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GENERALIZATION OF POPOVICIU TYPE INEQUALITIES VIA GREEN FUNCTION AND ABEL-GONTSCHAROFF INTERPOLATING POLYNOMIAL

SAAD IHSAN BUTT[1,*], KHURAM ALI KHAN [2], JOSIP PEČARIĆ[3]

[1]Department of Mathematics, COMSATS, Institute of Information Technology, Lahore, Pakistan

[2]Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan

[3]Faculty of Textile Technology, University of Zagreb, 10000 Zagreb, Croatia

Abstract. The inequality of Popoviciu for convex functions is generalized via Abel-Gontscharoff interpolating polynomial for higher order convex functions. Bounds are found for new identities, exponential convexity and Cauchy means are presented for linear functionals coming from the general inequality.

Keywords: convex function; exponentially convex function; Cauchy means.

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1. Introduction and preliminary results

A characterization of convex function established by T. Popoviciu [14] is studied by many people (see [15, 13] and references with in). For recent work, we refer [5, 8, 9, 10, 11]. The following form of Popoviciu's inequality is by Vasić and Stanković in [15] (see page 173 [13]):

*Corresponding author

E-mail address: saadihsanbutt@gmail.com

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Theorem 1.1. *Let* $m,k \in \mathbb{N},\ m \geq 3,\ 2 \leq k \leq m-1,\ [\alpha,\beta] \subset \mathbb{R},\ \mathbf{x} = (x_1,...,x_m) \in [\alpha,\beta]^m$, $\mathbf{p} = (p_1,...,p_m)$ *be a positive m-tuple such that* $\sum_{i=1}^{m} p_i = 1$. *Also let* $f : [\alpha,\beta] \to \mathbb{R}$ *be a convex function. Then*

$$p_{k,m}(\mathbf{x},\mathbf{p};f) \leq \frac{m-k}{m-1} p_{1,m}(\mathbf{x},\mathbf{p};f) + \frac{k-1}{m-1} p_{m,m}(\mathbf{x},\mathbf{p};f), \tag{1}$$

where

$$p_{k,m}(\mathbf{x},\mathbf{p};f) = p_{k,m}(\mathbf{x},\mathbf{p};f(x)) := \frac{1}{C_{k-1}^{m-1}} \sum_{1\leq i_1<\ldots<i_k\leq m} \left(\sum_{j=1}^{k} p_{i_j} \right) f \left(\frac{\sum\limits_{j=1}^{k} p_{i_j} x_{i_j}}{\sum\limits_{j=1}^{k} p_{i_j}} \right)$$

is the linear functional with respect to f.

By inequality (1), we write

$$\Upsilon(\mathbf{x},\mathbf{p};f) := \frac{m-k}{m-1} p_{1,m}(\mathbf{x},\mathbf{p};f) + \frac{k-1}{m-1} p_{m,m}(\mathbf{x},\mathbf{p};f) - p_{k,m}(\mathbf{x},\mathbf{p};f). \tag{2}$$

Remark 1.2. *It is important to note that under the assumptions of Theorem 1.1, if the function* f *is convex then* $\Upsilon(\mathbf{x},\mathbf{p};f) \geq 0$ *and* $\Upsilon(\mathbf{x},\mathbf{p};f) = 0$ *for* $f(x) = x$ *or* f *is constant function.*

The mean value theorems and exponential convexity of the linear functional $\Upsilon(\mathbf{x},\mathbf{p};f)$ are given in [8] for a positive m-tuple $\mathbf{p}$. Some special classes of convex functions are considered to construct the exponential convexity of $\Upsilon(\mathbf{x},\mathbf{p};f)$ in [8]. In [9] (see also [5]), the results related to $\Upsilon(\mathbf{x},\mathbf{p};f)$ are generalized with help of Green function and n-exponential convexity is proved instead of exponential convexity.

The presentation of the present paper is: in section 2 of this paper, we use Abel-Gontscharoff interpolating polynomial to generalize the Popoviciu inequality. In section 3, the Čebyšev functional is used to find the bounds for new identities. Grüss type and Ostrowski type inequalities related to generalized Popoviciu inequalities are constructed. In section 4, higher order convexity is used to produce exponential convexity of positive linear functionals coming from section 2. Last section is devoted to the respective Cauchy means. We employ the similar method as adopted in [7] for Steffensen's inequality. Hence the work in this paper is the extension of [9].

