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NEW KIND OF WEAKLY SET VALUED VECTOR F-IMPLICIT VARIATIONAL INEQUALITIES

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Abstract. In this work, we consider the weakly set valued vector F-implicit variational inequalities in real Hausdorff topological vector spaces. We use the Ferro Minimax Theorem to discussed the existence of strong solutions for set valued vector F-implicit variational inequalities.

Keywords: New kind of weakly set valued vector F-implicit variational inequalities, Hausdorff topological vector spaces, Ferro Minimax Theorem, cone.

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1. Introduction

Variational inequality was introduced by Stampacchia [15] in the early sixties. It has been shown that a wide class of linear, nonlinear problems arising in various branches of mathematical and engineering sciences can be studied within the unified and general framework of variational inequalities. Variational inequalities have been generalized and extended in several directions using novel techniques. Variational inequalities and generalized variational inequalities are powerful tools for studying nonconvex optimization

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problems, nonconvex and non differential optimization problems respectively, for example see the references [4, 5, 7, 12, 16].

Recently by using the combination of demicontinuity and pseudomonotonicity, Fang and Huang [7] studied a new class of vector F-complementarity problems with demipseudo monotone mappings in Banach spaces. They also presented the solvability of this class of vector F-complementarity problems with demipseudomonotone mappings and finite dimensional continuous mappings in reflexive Banach spaces. For some related works we refer to [10, 12, 13, 18].

2. Preliminaries

Let X, Y be two arbitrary real Hausdorff topological vector spaces, $L(X,Y)$ denotes the space of all continuous linear mapping from X to Y. Let K be a nonempty set of $X, C : K \to Y$ be a set valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and $intC(x) \neq \emptyset$. The mappings $F : K \to Y, g : K \to K, A : L(X,Y) \times L(X,Y) \to L(X,Y)$ and $S, T : K \to 2^{L(X,Y)}$ are given.

For each $x \in K$, we define the relations $\leq_{C(x)}$ and $\not\leq_{C(x)}$ as follows:

(i) $z \leq_{C(x)} y \Leftrightarrow y - z \in C(x)$

(ii) $z \not\leq_{C(x)} y \Leftrightarrow y - z \notin C(x)$.

If we replace the set $C(x)$ by $intC(x)$ we can define the relations $\leq_{intC(x)}$ and $\not\leq_{intC(x)}$. If the mapping $C(x)$ is constant then we write $C(x)$ as C.

Now consider the set valued vector F-implicit variational inequality problems (weak) : Find $x_0 \in K$ for some $s_0 \in S(x_0), t_0 \in T(x_0)$ such that

$$\langle A(s_0, t_0), y - g(x_0)\rangle + F(y) - F(g(x_0)) \not\leq_{intC(x_0)} 0, \ \forall\ y \in K. \tag{1}$$

If s_0 and t_0 does not depend on y, that is, to find an $x_0 \in K$ with $s_0 \in S(x_0), t_0 \in T(x_0)$ such that

$$\langle A(s_0, t_0), y - g(x_0)\rangle + F(y) - F(g(x_0)) \not\leq_{intC(x_0)} 0, \forall y \in K, \tag{2}$$

we call this solution a strong solutions of the set valued vector F-implicit variational inequality.

Lemma 2.1. [6] *Let K be a nonempty subset of the Hausdorff topological vector space X. Let $G : K \rightrightarrows X$ be a KKM mapping such that for any $y \in K, G(y)$ is closed and $G(y^*)$ is compact for some $y^* \in K$. Then there exists $x^* \in K$ such that $x^* \in G(y)$ for all $y \in K$.*

Definition 2.1. *Let Ω be a vector space, Σ a topological vector space, K a nonempty convex subset of $\Omega, C : K \rightrightarrows \Sigma$ a set valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and $intC(x) \neq \emptyset$. For any $x \in K, \psi : K \to \Sigma$ is said to be*

