) = 1$ the resolution of the identity of the operator L was investigated by Chakravarty and Roy Paladhi [5].

In this paper we consider the boundary-value problem (1)-(3) with

$$s(x)> 0,\ t(x)>0 \text{ for } a < x < b \tag{5}$$

and following Naimark([9], Pp - 13), Levitan and Sargsjan ([8], Pp. 128-129) we determine the resolution of the identity of the operator L_A generated by the matrix differential operator L as given in (2).

In what follows the notations $y_n(x), \phi(x,\lambda)$, $\theta(x,\lambda), A$, $G(.), \Phi(.), \alpha(.), \beta(.), \gamma(.), E(.), F(.)$, $\tilde{E}(.), \tilde{F}(.)$ etc. are those introduced in Sengupta [11].

2. SOME AUXILIARY RESULTS

Let $f(x) = (f_1(x), f_2(x))^T$ be a function such that $f^T(x)R(x)f(x) \in L(a,b)$. Then following Bhagat [1,2] the resolvent of $f(x)$, defined in (22) of Sengupta ([11] Pp. – 1570]), is given by

$$\Phi(a,b,x,z;f) = \int_a^b G(a,b,x,\xi,z)R(\xi)f(\xi)d\xi$$

$$= \sum_{n=-\infty}^{\infty} y_n(x)\int_a^b y_n^T(\xi)R(\xi)f(\xi)d\xi / A(z-\lambda_n) \tag{6}$$

Let us put $f(\xi) = Y_m(\xi) = \left(Y_{1m}(\xi), Y_{2m}(\xi)\right)^T$, ($m$ fixed) the eigenvector corresponding to the eigenvalue λ_m.

Then by the orthogonality of the eigenvectors, we have from (6) for the Green's matrix G(.),

$$\int_a^b G(a,b,x,\xi,z)R(\xi)Y_m(\xi)d\xi = Y_m(x)/(z-\lambda_m). \tag{7}$$

Therefore

$$\int_a^b G_r^T(a,b,x,\xi,z)R(\xi)\,Y_m(\xi)d\xi = \frac{Y_{rm}(x)}{z-\lambda_m}, r = 1,2 \tag{8}$$

i.e.; $Y_{rm}(x)$ are the Fourier Coefficient of $G_r(a,b,x,\xi,z), r = 1,2,$ considered as a vector function of ξ for fixed x, z.

Applying the Parseval equality (39) of Sengupta [11] to the vectors $G_r(a,b,x,\xi,z)$ and using (8) we obtain

$$\int_a^b G_r^T(a,b,x,\xi,z)R(\xi)G_r(a,b,x,\xi,\bar{z})d\xi$$

$$= \sum_{m=-\infty}^{\infty} \frac{Y_{rm}^2(x)}{A|z-\lambda_m|^2}, r = 1,2 \tag{9}$$

By using (21) of Sengupta [11] we have

$$\sum_{m=-\infty}^{\infty} \frac{Y_{rm}^2}{A|z-\lambda_m|^2} < \infty, r = 1,2, \tag{10}$$

Applying the inequality $(\sum a_n b_n)^2 \leq \sum a_n^2 . \sum b_n^2$ (11)

We obtain from (10) that

$$\sum_{m=-\infty}^{\infty} \frac{Y_{1m}(x)Y_{2m}(x)}{A|z-\lambda_m|^2} < \infty \tag{12}$$

Also for arbitrary but fixed μ, (-μ, μ) ⊂(a, b),

$$\sum_{-\mu\leq\lambda_m\leq\mu} \frac{Y_m(x,x)}{A|z-\lambda_m|^2} < \infty \tag{13}$$

where $Y_m(x,y) = \begin{pmatrix} Y_{1m}(x)Y_{1m}(y) & Y_{1m}(x)Y_{2m}(y) \\ Y_{2m}(x)Y_{1m}(y) & Y_{2m}(x)Y_{2m}(y) \end{pmatrix}$ (14)

and $Y_m^T(x,y) = Y_m(y,x)$.

Using the explicit representation for $y_m(x)$, as given in (37) of Sengupta [11] it follows from (10) after some manipulation that

$$\int_{-\mu}^{\mu} [\phi^T(x,\lambda)d\alpha(a,b,\lambda)\phi(x,\lambda) + \theta^T(x,\lambda)d\beta(a,b,\lambda)\theta(x,\lambda) + \phi^T(x,\lambda)d\gamma(a,b,\lambda)\theta(x,\lambda)$$

$$+\theta^T(x,\lambda)d\gamma(a,b,\lambda)\phi(x,\lambda)].|z-\lambda|^{-2} < \infty \tag{15}$$

Where α(a,b,λ), $\beta(a,b,\lambda)$, $\gamma(a,b,\lambda)$ tend to α(λ),β(λ),γ(λ) respectively as $a \to -\infty, b \to \infty$ (For detail ref. Sengupta [11]).

Hence by making $a \to -\infty, b \to \infty$ first and then μ→ ∞ we obtain from (15) the following theorem.

Theorem 1: For real λ≠ 0,

$$\int_{-\infty}^{\infty} [\phi^T(x,\lambda)d\alpha(\lambda)\phi(x,\lambda) + \theta^T(x,\lambda)d\beta(\lambda)\theta(x,\lambda) + \phi^T(x,\lambda)d\gamma(\lambda)\theta(x,\lambda)$$

$$+\theta^T(x,\lambda)d\gamma(\lambda)\phi(x,\lambda)].\,|z-\lambda|^{-2}<\infty \tag{16}$$

A consequence of Theorem-1 is the following. It is assumed that α(λ),β(λ),γ(λ) are continued to the negative λ axis as odd functions.

Theorem 2: For real λ≠ $0, \mathrm{m}>0$ the integrals

(i)$\int_m^\infty \lambda^{-2}d\alpha(\lambda)$, (ii) $\int_m^\infty \lambda^{-2}\, d\beta(\lambda)$ and

(iii)$\int_m^\infty \lambda^{-2}d\gamma(\lambda)$ are all convergent.