Consider the Green function $G:[\alpha,\beta]\times[\alpha,\beta]\to\mathbb{R}$ defined as

$$G(t,s)=\begin{cases}\frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha\le s\le t;\\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t\le s\le\beta.\end{cases}\tag{3}$$

The function G is convex and continuous w.r.t s and due to symmetry also w.r.t t.

For any function $\phi:[\alpha,\beta]\to\mathbb{R}$, $\phi\in C^2([\alpha,\beta])$, we have

$$\phi(x)=\frac{\beta-x}{\beta-\alpha}\phi(\alpha)+\frac{x-\alpha}{\beta-\alpha}\phi(\beta)+\int_\alpha^\beta G(x,s)\phi''(s)ds,\tag{4}$$

where the function G is defined in (3) (see [16]).

The following theorem is Abel-Gontscharoff theorem (see [1]) for two points with integral remainder.

Theorem 1.3. *Let $n,l\in\mathbb{N},\ n\ge 2,\ 0\le l\le n-1$ and $\phi\in C^n([\alpha,\beta])$. Then we have*

$$\phi(s)=T_{n-1}(\alpha,\beta,s;\phi)+R(s;\phi),\tag{5}$$

where $T_{n-1}(\alpha,\beta,s;\phi)$ is the Abel-Gontscharoff interpolating polynomial of degree $n-1$ for two points, i.e.

$$T_{n-1}(\alpha,\beta,s;\phi)=\sum_{v=0}^{l}\frac{(s-\alpha)^v}{v!}\phi^{(v)}(\alpha)+\sum_{w=0}^{n-l-2}\left[\sum_{v=0}^{w}\frac{(s-\alpha)^{l+1+v}(\alpha-\beta)^{w-v}}{(l+1+v)!(w-v)!}\right]\phi^{(l+1+w)}(\beta)$$

and the remainder is given by

$$R(s;\phi)=\int_\alpha^\beta G_n(s,t)\phi^{(n)}(t)dt,$$

where as $G_n(s,t)$ is defined by

$$G_n(s,t)=\frac{1}{(n-1)!}\begin{cases}\sum\limits_{v=0}^{l}\binom{n-1}{v}(s-\alpha)^v(\alpha-t)^{n-v-1}, & \alpha\le t\le s,\\ -\sum\limits_{v=l+1}^{n-l}\binom{n-1}{v}(s-\alpha)^v(\alpha-t)^{n-v-1}, & s\le t\le\beta.\end{cases}\tag{6}$$

Further, for $\alpha\le s,\ \ t\le\beta$ the following inequalities hold

$$(-1)^{n-l-1}\frac{\partial^v G_n(s,t)}{\partial s^v}\ge 0,\quad 0\le v\le l,\tag{7}$$

$$(-1)^{n-l}\frac{\partial^v G_n(s,t)}{\partial s^v}\ge 0,\quad l+1\le v\le n-1.\tag{8}$$

In the next section, we will present our main results using Green's function and Abel-Gontscharoff's theorem with the integral remainder.

2. Generalization of Popoviciu's Inequality

Motivated by identity (2), we construct the following identity with help of (4) and Abel-Gontscharoff's interpolating polynomial for two points.

Theorem 2.1. *Let $n, l \in \mathbb{N}$, $n \geq 4$, $0 \leq l \leq n-1$ and $\phi \in C^n([\alpha,\beta])$ and let $m,k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,...,x_m) \in [\alpha,\beta]^m$, $\mathbf{p} = (p_1,...,p_m)$ be a real m-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leq i_1 < ... < i_k \leq m$ and $\sum_{i=1}^{m} p_i = 1$. Also let $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for any $1 \leq i_1 < ... < i_k \leq m$ with G and G_n defined in* (3) *and* (6) *respectively. Then we have the following identity:*

$$
\begin{aligned}
(9)\quad \Upsilon(\mathbf{x},\mathbf{p};\phi(x)) &= \sum_{v=0}^{l} \frac{\phi^{(v+2)}(\alpha)}{v!} \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^v ds \\
&+ \sum_{w=0}^{n-l-4} \left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\phi^{(l+3+w)}(\beta)}{(l+1+v)!(w-v)!} \right] \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^{l+1+v} ds \\
&+ \int_\alpha^\beta \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))G_{n-2}(s,t)\phi^{(n)}(t)dtds.
\end{aligned}
$$