(i) *$C(x)$-convex iff $\psi(\alpha x_1 + (1-\alpha)x_2) \leq_{C(x)} \alpha\psi(x_1) + (1-\alpha)\psi(x_2)$ for $x_1, x_2 \in K$ and $\alpha \in [0,1]$,*

(ii) *Properly quasi $C(x)$-convex iff we have either*
$\psi(\alpha x_1 + (1-\alpha)x_2) \leq_{C(x)} \psi(x_1)$ or
$\psi(\alpha x_1 + (1-\alpha)x_2) \leq_{C(x)} \psi(x_2)$ for $x_1, x_2 \in K$ and $\alpha \in [0,1]$.

Definition 2.2.[12] *Let Ω be a vector space, Σ a topological vector space, K a nonempty convex subset of $\Omega, C : K \rightrightarrows \Sigma$ a set valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and $intC(x) \neq \emptyset$. If A is a nonempty subset of Σ, therefore any $x \in K$:*

(i) *a point $z \in A$ is called a minimal point of A with respect to the cone $C(x)$ iff $A \cap (z - C(x)) = \{z\}; Min^{C(x)}A$ is the set of all minimal points of A with respect to the cone $C(x)$;*

(ii) *a point $z \in A$ is called a maximal point of A with respect to the cone $C(x)$ iff $A \cap (z + C(x)) = \{z\}; Max^{C(x)}A$ is the set of all maximal points of A with respect to the cone $C(x)$;*

(iii) *a point $z \in A$ is called a weakly minimal point of A with respect to the cone $C(x)$ iff $A \cap (z - intC(x)) = \emptyset; Min_w^{C(x)}A$ is the set of all weakly minimal points of A with respect to the cone $C(x)$;*

(iv) *a point $z \in A$ is called a weakly maximal point of A with respect to the cone $C(x)$ iff $A \cap (z + intC(x)) = \emptyset; Max_w^{C(x)} A$ is the set of all weakly maximal point of A with respect to the cone $C(x)$.*

Lemma 2.2. [14] *Let X, Y, Z be the real topological vector spaces and K, C be two nonempty subsets of X and Y respectively. Let $F : K \times C \rightrightarrows Z$, $S : K \rightrightarrows Y$ be set valued mappings, F and S be the upper semicontinuous with nonempty compact values, then the multivalued mapping $T : K \rightrightarrows Z$ defined by*

$$T(x) = \bigcup_{y \in S(x)} F(x, y) = F(x, S(x))$$

is upper semicontinuous with nonempty compact values.

Definition 2.3. *Let X and Y be real topological vector spaces. The set valued mapping $T : X \rightrightarrows Y$ is a closed mapping iff the following condition holds:*
the net $\{x_\alpha\} \to x_0, \{y_\alpha\} \to y_0, y_\alpha \in T(x_\alpha) \Rightarrow y_0 \in T(x_0)$.

3. Main results

1. Existence of weak solution for set valued vector F-implicit variational inequality (Weak)

Theorem 3.1. *Let X, Y be real Hausdorff topological vector spaces, K a nonempty closed convex subset of $X, C : K \rightrightarrows Y$ a set valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and $intC(x) \neq \emptyset$. Let $A : L(X, Y) \times L(X, Y) \to L(X, Y)$, $v : K \times K \to Y$, $F : K \to Y$, $g : K \to K$ and $S, T : K \rightrightarrows 2^{L(X,Y)}$ be the mappings. Suppose that*

(i) *$0 \leq_{C(x)} v(x, x)$ for all $x \in K$,*

(ii) *for each $x \in K$, $t \in T(x), s \in S(x)$ such that*

$$v(x, y) - \langle A(s, t), y - g(x)\rangle \leq_{C(x)} F(y) - F(g(x)), \text{ for all } y \in K;$$

(iii) *for each $x \in K$, the set*
$\{y \in K : 0 \nleq_{C(x)} v(x, y)\}$ is convex;

(iv) *there is a nonempty compact convex subset D of K such that for every $x \in K\backslash D$, there is $y \in D$ such that for all $t \in T(x), s \in S(x)$*

$$\langle A(s,t), y - g(x)\rangle \leq_{intC(x)} F(g(x)) - F(y);$$

(v) *for each $y \in K$, the set*

$$\{x \in K : \langle A(s,t), y - g(x)\rangle \leq_{intC(x)} F(g(x)) - F(y),\ \textit{for all } s \in S(x), t \in T(x)\}$$

is open in K.