Proof: Putting $x=0$ in (16) and making use of the initial conditions (5) and (6) of Sengupta [11], the theorem for α(λ) follows easily.

Differentiating both sides of the relation (8) with reference to x we obtain

$$\int_a^b \frac{\delta}{\delta x}[G_r^T(a,b,x,\xi,\lambda)R(\xi)y_m(\xi)]d\xi=\frac{y'_{rm}(x)}{z-\lambda_m}, r=1,2.$$

Applying the Parseval equality (39) of Sengupta [11] to the functions $\frac{\delta}{\delta x}G_r^T(a,b,x,\xi,\lambda)$ and arguing in exactly the same way as before for α(λ), the theorem for β(λ) follows.

Since $\left|d\gamma_{ij}(\lambda)\right|^2 \le |d\alpha_{ii}(\lambda)|\left|d\beta_{jj}(\lambda)\right|, for\ i,j=1,2$ the theorem for γ(λ) also follows.

Let us now put $H_\Delta(x,y,a,b)=\left(H_{ij\Delta}(x,y,a,b)\right), i,j=1,2$

$$=\int_\lambda^{\lambda+\Delta}[\phi^T(x,\lambda)d\alpha(a,b,\lambda)\phi(y,\lambda)+\theta^T(x,\lambda)d\beta(a,b,\lambda)\theta(y,\lambda)+\phi^T(x,\lambda)d\gamma(a,b,\lambda)\theta(y,\lambda)$$
$$+\theta^T(x,\lambda)d\gamma(a,b,\lambda)\phi(y,\lambda)] \tag{17}$$

where α(.), β(.), γ(.) are continuous at the end points λ and λ+Δ .

Let $H_\Delta(x,y,a,b)$ tend to $H_\Delta(x,y)$ and as before $\alpha(a,b,\lambda), \beta(a,b,\lambda), \gamma(a,b,\lambda)$ tend to α(λ), β(λ), γ(λ) as $a\to-\infty, b\to\infty$. Then by making $a\to-\infty, b\to\infty$ it follows from (17) that

$$H_\Delta(x,y)=\left(H_{ij\Delta}(x,y)\right), i,j=1,2$$
$$=\int_\lambda^{\lambda+\Delta}[\phi^T(x,\lambda)d\alpha(\lambda)\phi(y,\lambda)+\theta^T(x,\lambda)d\beta(\lambda)\theta(y,\lambda)$$
$$+\phi^T(x,\lambda)d\gamma(\lambda)\theta(y,\lambda)+\theta^T(x,\lambda)d\gamma(\lambda)\phi(y,\lambda)] \tag{18}$$

We prove the following theorem.

Theorem 3: For every fixed y, $H_\Delta^T(x,y)R(y)H_\Delta(x,y)\in L(-\infty,\infty)$ (19)

Proof: From the explicit representation of the normalized eigenvector $y_n(x)/\sqrt{A}$ (Ref. equation (38) of Sengupta [11]) we obtain by using (14) that

$$\sum_{\lambda \le \lambda_n \le \lambda+\Delta} \frac{y_n(x,y)}{A} = H_\Delta(x,y,a,b) \tag{20}$$

Using (20) and the orthogonality conditions for $y_n(x)$ it follows that

$$\int_a^b H_\Delta^T(x,y,a,b)R(x)H_\Delta(x,y,a,b)dx < \sum_{\lambda \le \lambda_n \le \lambda+\Delta} y_n(y,y)/A \tag{21}$$

which is finite.

For arbitrary but fixed $a_1, b_1, (a_1, b_1) \subset (a,b)$ it follows from (21) that

$$\int_{a_1}^{b_1} H_\Delta^T(x,y,a,b)R(x)H_\Delta(x,y,a,b)dx < \sum_{\lambda \le \lambda_n \le \lambda+\Delta} y_n(y,y)/A \tag{22}$$

Passing to the limit as $a \to -\infty, b \to \infty$ we obtain from (22) that

$$\int_{a_1}^{b_1} H_\Delta^T(x,y)R(x)H_\Delta(x,y)dx < \sum_{\lambda \le \lambda_n \le \lambda+\Delta} y_n(y,y)/A \tag{23}$$

As a_1, b_1 are arbitrary, the theorem therefore follows.

Let us now put $H_\Delta(x,f) = (H_{1\Delta}(x,f), H_{2\Delta}(x,f))^T$

$$= \int_{-\infty}^{\infty} H_\Delta(x,y)R(y)f(y)dy \tag{24}$$

where $f(x) = (f_1(x), f_2(x))^T$ is a vector such that $f^T(x)R(x)f(x) \in L(-\infty,\infty)$

The existence of $H_\Delta(x,f)$ is ensured by the Schwarz inequality, the Theorem-3 and the conditions on $f(x)$.

In what follows we say that $f(x) \in \alpha^2(-\infty,\infty)$ or $f \in L^2$ if $f^T(x)R(x)f(x) \in L^2(-\infty,\infty)$.

Theorem 4: If $f(x) \in L^2(-\infty,\infty)$ and $(\lambda, \lambda+\Delta)$ is any finite interval, then

$$H_\Delta(x,f) = \int_\lambda^{\lambda+\Delta} [\phi^T(x,\lambda)\alpha\alpha(\lambda)E(\lambda) + \theta^T(x,\lambda)\alpha\beta(\lambda)F(\lambda) + \phi^T(x,\lambda)d\gamma(\lambda)F(\lambda) + \theta^T(x,\lambda)d\gamma(\lambda)E(\lambda)] \tag{25}$$

Proof: Let $f(x) = f_n(x) \equiv (f_{1n}(x), f_{2n}(x))^T$ be a vector with compact support i.e.; $f(x)$ defined on $(-n,n)$ vanish outside the interval, where n<min$\{|a|,|b|\}$a<0, b>0.