Proof. Using (4) in (2) and following Remark 1.2, we have

$$
(10)\qquad \Upsilon(\mathbf{x},\mathbf{p};\phi(x)) = \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))\phi''(s)ds.
$$

By Theorem 1.3, $\phi''(s)$ can be expressed as:

$$
\begin{aligned}
(11)\quad \phi''(s) &= \sum_{v=0}^{l} \frac{(s-\alpha)^v}{v!}\phi^{(v+2)}(\alpha) + \sum_{w=0}^{n-l-4} \left[\sum_{v=0}^{w} \frac{(s-\alpha)^{l+1+v}(\alpha-\beta)^{w-v}}{(l+1+v)!(w-v)!} \right] \phi^{(l+3+w)}(\beta) \\
&+ \int_\alpha^\beta G_{n-2}(s,t)\phi^n(t)dt.
\end{aligned}
$$

Using (11) in (10), we get (9). □

In the following theorem we obtain generalizations of Popoviciu's inequality for $n-$convex functions.

Theorem 2.2. *Let $n,l \in \mathbb{N}$, $n \geq 4$, $0 \leq l \leq n-1$ and let $m,k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,...,x_m) \in [\alpha,\beta]^m$, $\mathbf{p} = (p_1,...,p_m)$ be a real m-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leq i_1 < ... < i_k \leq m$ and $\sum_{i=1}^{m} p_i = 1$. Also let $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for any $1 \leq i_1 < ... < i_k \leq m$ with G and G_n defined in* (3) *and* (6) *respectively. If $\phi : [\alpha,\beta] \to \mathbb{R}$ is n-convex function and*

$$\int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s))G_{n-2}(s,t)ds \geq 0,\ t \in [\alpha,\beta]. \tag{12}$$

Then

$$\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) \geq \sum_{v=0}^{l} \frac{\phi^{(v+2)}(\alpha)}{v!} \int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^v ds \tag{13}$$
$$+ \sum_{w=0}^{n-l-4} \left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\phi^{(l+3+w)}(\beta)}{(l+1+v)!(w-v)!} \right] \int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^{l+1+v} ds.$$

Proof. Since the function ϕ is n-convex, therefore without loss of generality we can assume that ϕ is n-times differentiable and $\phi^{(n)}(x) \geq 0$ for all $x \in [\alpha,\beta]$ (see [13], p. 16). Hence we can apply Theorem 2.1 to obtain (13). □

Remark 2.3. *As from* (7) *we have $(-1)^{n-l-3}G_{n-2}(s,t) \geq 0$, therefore for the cases when $(n = even,\ l = odd)$ and $(l = even,\ n = odd)$, it is sufficient to assume that $\Upsilon(\mathbf{x},\mathbf{p};G(x,s)) \geq 0,\ s \in [\alpha,\beta]$, instead of assumption* (12) *in Theorem 2.2.*

Now, we give generalization of Popoviciu's inequality for m-tuples. As the weights are positive in Theorem 1.1, therefore in next theorem $\mathbf{p} = (p_1,...,p_m)$ be a positive m-tuple such that $\sum_{i=1}^{m} p_i = 1$.

Theorem 2.4. *Let $n,l,m,k \in \mathbb{N}$ such that $n \geq 4$, $0 \leq l \leq n-1$, $m \geq 3$ and $2 \leq k \leq m-1$, also let $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,...,x_m) \in [\alpha,\beta]^m$ and $\phi : [\alpha,\beta] \to \mathbb{R}$ be n-convex function, with G and G_n defined in* (3) *and* (6) *respectively.*