Then there exists $x_0 \in K$ which is a solution of the set valued vector F-implicit variational inequality (weak). That is there is $x_0 \in K$ such that

$$\langle A(s_0,t_0), y - g(x_0)\rangle + F(y) - F(g(x_0)) \nleq_{intC(x_0)} 0,\ \textit{for all } y \in K \textit{ and } s_0 \in S(x_0), t_0 \in T(x_0).$$

Proof. Define $Q : K \rightrightarrows D$ by

$$Q(y) = \{x \in D : \langle A(s,t), y - g(x)\rangle \nleq_{intC(x)} F(g(x)) - F(y) \textit{ for } s \in S(x), t \in T(x)\}, \forall\, y \in K.$$

From condition (v) the set $Q(y)$ is closed in K and compact in D because of the compactness of D. Next we claim that the family $\{Q(y) : y \in K\}$ has the finite intersection property, then the whole intersection $\bigcap_{y\in K} Q(y)$ is nonempty and any element in the intersection $\bigcap_{y\in K} Q(y)$ is a solution of (1). For any given nonempty finite subset N of K, let $D_N = conv\{D \cup N\}$ the convex hull of $D \cup N$. Then D_N is a compact convex subset of K. Define $P, R : D_N \rightrightarrows D_N$ by

$$P(y) = \{x \in D_N : \langle A(s,t), y - g(x)\rangle \nleq_{intC(x)} F(g(x)) - F(y),\ \text{for } s \in S(x), t \in T(x)\}.$$

$$R(y) = \{x \in D_N : 0 \leq_{C(x)} v(x,y)\} \text{ for each } y \in D_N.$$

From (i) and (ii) we have

$$0 \leq_{C(y)} v(y,y),\ \text{for all } y \in D_N, \tag{3}$$

and each $y \in K$, there are $s \in S(y), t \in T(y)$ such that

$$v(y,y) - \langle A(s,t), y - g(y)\rangle \leq_{C(y)} F(y) - F(g(y)).$$

Hence

$$0 \leq_{C(y)} \langle A(s,t), y - g(y)\rangle + F(y) - F(g(y))$$

and then $y \in P(y)$ for all $y \in D_N$.

Since P has closed valued in D_N, for each $y \in D_N, Q(y) = P(y) \bigcap D$. Next show that the whole intersection of the family $\{P(y) : y \in D_N\}$ is nonempty. Now we can deduce that the family $\{Q(y) : y \in K\}$ has the finite intersection property because $N \subset D_N$ and (iv). In order to deduce the conclusion of our theorem, we can apply Lemma 2.1, if we claim that P is a KKM mapping. Indeed if P is not a KKM mapping, neither is R, since $R(y) \subset P(y)$ for each $y \in D_N$, then there is a nonempty finite subset M of D_N such that

$$convM \not\subset \bigcup_{u \in M} R(u).$$

Thus there is an element $\bar{u} \in convM \subset D_N$ such that $\bar{u} \notin R(u)$ for all $u \in M$, that is $0 \nleq_{C(\bar{u})} v(\bar{u}, u)$ for all $u \in M$. By (iii)

$$\bar{u} \in convM \subset \{y \in K : 0 \nleq_{C(\bar{u})} v(\bar{u}, y)\}$$

and hence $0 \nleq_{C(\bar{u})} v(\bar{u}, \bar{u})$ which contradict (3). Hence R is a KKM mapping and so is P. Therefore there exists $x_0 \in K, s_0 \in S(x_0), t_0 \in T(x_0)$ which is a solutions of the set valued vector F-implicit variational inequality (weak). This completes the proof.