Then

$$\int_{-n}^{n} H_\Delta(x,y,a,b)R(y)f_n(y)dy$$
$$= \int_\lambda^{\lambda+\Delta} [\phi^T(x,\lambda)d\alpha(a,b,\lambda)E_n(\lambda) + \theta^T(x,\lambda)d\beta(a,b,\lambda)F_n(\lambda) + \phi^T(x,\lambda)d\gamma(a,b,\lambda)F_n(\lambda) + \theta^T(x,\lambda)d\gamma(a,b,\lambda)E_n(\lambda)] \tag{26}$$

Where $E_n(\lambda)$, $F_n(\lambda)$ are explicitly given in (82) of Sengupta [11].

Making $a \to -\infty$, $b \to \infty$ from (26) we obtain

$$H_\Delta(x,f_n) \equiv \int_{-n}^{n} H_\Delta(x,y)R(y)f_n(y)dy = \int_\lambda^{\lambda+\Delta} [\phi^T(x,\lambda)d\alpha(\lambda)E_n(\lambda) + \theta^T(x,\lambda)d\beta(\lambda)F_n(\lambda) + \phi^T(x,\lambda)d\gamma(\lambda)F_n(\lambda) + \theta^T(x,\lambda)d\gamma(\lambda)E_n(\lambda)] \tag{27}$$

Now let $f(x) = (f_1(x), f_2(x))^T$ be an arbitrary vector such that $f(x) \in L^2(-\infty, \infty)$. We approximate in mean to $f(x)$ by the sequence $\{f_n(x)\}$.

From (25) it follows that for $r = 1, 2$

$$H_{r\Delta}(x, f_n) = \int_{-n}^{n} H_{r\Delta}^T(x, y)R(y)(f_n(y) - f(y))dy + \int_{-n}^{n} H_{r\Delta}^T(x, y)R(y)f(y)dy = J_1 + J_2 \text{ , say.} \quad (28)$$

Now $|J_1| \leq \left(\int_{-n}^{n} \left|H_{r\Delta}^T(x, y)R(y)H_{r\Delta}(x, y)/dy\right|\right)^{1/2} . (\int_{-n}^{n} |(f_n(y) - f(y))^T R(y)(f_n(y) - f(y))|dy)^{1/2}$ (29)

As $n \to \infty, J_1 \to 0$ and similarly $J_2 \to H_\Delta(x, f)$.

Therefore we obtain

$$H_{r\Delta}(x, f_n) \to H_{r\Delta}(x, f) \equiv \int_{-\infty}^{\infty} H_{r\Delta}^T(x, y)R(y)f(y)dy, \;\; r = 1, 2 \quad (30)$$

Thus $H_\Delta(x, f_n) \to H_\Delta(x, f)$ as $n \to \infty$.

Also in the right side of (27), $E_n(\lambda), F_n(\lambda)$ converges in mean to $E(\lambda), F(\lambda)$ as $n \to \infty$.

(See Theorem-2 of Sengupta [11]).

Hence the theorem follows from (27).

Theorem 5: If $f(x) \in L^2(-\infty, \infty)$ then for any finite interval (λ,λ+ Δ) as a function of $x, H_\Delta^T(x, f)R(x)H_\Delta(x, f) \in L(-\infty, \infty)$ (31)

Proof With $f_n(x)$ defined in Theorem-4 we obtain by making use of (20), (21) that

$$\int_{a_1}^{b_1} \left|\left(\int_{-n}^{n} f_n^T(y)R(y)H_\Delta^T(x, y, a, b)dy\right)R(x)\left(\int_{-n}^{n} H_\Delta(x, y, a, b)R(y)f_n(y)dy\right)dx\right| <$$

$$\int_{a}^{b} \left|\left(\int_{-n}^{n} f_n^T(y)R(y)H_\Delta^T(x, y, a, b)dy\right)R(x)\left(\int_{-n}^{n} H_\Delta(x, y, a, b)R(y)f_n(y)dy\right)dx\right| =$$

$\sum_{\lambda \leq \lambda_k \leq \lambda + \Delta} \frac{1}{A} (\int_{-n}^{n} y_k^T(y)R(y)f_n(y)dy)^2$ (by (20) and the orthogonality of the eigenvectors)

$\leq \int_{-n}^{n} f_n^T(y)R(y)f_n(y)dy$ (by Bessel's inequality) (32)

where a_1,b_1 are arbitrary but fixed and $(a_1, b_1) \subset (a, b)$.

Making $a \to -\infty, b \to \infty$ first and then $a_1 \to -\infty$, $b_1 \to \infty$ the theorem is established for the function $f_n(x)$.

The general result for arbitrary $f(x) = (f_1(x), f_2(x))^T$ such that $f(x) \in L^2(-\infty, \infty)$ follows by approximating in mean to $f(x)$ by the sequence $\{f_n(x)\}$ for which we take note of the fact that

$\left|\int_{-n}^{n} f_n^T(x)R(x)f_n(x)dx\right| \le \left|\int_{-n}^{n}\left(f_n(x) - f(x)\right)^T R(x)(f_n(x) - f(x))dx\right| + \left|\int_{-n}^{n}\left(f_n(x) - f(x)\right)^T R(x)f(x)dx\right| + \left|\int_{-n}^{n} f_n^T(x)R(x)(f_n(x) - f(x))dx\right| + \left|\int_{-n}^{n} f^T(x)R(x)f(x)dx\right|)$

(33)

Theorem 6:

If $f(x) \in L^2(-\infty,\infty)$, then for any non-real z, $\Phi(x, z; H_\Delta(x, f)) \equiv \int_{-\infty}^{\infty} G(x, y, z)R(y)H_\Delta(y, f)dy$

$=\int_{\lambda}^{\lambda+\Delta}[\phi^T(x,\lambda)d\alpha(\lambda)E(\lambda) + \theta^T(x,\lambda)d\beta(\lambda)F(\lambda) + \phi^T(x,\lambda)d\gamma(\lambda)F(\lambda)$

$+\theta^T(x,\lambda)d\gamma(\lambda)E(\lambda)](z-\lambda)^{-1}$ (34)