(i) If $(n = even,\ l = odd)$ or $(l = even,\ n = odd)$. Then

$$\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) \geq \sum_{v=0}^{l} \frac{\phi^{(v+2)}(\alpha)}{v!} \int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^v ds \tag{14}$$
$$+ \sum_{w=0}^{n-l-4} \left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\phi^{(l+3+w)}(\beta)}{(l+1+v)!(w-v)!} \right] \int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^{l+1+v} ds.$$

Moreover if $\phi^{(v+2)}(\alpha) \geq 0$ for $v = 0,1,2,...,l$ and $\phi^{(l+3+w)}(\beta) \geq 0$ if $w-v$ is even and $\phi^{(l+3+w)}(\beta) \leq 0$ if $w-v$ is odd, for $v = 0,\ldots,w$ and $w = 0,\ldots,n-l-4$, then the R.H.S. of (14) will be non negative, that is (1) holds.

(ii) If $(n = even,\ l = even)$ or $(l = odd,\ n = odd)$. Then (14) holds in reverse direction. Moreover if $\phi^{(v+2)}(\alpha) \leq 0$ for $v = 0,1,2,...,l$ and $\phi^{(l+3+w)}(\beta) \leq 0$ if $w-v$ is even and $\phi^{(l+3+w)}(\beta) \geq 0$ if $w-v$ is odd, for $v = 0,\ldots,w$ and $w = 0,\ldots,n-l-4$, then the R.H.S. of the reverse inequality in (14) will be non positive, that is reverse inequality in (1) holds.

Proof. By using (7) we have $(-1)^{n-l-3}G_{n-2}(s,t) \geq 0,\ \alpha \leq s,\ t \leq \beta$, therefore if $(n = even,\ l = odd)$ or $(l = even,\ n = odd)$ then $G_{n-2} \geq 0$. Also as G is convex so by Remark 1.2 and non negativity of G_{n-2}, the inequality (12) holds for m-tuples. Hence by Theorem 2.2, the inequality (14) holds. By using other conditions the non negativity of the R.H.S. (14) is quite understandable.

Similarly, we can prove (ii). □

3. Bounds for Identities Related to Generalization of Popoviciu's Inequality

For two Lebesgue integrable functions $f,h : [\alpha,\beta] \to \mathbb{R}$, we consider the Čebyšev functional

$$\Delta(f,h) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)h(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt.\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)dt.$$

In [4] the authors proved the following theorems:

Theorem 3.1. *Let $f : [\alpha,\beta] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [\alpha,\beta] \to \mathbb{R}$ be an absolutely continuous function with $(.-\alpha)(\beta-.)[h']^2 \in L[\alpha,\beta]$. Then we have the inequality*

$$|\Delta(f,h)| \leq \frac{1}{\sqrt{2}}[\Delta(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (x-\alpha)(\beta-x)[h'(x)]^2dx\right)^{\frac{1}{2}}. \tag{15}$$

The constant $\frac{1}{\sqrt{2}}$ in (15) is the best possible.

Theorem 3.2. *Assume that $h:[\alpha,\beta]\to\mathbb{R}$ is monotonic nondecreasing on $[\alpha,\beta]$ and $f:[\alpha,\beta]\to\mathbb{R}$ be an absolutely continuous with $f'\in L_\infty[\alpha,\beta]$. Then we have the inequality*

$$|\Delta(f,h)|\leq\frac{1}{2(\beta-\alpha)}||f'||_\infty\int_\alpha^\beta(x-\alpha)(\beta-x)dh(x). \tag{16}$$

The constant $\frac{1}{2}$ in (16) is the best possible.

In the sequel, we consider above theorems to derive generalizations of the results proved in the previous section. In order to avoid many notions let us denote

$$\mathfrak{R}(t)=\int_\alpha^\beta\Upsilon(\mathbf{x},\mathbf{p};G(x,s))G_{n-2}(s,t)ds\geq 0,\ t\in[\alpha,\beta], \tag{17}$$

Consider the Čebyšev functional $\Delta(\mathfrak{R},\mathfrak{R})$ given as:

$$\Delta(\mathfrak{R},\mathfrak{R})=\frac{1}{\beta-\alpha}\int_\alpha^\beta\mathfrak{R}^2(t)dt-\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta\mathfrak{R}(t)dt\right)^2, \tag{18}$$