Theorem 3.2. *Let $X, Y, K, C, A, F, g, S, T$ be as in Theorem 3.1. Assume that for each $x \in K, F$ is $C(x)$-convex on K such that*

(i) for each $x \in K$, there are $s \in S(x), t \in T(x)$ such that

$$\langle A(s,t), x - g(x)\rangle + F(x) - F(g(x)) \nleq_{intC(x)} 0;$$

(ii) there is a nonempty compact convex subset D of K such that for every $x \in K \backslash D, y \in D, s \in S(x), t \in T(x)$ and

$$\langle A(s,t), y - g(x)\rangle \leq_{intC(x)} F(g(x)) - F(y);$$

(iii) for each $y \in K$, the set

$$\{x \in K : \langle A(s,t), y - g(x)\rangle \leq_{intC(x)} F(g(x)) - F(y), \text{ for all } s \in S(x), t \in T(x)\}$$

is open in K. Then there are $x_0 \in K, s_0 \in S(x_0), t_0 \in T(x_0)$, which is a weak solutions of problem (1).

Proof. For any given nonempty finite subset N of K, let $D_N = conv(D \cup N)$, then D_N is a nonempty compact convex subset of K. Define $P : D_N \rightrightarrows D_N$ as in the proof of Theorem 3.1 and for each $y \in K$ let

$$Q(y) = \{x \in D : \langle A(s,t), y - g(x)\rangle \nleq_{intC(x)} F(g(x)) - F(y) \text{ for some } s \in S(x), t \in T(x)\}.$$

We remark that for each $x \in D_N, P(x)$ is nonempty and closed since $x \in P(x)$ by conditions (i) and (ii). For each $y \in K, Q(y)$ is compact in D. Next we claim that P is a KKM-mapping. Indeed if not there is a nonempty finite subset M of D_N such that conv $M \not\subset \bigcup\limits_{x \in M} P(x)$. There is an $x^* \in convM \subset D_N$ such that

$$\langle A(s,t), x - g(x^*)\rangle \leq_{intC(x^*)} F(g(x^*)) - F(x) \text{ for all } x \in M, s \in S(x^*), t \in T(x^*).$$

Since F is $C(x^*)$-convex, the mapping

$$x \to \langle A(s,t), x - g(x^*)\rangle + F(x) - F(g(x))$$

is $C(x^*)$-convex on D_N. Hence we can deduce that

$$\langle A(s,t), x^* - g(x^*)\rangle \leq_{intC(x^*)} F(g(x^*)) - F(x^*),$$

for all $s \in S(x^*), t \in T(x^*)$, this contradicts the condition (i) therefore P is a KKM-mapping and by Lemma 2.1, we have

$$\bigcap_{x \in D_N} P(x) \neq \emptyset.$$

Note that for any $u \in \bigcap\limits_{x \in D_N} P(x)$ we have $u \in D$ by condition (ii). Hence we have

$$\bigcap_{y \in N} Q(y) = \bigcap_{y \in N} (P(y) \bigcap D) \neq \emptyset,$$

for each nonempty finite subset N of K. Therefore, the whole intersection $\bigcap\limits_{y \in K} Q(y)$ is nonempty. Let $x_0 \in \bigcap\limits_{y \in K} Q(y)$. Then $x_0 \in K, s_0 \in S(x_0), t_0 \in T(x_0)$ is a solution of problem (1)(weak). This completes the proof.

2. Existence of strong solutions for set valued vector F-implicit variational inequality (Weak)

In this section we discuss the existence for the strong solutions of set valued vector F-implicit variational inequality. We define the condition (**), it is obviously full filled if $y \in R$ and $C(x) = [0, \infty)$ for all $x \in K$.