Proof. With $f_n(x)$ defined in Theorem-4, we have for any non-real z

$\Phi(a, b, x, z;\ H_\Delta(x, f_n))$

$= \int_a^b G(a, b, x, y; z)R(y)H_\Delta(y, f_n)dy = \sum_{n=-\infty}^{\infty} Y_n(x)\int_a^b Y_n^T(\xi)R(\xi)H_\Delta(\xi, f_n)d\xi \ / \ A(z-\lambda_n)$

(35)

Using (20) and (24) it now follows from (35) that

$\int_a^b G(a, b, x, y; z)R(y)H_\Delta(y, f_n)dy = \sum_{\lambda\le\lambda_n\le\lambda+\Delta} Y_n(x)(\int_{-n}^{n} Y_n^T(\xi)R(\xi)f_n(\xi)d\xi) / A(z-\lambda_n)$

(36)

Replacing $y_n(x)$ by that given in (37) of Sengupta [11] we obtain from (36) that

$\int_a^b G(a, b, x, y; z)R(y)H_\Delta(y, f_n)dy$

$=\int_{\lambda}^{\lambda+\Delta}[\phi^T(x,\lambda)d\alpha(a,b,\lambda)E_n(\lambda) + \theta^T(x,\lambda)d\beta(a,b,\lambda)F_n(\lambda) + \phi^T(x,\lambda)d\gamma(a,b,\lambda)F_n(\lambda)$

$+\theta^T(x,\lambda)d\gamma(a,b,\lambda)E_n(\lambda)](z-\lambda)^{-1}$ (37)

where $E_n(\lambda), F_n(\lambda)$ are given in (82) of Sengupta [11].

The convergence to the limit of the right side of the equality (37) as $a \to -\infty, b \to \infty$ is obvious. By using (27) of Sengupta [11] and (31) and closely following Chakravarty ([4] Pp-410) we obtain that as $a \to -\infty, b \to \infty$, $\Phi(a, b, x, z; H_\Delta(x, f_n))$ and $G(a, b, x, y; z)$ tend to $\Phi(x, z; H_\Delta(x, f_n))$ and $G(x, y; z)$ respectively. Since $\alpha(a, b, \lambda)$, $\beta(a, b, \lambda)$, $\gamma(a, b, \lambda)$ tend to $\alpha(\lambda)$, $\beta(\lambda)$, $\gamma(\lambda)$ respectively as $a \to -\infty, b \to \infty$ it follows from (37) by making $a \to -\infty, b \to \infty$ that

$\Phi(x, z; H_\Delta(x, f_n)) = \int_{-\infty}^{\infty} G(x, y; z)R(y)H_\Delta(x, f_n)dy$

$=\int_{\lambda}^{\lambda+\Delta}[\phi^T(x,\lambda)d\alpha(\lambda)E_n(\lambda) + \theta^T(x,\lambda)d\beta(\lambda)F_n(\lambda) + \phi^T(x,\lambda)d\gamma(\lambda)F_n(\lambda)$

$+\theta^T(x,\lambda)d\gamma(\lambda)E_n(\lambda)](z-\lambda)^{-1}$ (38)

Let $f(x) = (f_1(x), f_2(x))^T$ be such that $f(x) \in L^2(-\infty, \infty)$. We approximate in mean to $f(x)$ by means of the sequence $\{f_n(x)\}$.

By Theorem-5 and inequality (28) of Sengupta [11] as before it follows that $\Phi(x, z; H_\Delta(x, f_n))$ tend to $\Phi(x, z; H_\Delta(x, f_n))$ as $n \to \infty$. Also the sequences $\{E_n(\lambda)\}$,$\{F_n(\lambda)\}$ converge in mean to $E(\lambda), F(\lambda)$ respectively as $n \to \infty$ (See Theorem-2 of Sengupta [11]). Hence by the mean convergence theorems (Stated explicitly in Sengupta [11]) the theorem follows completely.

Let $f(x), g(x) \in L^2(-\infty, \infty)$. Then from (34) we have

$$\int_{-\infty}^{\infty} \Phi^T\big(x, z; H_\Delta(x, f)\big) R(x) g(x) d(x)$$

$$= \int_{\lambda}^{\lambda+\Delta} [E^T(\lambda) d\alpha(\lambda) E(\lambda) + F^T(\lambda) d\beta(\lambda) F(\lambda) + E^T(\lambda) d\gamma(\lambda) F(\lambda)$$

$$+ F^T(\lambda) d\gamma(\lambda) E(\lambda)] (z - \lambda)^{-1} \tag{39}$$

(The convergence problem being settled by (29) of Sengupta [11] and (31)).

3. INTEGRAL REPRESENTATION OF THE RESOLVENT

In what follows let us put

$$H_\lambda(x, y = \int_0^\lambda [\phi^T(x, \lambda) d\alpha(\lambda) \phi(y, \lambda) + \theta^T(x, \lambda) d\beta(\lambda) \theta(y, \lambda) + \phi^T(x, \lambda) d\gamma(\lambda) \theta(\gamma, \lambda) +$$

$$\theta^T(x, \lambda) d\gamma(\lambda) \phi(y, \lambda)], \text{ for } \lambda > 0$$

$$= -\int_\lambda^0 [\phi^T(x, \lambda) d\alpha(\lambda) \phi(y, \lambda) + \theta^T(x, \lambda) d\beta(\lambda) \theta(y, \lambda) + \phi^T(x, \lambda) d\gamma(\lambda) \theta(y, \lambda) +$$

$$\theta^T(x, \lambda) d\gamma(\lambda) \phi(y, \lambda)], \text{ for } \lambda < 0$$

$$= 0, \text{ for } \lambda = 0 \tag{40}$$

and $H_\lambda(x, f) = \int_{-\infty}^{\infty} H_\lambda(x, y) R(y) f(y) dy$ (41)

where $f(x) = (f_1(x), f_2(x))^T$ be such that $f(x) \in L^2(-\infty, \infty)$.