Theorem 3.3. *Let X be a real Banach space, Y, K, C, D, A, F, g and v be as in Theorem 3.1, under the assumptions of Theorem 3.1, we have a weak solution x_0 of the (1) (weak) with $s_0 \in S(x_0), t_0 \in T(x_0)$. In addition if K is compact, $x \to Y\backslash(-intC(x))$ is a closed mapping on K, F is $C(x_0)$-convex and continuous on K, the mappings $A : L(X,Y) \times L(X,Y) \to L(X,Y)$, $g : K \to K$ are continuous, $S, T : K \rightrightarrows 2^{L(X,Y)}$ is upper semicontinuous with nonempty compact valued mapping. Assume that*

$$(**)\ Max^{C(x_0)} \bigcup_{s\in S(x_0), t\in T(x_0)} Min_w^{C(x_0)} \bigcup_{x\in K} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\} \subset$$

$$Min_w^{C(x_0)} \bigcup_{x\in K} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\} + C(x_0), \forall s \in S(x_0), t \in T(x_0).$$

Assume also that

(i) for any $x \in K$, if

$$\delta \in Max^{C(x_0)} \bigcup_{s\in S(x_0), t\in T(x_0)} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\}$$

and δ can not be compared with

$$\langle A(s_0,t_0), x - g(x_0)\rangle - F(g(x_0)) + F(x)$$

which does not equal to δ, then

$$\delta \not\leq_{intC(x_0)} 0,$$

(ii) if

$$Max^{C(x_0)} \bigcup_{s\in S(x_0), t\in T(x_0)} Min_w^{C(x_0)} \bigcup_{x\in K} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\} \subset Y\backslash(-intC(x_0)),$$

there exists $s \in S(x_0), t \in T(x_0)$ such that

$$Min_w^{C(x_0)} \bigcup_{x\in K} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\} \subset Y\backslash(-intC(x_0)).$$

Then x_0 is a strong solution of the problem (1)(weak) that is there exists $s_0 \in S(x_0), t_0 \in T(x_0)$ such that

$$\langle A(s_0,t_0), x - g(x_0)\rangle + F(x) - F(g(x_0)) \nleq_{intC(x_0)} 0, \forall x \in K.$$

Further more, the set of all strong solutions of (1)(weak) is compact.

Proof.

Since F is $C(x_0)$-convex on K, the mapping

$$x \to \langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)$$

is $C(x_0)$-convex on K. From Theorem 3.1, we know that $x_0 \in K$ such that (1) holds for all $x \in K$ and for some $s_0 \in S(x_0), t_0 \in T(x_0)$. Then

$$\forall \gamma \in Min^{C(x_0)} \bigcup_{x\in K} Max^{C(x_0)} \bigcup_{s\in S(x_0), t\in T(x_0)} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\},$$

by (i) we have $\gamma \nleq_{intC(x)} 0$.

From condition (**), Ferro Minimax Theorem [8] tells us, for every

$$\alpha \in Max^{C(x_0)} \bigcup_{s\in S(x_0), t\in T(x_0)} Min_w^{C(x_0)} \bigcup_{x\in K} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\}, \alpha \nleq_{intC(x_0)} 0.$$

This implies that

$$Max^{C(x_0)} \bigcup_{s\in S(x_0), t\in T(x_0)} Min_w^{C(x_0)} \bigcup_{x\in K} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\} \subset Y \backslash (-intC(x_0)).$$

From (ii) there is $s_0 \in S(x_0), t_0 \in T(x_0)$ such that

$$Min_w^{C(x_0)} \bigcup_{x\in K} \{\langle A(s_0,t_0), x - g(x_0)\rangle - F(g(x_0)) + F(x)\} \subset Y \backslash (-intC(x_0)).$$