Then $H_\lambda(x, f) = \int_{-\infty}^{\infty} H_\lambda(x, y) R(y) f(y) dy$

$$= \int_0^\lambda [\phi^T(x, \lambda) d\alpha(\lambda) E(\lambda) + \theta^T(x, \lambda) d\beta(\lambda) F(\lambda) + \phi^T(x, \lambda) d\gamma(\lambda) F(\lambda) + \theta^T(x, \lambda) d\gamma(\lambda) E(\lambda)] \tag{42}$$

[Compare Theorem-4 and Theorem-5].

We prove the following theorems.

Theorem-7 : Let $f(x) \in L^2(-\infty,\infty)$. Then as a function of λ, $H_\lambda(x,f)$ is of bounded variation in every finite λ-interval.

Proof : For $\lambda \neq \lambda_n$, write the system (1) in the form

$$(L+\lambda^2 R(x))y_n(x) = (\lambda^2-\lambda_n{}^2)R(x)y_n(x) \quad (43)$$

where $y_n(x)$ is the eigenvector corresponding to the eigenvalue λ_n.

Hence $y_n(x) = (\lambda^2-\lambda_n^2)\int_a^b G(a,b,x,\xi;\lambda)R(\xi)y_n(\xi)d\xi$ (44)

satisfies the differential system (43).

For definiteness let λ>0 and we prove the result for the vector $f_n(x)$ as defined in Theorem-4. By making use of (20) and (24) it follows that as $a\to-\infty, b\to\infty, H_\lambda(x,f_n)$ is the limit of the function

$$s=(s_1,s_2)^T=\sum_{0\le\lambda_n\le\lambda}\frac{1}{A}\int_a^b y_k(x,\xi)R(\xi)F_n(\xi)d\xi$$

$$=\sum_{0\le\lambda_k\le\lambda}\frac{1}{A}y_k(x)\int_a^b y_k^T(\xi)R(\xi)f_n(\xi)d\xi \quad (45)$$

where λ_k is any point on the finite λ-interval.

By (44) it follows from (45) that

$$s=\sum_{0\le\lambda_k\le\lambda}\frac{1}{A}(\lambda^2-\lambda_k^2)\left(\int_a^b G(a,b,x,\xi;\lambda)R(\xi)y_k(\xi)d\xi\right).(\int_a^b y_k^T(\xi)R(\xi)f_n(\xi)d\xi) \quad (46)$$

Now, $\left|\int_a^b G_r^T(a,b,x,\xi;\lambda)R(\xi)y_k(\xi)d\xi\right|$

$$\le(\int_a^b|G_r^T(a,b,x,\xi;\lambda)R(\xi)G_r(a,b,x,\xi;\lambda)|\,d\xi)^{1/2}.(\int_a^b|y_k^T(\xi)R(\xi)y_k(\xi)|d\xi)^{1/2}. \quad (47)$$

Thus from (46) by using (21) of Sengupta [11] we obtain that s_1, s_2 and consequently S is bounded uniformly in any finite λ-interval.

Hence, the limit function is of bounded variation. This completes the proof of the Theorem.

Theorem 8: Let $f(x) \in L^2(-\infty,\infty)$. Then for any non-real $Z=\sigma_1+i\gamma_1, \gamma_1>0$ and

$$\Phi(x,z;f)=\int_{-\infty}^{\infty} d_\lambda(H_\lambda(x,f))/(z-\lambda)) \quad (48)$$

where $H_\lambda(x,f)$ is given by (42). The integral in the right side of (48) converge absolutely.

Proof: Writing $Y_n(x)$ explicitly as given in (37) of Sengupta [11] we obtain from (6) that

$$\Phi(a,b,x,z;f_n)=\int_{-\infty}^{\infty}[\phi^T(x,\lambda)d\alpha(a,b,\lambda)E_n(\lambda)+\theta^T(x,\lambda)d\beta(a,b,\lambda)F_n(\lambda)$$

$$+\phi^T(x,\lambda)d\gamma(a,b,\lambda)F_n(\lambda)+\theta^T(x,\lambda)d\gamma(a,b,\lambda)E_n(\lambda)](z-\lambda)^{-1} \quad (49)$$

where the vector $f_n(x) = (f_{1n}(x)f_{2n}(x))^T$ is the same as that defined in Theorem-4.

Passing to the limit as $a \to -\infty, b \to \infty$ we obtain from (42) and (49) that

$\Phi(x, z; f_n) = \int_{-\infty}^{\infty} d_\lambda(H_\lambda(x, f_n))/(z - \lambda)$

$= \int_{-\infty}^{\infty}[\phi^T(x,\lambda)d\alpha(\lambda)E_n(\lambda) + \theta^T(x,\lambda)d\beta(\lambda)F_n(\lambda) + \phi^T(x,\lambda)d\gamma(\lambda)F_n(\lambda)$

$+\theta^T(x,\lambda)d\gamma(\lambda)E_n(\lambda)](z-\lambda)^{-1}$ (50)

We now approximate in mean to the vector $f(x) = (f_1(x), f_2(x))^T$ which satisfy that $f(x) \in L^2(-\infty, \infty)$ by means of the sequence $\{f_n(x)\}$.

By using (24) of Sengupta [11], $\Phi(x, z; f_n) \to \Phi(x, z; f)$. Let us now consider the right side of the equality (50). Let us put

$\Phi(x, z; f_n) = \int_{-\infty}^{-m} \frac{d_\lambda(H_\lambda(x,f_n))}{(z-\lambda)} + \int_{-m}^{m} \frac{d_\lambda(H_\lambda(x,f_n))}{(z-\lambda)} + \int_{m}^{\infty} \frac{d_\lambda(H_\lambda(x,f_n))}{(z-\lambda)}$

$= K_1 + K_2 + K_3$, say (51)

where m is an arbitrary positive number.