Hence

$$\forall \rho \in \bigcup_{x\in K} \{\langle A(s_0,t_0), x - g(x_0)\rangle - F(g(x_0)) + F(x)\},$$

we have that $\rho \nleq_{intC(x_0)} 0$. Hence there exists $s_0 \in S(x_0), t_0 \in T(x_0)$ such that

$$\langle A(s_0,t_0), x - g(x_0)\rangle - F(g(x_0)) + F(x) \nleq_{intC(x_0)} 0, \forall x \in K$$

such that x_0 is a strong solution of the problem (1)(weak). Finally, to see that the solution set of problem (1) (weak) is compact. It is sufficient to prove that the solution set is closed due to the coercivity condition (iv) of Theorem 3.1. To this end, let Γ denote the solution set of (1)(weak). Suppose $\{x_n\} \subset \Gamma$, which converges to some ρ. Fix any $y \in K$, each n there is $s_n \in S(x_n), t_n \in T(x_n)$ such that

$$\langle A(s_n, t_n), y - g(x_n)\rangle \not\leq_{intC(x_n)} F(g(x_n)) - F(y). \tag{4}$$

Since S, T are upper semicontinuous with nonempty compact values and the set $\{x_n\} \cup \{p\}$ is compact, therefore, without loss of generality, we may assume that the sequence $\{s_n\}$ converges to s and also $\{t_n\}$ converges to t. Then $s \in S(p), t \in T(p)$ and

$$F(g(x_n)) - F(y) - \langle A(s_n, t_n), y - g(x_n)\rangle \notin intC(x_n).$$

This implies that

$$F(g(x_n)) - F(y) - \langle A(s_n, t_n), y - g(x_n)\rangle \in Y \backslash intC(x_n)$$

we note that

$$F(g(x_n)) - F(y) - \langle A(s_n, t_n), y - g(x_n)\rangle$$

$$= F(g(x_n)) - F(y) - \langle A(s_n, t_n) - A(s, t), y - g(x_n)\rangle - \langle A(s, t), y - g(x_n)\rangle \tag{5}$$

$= F(g(x_n)) - F(y) - \langle A(s_n, t_n) - A(s, t), y - g(x_n)\rangle - \langle A(s, t), (y - g(x_n)) - (y - g(p))\rangle -$
$\langle A(s, t), y - g(p)\rangle.$

Since $\{x_n\} \cup \{p\}$ is compact and g is continuous $g(\{x_n\} \bigcup \{p\})$ is also compact. Hence it is bounded. Thus

$$\langle A(s_n, t_n) - A(s, t), y - g(x_n)\rangle \to 0 \text{ as } n \to \infty.$$

$$\langle A(s, t), (y - g(x_n)) - (y - g(p))\rangle = \langle A(s, t), g(p) - g(x_n)\rangle \to 0 \ as \ n \to \infty$$

by continuity of g. Since F is continuous and $x \to Y \backslash intC(x)$ is a closed mapping on K, from (5) we have

$$F(g(p)) - F(y) - \langle A(s, t), y - g(p)\rangle$$

$$= \lim_{n\to\infty} F(g(x_n)) - F(y) - \langle A(s_n,t_n), y - g(x_n)\rangle \in Y\backslash intC(p),$$

we obtain

$$\langle A(s,t), y - g(p)\rangle + F(y) - F(g(p)) \nleq_{intC(p)} 0.$$

Hence $p \in \Gamma$ and Γ is closed. This completes the proof.

Next, we consider the result of existence theorem for the strong solutions of (1)(weak) with the set K without compactness.