From the Parseval relation as given in (72) of Sengupta [11] it follows that as m→ ∞,

$\int_m^{\infty}[E_n^T(\lambda)d\alpha(\lambda)E_n(\lambda) + F_n^T(\lambda)d\beta(\lambda)F_n(\lambda) + E_n^T(\lambda)d\gamma(\lambda)F_n(\lambda) + F_n^T(\lambda)d\gamma(\lambda)E_n(\lambda)] = 0(1)$

(52)

and from (16) we obtain as m→ ∞

$\int_m^{\infty}[\phi^T(x,\lambda)d\alpha(\lambda)\phi(x,\lambda) + \theta^T(x,\lambda)d\beta(\lambda)\theta(x,\lambda) + \phi^T(x,\lambda)d\gamma(\lambda)\theta(x,\lambda)$

$+\theta^T(x,\lambda)d\gamma(\lambda)\phi(x,\lambda)]|z-\lambda|^{-2} = 0(1)$ (53)

To each of the integrals $K_i, i = 1,2,3$,we apply the inequality Hardy <u>et</u> <u>al.</u> ([7], Section 29, Pp – 33).

$\sum a_{\mu\gamma}x_\mu y_\gamma \le (\sum a_{\mu\gamma}x_\mu x_\gamma)^{1/2}.(\sum a_{\mu\gamma}y_\mu y_\gamma)^{1/2}$.

where $a_{\mu\gamma} = a_{\gamma\mu}, \sum a_{\mu\gamma}x_\mu x_\gamma$ is a positive quadratic form (with real but not necessarily positive coefficients). Then using (52), (53) we obtain as in Chakravarty and Roy Paladhi ([6], Pp-150) that $K_3 \to 0$ as $m \to \infty$.

Similarly, $K_1 \to 0$ as $m \to \infty$.

Integrating by parts we get

$K_2 = \frac{H_m(x,f_n)}{(z-m)} - \frac{H_{-m}(x,f_n)}{z+m} - \int_{-m}^{m} H_\lambda(x,f_n).(z-\lambda)^{-2}d\lambda$ (54)

By Theorem-7 we have

$$\lim_{n\to\infty} \int_{-m}^{m} \frac{d_\lambda(H_\lambda(x,f_n))}{z-\lambda} = \int_{-m}^{m} \frac{d_\lambda(H_\lambda(x,f))}{z-\lambda} \tag{55}$$

Since m is arbitrary, the theorem follows from (55).

Now from (50) we obtain that for any non-real z and vectors $f(x), g(x) \in L^2(-\infty,\infty)$

$$\int_{-\infty}^{\infty} \Phi^T(x,z;f)R(x)g(x)dx = \int_{-\infty}^{\infty}[E^T(\lambda)d\alpha(\lambda)\tilde{E}(\lambda) + F^T(\lambda)d\beta(\lambda)\tilde{F}(\lambda) + E^T(\lambda)d\gamma(\lambda)\tilde{F}(\lambda) + F^T(\lambda)d\gamma(\lambda)\tilde{E}(\lambda)]/(z-\lambda) \tag{56}$$

4. RESOLUTION OF THE IDENTITY

Let $f(x = (f_1(x), f_2(x))^T$, $g(x) = (g_1(x), g_2(x))^T$ be two vectors such that $f(x), g(x) \in L^2(-\infty,\infty)$. For λ>0. put

$$J(f,g,\lambda) = \int_{-\infty}^{\infty} H_\lambda^T(x,f)R(x)g(x)dx \tag{57}$$

and $J(f,\lambda) = J(f,f,\lambda)$; where λ>0 (58)

(For notation compare Titchmarsh ([12]Pp-50))

where $H_\lambda(x,f)$ is given by (41).

Using the expressions for $H_\lambda(x,f)$ given in (42) in the usual manner we obtain

$$J(f,g,\lambda) = \int_0^\lambda [E^T(\lambda)d\alpha(\lambda)E(\lambda) + F^T(\lambda)d\beta(\lambda)F(\lambda) + E^T(\lambda)d\gamma(\lambda)F(\lambda) + F^T(\lambda)d\gamma(\lambda)E(\lambda)] \tag{59}$$

Now the equation (39) can be expressed as

$$\int_{-\infty}^{\infty} \phi^T(x,z;H_\Delta(x,f))R(x)g(x)dx = \int_\lambda^{\lambda+\Delta} d_\lambda(J(f,g,\lambda))/(z-\lambda) \tag{60}$$

Putting $H_\Delta(x,f)$ for $f(x)$ in (56) and using (59) we also obtain

$$\int_{-\infty}^{\infty} \Phi^T(x,z;H_\Delta(x,f))R(x)g(x)dx = \int_{-\infty}^{\infty} d_\lambda(J(H_\Delta f,g,\lambda))/(z-\lambda) \tag{61}$$

From (57), (60), (61) and the uniqueness theorem for the Stieltjes transforms it follows that

$$H_\lambda . H_\Delta = H_{\lambda\cap\Delta} \tag{62}$$

where λ∩ Δ = (-∞, λ)∩ $(\lambda, \lambda+\Delta)$

(See Levitan and Sargsjan ([8], Pp-129 and Pp-503))

Let Δ *and* Δ' denote the intervals (λ, λ+Δ) and $(\lambda', \lambda'+\Delta')$ respectively. Then

$$H_{\Delta'}.H_\Delta = \{H_{(-\infty,\lambda'+\Delta')} - H_{(-\infty,\lambda')}\}H_\Delta$$

$$= H_{(-\infty,\lambda'+\Delta')}H_\Delta - H_{(-\infty,\lambda')}H_\Delta$$

$= H_{(-\infty,\lambda'+\Delta')\cap\Delta} - H_{(-\infty,\lambda')\cap\Delta}$

$= H_{[(-\infty,\lambda'+\Delta')-(-\infty,\lambda')]\Delta} = H_{\Delta'\cap\Delta}$ (63)

We obtain the following theorem.