Theorem 3.4. *Let X be a finite dimensional real Banach space, $Y, K, C, D, A, F, g, S, T$ and v be as in Theorem 3.1. Under the assumptions of Theorem 3.1, we have a weak solution x_0 of the problem (1)(weak) with $s_0 \in S(x_0), t_0 \in T(x_0)$. In addition, if F is $C(x_0)$-convex, $x \to Y\backslash(-intC(x))$ is a closed mapping on K, the mappings $A : L(X,Y) \times L(X,Y) \to L(X,Y), g : K \to K$ are continuous, $S, T : K \rightrightarrows 2^{L(X,Y)}$ is upper semicontinuous with nonempty compact values. Assume for any nonempty compact subset M of K:*

$$(***)\ Max^{C(x_0)} \bigcup_{s\in S(x_0), t\in T(x_0)} Min_w^{C(x_0)} \bigcup_{x\in M} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(y)\}$$

$$\subset Min_w^{C(x_0)} \bigcup_{x\in M} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\} + C, \forall s \in S(x_0), t \in T(x_0).$$

Assume also that:

(i) for any fixed $x \in M$, if

$$\delta \in Max^{C(x_0)} \bigcup_{s\in S(x_0), t\in T(x_0)} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\}$$

and δ can not be compared with

$$\langle A(s_0,t_0), x - g(x_0)\rangle - F(g(x_0)) + F(x)$$

which is not equal to δ, then

$$\delta \nleq_{intC(x_0)} 0,$$

(ii) if

$$Max^{C(x_0)} \bigcup_{s\in S(x_0), t\in T(x_0)} Min_w^{C(x_0)} \bigcup_{x\in M} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\} \subset Y\backslash(-intC(x_0)),$$

there exists $s \in S(x_0), t \in T(x_0)$ *such that*

$$Min_w^{C(x_0)} \bigcup_{x \in M} \{\langle A(s,t), x - g(x_0)\rangle - F(g(x_0)) + F(x)\} \subset Y \backslash (-intC(x_0)).$$

Then, x_0 *is a strong solution of the problem (1)(weak), that is there exists* $s_0 \in S(x_0), t_0 \in T(x_0)$ *such that*

$$\langle A(s_0,t_0), x - g(x_0)\rangle + F(x) - F(g(x_0)) \nleq_{intC(x_0)} 0, \ \forall \ x \in K.$$

Furthermore, the set of all strong solutions of the problem (1)(weak) is compact.

Proof. Let $\bar{B}(0,r) = \{x \in X : \|x\| \leq r\}$ for each $r > 0$; then the set $K_r = \bar{B}(0,r) \bigcap K$ is compact in X. If $K_r \neq \emptyset$ and we replace K by K_r in Theorem 3.3, all the conditions of Theorem 3.3 hold. Hence by Theorem 3.3, there exists $s_0 \in S(x_0), t_0 \in T(x_0)$ such that

$$\langle A(s_0,t_0), z - g(x_0)\rangle + F(z) - F(g(x_0)) \nleq_{intC(x_0)} 0, \ \forall \ z \in K_r. \tag{6}$$

Let us choose $r > \|g(x_0)\|$. For any $x \in K$, choose $\alpha \in (0,1]$ small enough such that $(1-\alpha)g(x_0) + \alpha x \in K_r$. Putting $z = (1-\alpha)g(x_0) + \alpha x$ in (6), we have

$$\langle A(s_0,t_0), \alpha(x - g(x_0))\rangle + F((1-\alpha)g(x_0) + \alpha x) - F(g(x_0)) \nleq_{intC(x_0)} 0.$$

We note that

$$\langle A(s_0,t_0), \alpha(x - g(x_0))\rangle + F((1-\alpha)g(x_0) + \alpha x) - F(g(x_0)) \leq_{C(x_0)}$$

$$\alpha\langle A(s_0,t_0), x - g(x_0)\rangle + (1-\alpha)F(g(x_0)) + \alpha F(x) - F(g(x_0))$$

$$= \alpha\{\langle A(s_0,t_0), x - g(x_0)\rangle + F(x) - F(g(x_0))\}$$

implies that

$$\langle A(s_0,t_0), x - g(x_0)\rangle + F(x) - F(g(x_0)) \nleq_{intC(x_0)} 0, \forall x \in K.$$

This completes the proof.

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