Theorem 9: Let $\Delta \equiv (\lambda, \lambda + \Delta), \Delta' \equiv (\lambda', \lambda' + \Delta')$, then

$\int_{-\infty}^{\infty} H_\Delta(x,\xi) R(\xi) H_{\Delta'}(\xi, y) d\xi = H_{\Delta\cap\Delta'}(x,y)$

(64)

Proof From the representation of $H_\Delta(x,y)$ given by (20) and by the orthogonality conditions for the eigenvectors it follows that

$\int_{-\infty}^{\infty} H_\Delta(x,\xi) R(\xi) H_{\Delta'}(\xi, y) d\xi$

$= \int_{-\infty}^{\infty} \sum_{\lambda \le \lambda_n \le \lambda+\Delta} \frac{y_n(x,\xi)}{A} . R(\xi) . \sum_{\lambda' \le \lambda_n \le \lambda'+\Delta'} \frac{y_n(\xi,y)}{A} . d\xi$

$= \sum_{\lambda_n \in \Delta\cap\Delta'} {y_n(x,\xi)}/{A} \quad = H_{\Delta\cap\Delta'}(x,y)$.

Hence the theorem is proved.

Let $J(f,g,\infty) = \lim_{\lambda\to\infty} J(f,g,\lambda)$ (65)

The generalized Parseval formula (90) of Sengupta[11] now takes the form

$\int_{-\infty}^{\infty} f^T(x) R(x) g(x) dx = J(f,g,\infty) - J(f,g,-\infty)$ (66)

By (59) we also obtain

$J^T(f,g,\lambda) = J(g,f,\lambda)$ (67)

Now we apply the Stieltjes inversion formula (See Levitan and Sargsjan [8] Pp-502) to each of the elements of $\Phi(x,z;f)$ (Z being non-real) given by (48) and obtain

$H_\mu(x,f) = \lim_{\gamma\to 0} \int_0^\mu Im\Phi(x, \sigma + i\gamma, f) d\sigma, \lambda = \sigma + i\gamma, \gamma > 0$ (68)

By making use of the definition of $H_\lambda(x,f)$ given by (41) and that of $\Phi(x,\lambda,f)$ by

$\Phi(x,\lambda,f) = \int_{-\infty}^{\infty} G(x,y,\lambda) R(y) f(y) dy$ (69)

($G(x,y,\lambda)$ being the Green's matrix) which follows by making $a \to -\infty, b \to \infty$ in (22) of Sengupta [11], we obtain by proceeding as in Chakravarty and Roy Paladhi ([6], Pp-141) that

$H_\lambda(x,y) = -\lim_{\gamma\to 0} \int_0^\mu ImG(x,y,\sigma + i\gamma) d\sigma$ (70)

From (6)

$\int_a^b \Phi^T(a,b,x,z;f) R(x) g(x) dx$

$= \int_a^b \int_a^b f^T(\xi)R(\xi)G^T(a,b,\eta,\xi,z)\, R(\eta)f(\eta)d\xi d\eta$

$= \sum_{n=-\infty}^{\infty} \frac{C_n d_n}{(Z-\lambda_n)}$ (71)

Where $C_n = \frac{1}{\sqrt{A}}\int_a^b y_n^T(\xi)R(\xi)f(\xi)d\xi$

$d_n = \frac{1}{\sqrt{A}}\int_a^b y_n^T(\xi)R(\xi)g(\xi)d\xi$ (72)

the Fourier coefficients of$f(x)$ and $g(x)$ respectively.

Therefore,

$-\lim_{\gamma_1\to 0}\int_a^b \int_a^b \int_\alpha^\beta \left(R(\xi)f(\xi)\right)^T \left(ImG^T\left(a,b,\eta,\xi;\sigma_1+i\gamma_1\right)\right) R(\eta)g(\eta)d\xi d\eta\, d\sigma_1$

$=\sum_{n=-\infty}^{\infty} C_n d_n \int_\alpha^\beta \frac{\gamma_1 d\sigma_1}{(\lambda_n-\sigma_n)^2+\gamma_1^2}, \quad (z=\sigma_1+i\gamma_1)$ (73)

Where the integral $\int_\alpha^\beta \frac{\gamma_1 d\sigma_1}{(\lambda_n-\sigma_1)^2+\gamma_1^2}$ does not exceed Π.

On taking $g(x)=f(x), d_n = C_n$ we have from (73)

$\int_a^b \int_a^b \left(R(\xi)f(\xi)\right)^T \left(H_\beta^T(a,b,\eta,\xi) - H_\alpha^T(a,b,\eta,\xi)\right) R(\eta)f(\eta)d\xi d\eta \geq 0$ (74)

Hence if $f(x) = f_n(x)$ vanish outside $(a_1,b_1) \subset (a,b)$ we have

$\int_{a_1}^{b_1} \int_{a_1}^{b_1} \left(R(\xi)f_n(\xi)\right)^T \left(H_\beta^T(a,b,\eta,\xi) - H_\alpha^T(a,b,\eta,\xi)\right) R(\eta)f_n(\eta)d\xi d\eta \geq 0$

By making $a\to -\infty, b\to\infty$ we obtain

$\int_{a_1}^{b_1} \left(H_\beta(\eta,f_n) - H_\alpha(\eta,f_n)\right)^T R(\eta)f_n(\eta)d\eta \geq 0$ (75)

From this it follows that in the usual way by a mean square approximation that for any $f(x) \in L^2(-\infty,\infty)$

$J(f,\beta) \geq J(f,\alpha)$ for $\beta \geq \alpha$ (76)

(Compare Titchmarsh [12] Pp- 51-53).

From the relations (63), (66), (67) and (76) it follows that the family of operators $H_\lambda(x,y)$ defined by (40) satisfy the properties of (i) orthogonality (ii) completeness (iii) self-adjointness and (iv) monotonicity. $H_\lambda(x,y)$ thus plays on essential role in deriving the resolution of the identity of the operator L_A (See Levitan and Sargsjan [8] Pp - 129). Also compare Chakravarty and Roy Paladhi ([5]).

We can define $H_\lambda(x,y)$ by (68) as in Chakravarty and Roy Paladhi [[5]] and obtain results of the forgoing section.

5. INTERPRETATION IN TERMS OF THE THEORY OF LINEAR OPERATORS

Our analysis now closely follows Titchmarsh [12]. We simply outline the procedure giving details only when we considerably differ.

From (68) it follows that

$$dH_\mu(x,f) = -\lim_{\gamma\to 0} Im\Phi(x,\mu+i\gamma,f)d\mu \tag{77}$$

Let the vectors $f(x) = (f_1(x), f_2(x))$; and $\tilde{f}(x) = Lf(L$ given by (2)) which satisfy the equation (30) of Sengupta[11] and $\tilde{f}(x) \in L^2(-\infty,\infty)$. Then

$$Im\Phi(x,\lambda,\tilde{f}) = Im\{\lambda\Phi(x,\lambda,f)\} \tag{78}$$

and $H_\mu(x,\tilde{f}) = \lim_{\gamma\to 0}\int_0^\mu Im\Phi(x,\sigma+i\gamma,f)d\sigma$

$$= -\lim_{\gamma\to 0}\int_0^\mu \sigma Im\Phi(x,\sigma+i\gamma,f)d\sigma + \lim_{\gamma\to 0}\int_0^\mu \gamma Re\Phi(x,\sigma+i\gamma,f)d\sigma$$

$$= H_1 + H_2 \text{ , say} \tag{79}$$

By (70), $H_1 = \int_0^\mu \sigma dH_\sigma(x,f)$.

By Theorem-8, $\int_0^\mu Re\Phi(x,\sigma+i\gamma,f)d\sigma$ is finite.

Hence $H_2 \to 0$ as $\gamma\to 0$.

Thus $H_\mu(x,\tilde{f}) = \int_0^\mu \sigma dH_\sigma(x,f)$ (80)

Therefore , $J(\tilde{f},g,\mu) = \int_{-\infty}^{\infty} H_\mu^T(\xi,\tilde{f})R(\xi)g(\xi)d\xi$

$$= \int_{-\infty}^{\infty}\left(\int_0^\mu \sigma dH_\sigma(\xi,f)\right)^T R(\xi)g(\xi)d\xi$$

$$= \int_0^\mu \sigma d\left(\int_{-\infty}^{\infty} H_\sigma^T(\xi,f)\right)R(\xi)g(\xi)d\xi$$

$$= \int_0^\mu \sigma dJ(f,g,\sigma) \tag{81}$$

In view of the relation (41) the expansion formula as given in (91) of Sengupta [11] for the function $\tilde{f}(x)$ takes the form

$$\tilde{f}(x) = \lim_{\mu\to\infty}\int_{-\infty}^{\infty}\left(H_\mu(x,\xi) - H_{-\mu}(x,\xi)\right)^T R(\xi)\tilde{f}(\xi)d\xi, \ (\lambda \text{ real})$$

$$= \lim_{\mu\to\infty}(H_\mu(x,\tilde{f}) - H_{-\mu}(x,\tilde{f})) \tag{82}$$

Therefore, $\int_{-\infty}^{\infty}\tilde{f}^T(x)R(x)g(x)dx$

$$= \lim_{\mu\to\infty}\int_{-\infty}^{\infty}\left(H_\mu(x,\tilde{f}) - H_{-\mu}(x,f)\right)^T R(x)g(x)dx$$

$$= \lim_{\mu\to\infty}[J(\tilde{f},g,\mu) - J(\tilde{f},g,-\mu)]$$

$= \lim_{\mu\to\infty} \int_{-\mu}^{\mu} \sigma dJ(f.g,\sigma)$ (83)

For real λ, from (75) it follows that

$J(\tilde{f},\lambda) = \int_0^{\lambda} \sigma dJ(\tilde{f},f,\sigma) = \int_0^{\lambda} \sigma d\{\int_0^{\sigma} \mu J(f,\mu)\}$

$= \int_0^{\lambda} \sigma^2 dJ(f,\sigma)$ (84)

Hence, $\int_{-\infty}^{\infty} \tilde{f}^T(x)R(x)\tilde{f}(x)dx$

$= J(\tilde{f},\infty) - J(\tilde{f},-\infty)$ by (66)

$= \int_{-\infty}^{\infty} \sigma dJ(\tilde{f},\sigma)$, by (83)

$= \int_{-\infty}^{\infty} \sigma^2 dJ(f,\sigma)$ (85)

As $p(x), q(x), r(x)$ are real-valued twice differentiable functions of x over $(-\infty,\infty)$, the differential operator L_A generated by (2) is a symmetric operator on $L^2(-\infty,\infty)$.

Put $K(x,y,\lambda) = H_{\lambda-0}(x,y) - H_{-\infty}(x,y)$, (λ real) (86)

(For notation See Chakravarty and Roy Paladhi [5])

Then the operator

$G(\lambda): f(x) \to \int_{-\infty}^{\infty} K^T(x,\xi,\lambda)R(\xi)f(\xi)d\xi$ (87)

(i.e; $G(\lambda)f(x) = \int_{-\infty}^{\infty} K^T(x,\xi,\lambda)R(\xi)f(\xi)d\xi$)

is a linear symmetric operator on $L^2(-\infty,\infty)$

(See Chakravarty and Roy Paladhi [5]), $K(x,y,\lambda)$ being a Carleman type kernel.

We now argue as in Titchmarsh ([12], Pp-55). (Also Ref Chakravarty and Roy Paladhi [5], Pp-160-161) so as to obtain ultimately

$L_A = \int_{-\infty}^{\infty} \lambda dG(\lambda)$ (88)

where $G(\lambda)$ is the resolution of the identity of the self-adjoint differential operator L_A generated by the given differential equation (1).

Thus we obtain the follow theorem.

Theorem 10: The matrix $H_\lambda(x,y)$ (λ-real) defined by (40) generates an operator $G(\lambda)$ given by (87) which is associated with the differential operator L given by (2). L_A generated by the differential expression (1) is associated in the same way as the resolution of the identity of a given operator T is associated with T. $G(\lambda)$ is the resolution of the identity of the operator L_A.

The matrix $H_\lambda(x, y)$ generating the operator $G(\lambda)$ may be called the resolution matrix of the operatorL_A.

Conflict of Interests

The authors declare that there is no conflict of interests.

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