Theorem 3.3. *Let $n,l\in\mathbb{N},\ n\geq 4,\ 0\leq l\leq n-1,\ \phi\in C^n([\alpha,\beta])$ with $(.-\alpha)(\beta-.)[\phi^{(n+1)}]^2\in L[\alpha,\beta]$ and let $m,k\in\mathbb{N},\ m\geq 3,\ 2\leq k\leq m-1,\ [\alpha,\beta]\subset\mathbb{R},\ \mathbf{x}=(x_1,...,x_m)\in[\alpha,\beta]^m,\ \mathbf{p}=(p_1,...,p_m)$ be a real m-tuple such that $\sum_{j=1}^k p_{i_j}\neq 0$ for any $1\leq i_1<...<i_k\leq m$ and $\sum_{i=1}^m p_i=1$. Also let $\frac{\sum_{j=1}^k p_{i_j}x_{i_j}}{\sum_{j=1}^k p_{i_j}}\in[\alpha,\beta]$ for any $1\leq i_1<...<i_k\leq m$ with G, G_n and $\mathfrak{R}$ defined in (3), (6) and (17) respectively. Then*

$$\begin{aligned}\Upsilon(\mathbf{x},\mathbf{p};\phi(x))&=\sum_{v=0}^{l}\frac{\phi^{(v+2)}(\alpha)}{v!}\int_\alpha^\beta\Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^v ds\\&+\sum_{w=0}^{n-l-4}\left[\sum_{v=0}^{w}\frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\phi^{(l+3+w)}(\beta)}{(l+1+v)!(w-v)!}\right]\int_\alpha^\beta\Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^{l+1+v}ds\\&+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)}\int_\alpha^\beta\mathfrak{R}(t)dt+\mathfrak{K}_n(\alpha,\beta;\phi),\end{aligned} \tag{19}$$

where the remainder $\mathfrak{K}_n(\alpha,\beta;\phi)$ satisfies the bound

$$|\mathfrak{K}_n(\alpha,\beta;\phi)|\leq\frac{\sqrt{\beta-\alpha}}{\sqrt{2}}[\Delta(\mathfrak{R},\mathfrak{R})]^{\frac{1}{2}}\left|\int_\alpha^\beta(t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}. \tag{20}$$

Proof. If we apply Theorem 3.1 for $f \mapsto \mathfrak{R}$ and $h \mapsto \phi^{(n)}$, we get

$$\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}(t)\phi^{(n)}(t)dt - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}(t)dt.\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\phi^{(n)}(t)dt\right| \leq \frac{1}{\sqrt{2}}[\Delta(\mathfrak{R},\mathfrak{R})]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left|\int_{\alpha}^{\beta}(t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^{2}dt\right|^{\frac{1}{2}}.$$

Hence, we have

$$\int_{\alpha}^{\beta}\mathfrak{R}(t)\phi^{(n)}(t)dt = \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)}\int_{\alpha}^{\beta}\mathfrak{R}(t)dt + \mathfrak{K}_{n}(\alpha,\beta;\phi),$$

where the remainder $\mathfrak{K}_n(\alpha,\beta;\phi)$ satisfies the estimation (20). Now from identity (9) , we obtain (19).

□

The following Grüss type inequalities can be obtained by using Theorem 3.2

Theorem 3.4. *Let $n,l \in \mathbb{N},\ n \geq 4,\ 0 \leq l \leq n-1$ with $\phi \in C^{n}([\alpha,\beta])$ such that $\phi^{(n+1)} \geq 0$ on $[\alpha,\beta]$ and let the functions G, G_n and $\mathfrak{R}$ defined in (3), (6) and (17) respectively. Then the representation (19) and the remainder $\mathfrak{K}_n(\alpha,\beta;\phi)$ satisfies the estimation*

$$|\mathfrak{K}_n(\alpha,\beta;\phi)| \leq ||\mathfrak{R}'||_{\infty}\left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{21}$$

Proof. Applying Theorem 3.2 for $f \mapsto \mathfrak{R}$ and $h \mapsto \phi^{(n)}$, we get

$$\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}(t)\phi^{(n)}(t)dt - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}(t)dt.\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\phi^{(n)}(t)dt\right| \leq \frac{1}{2(\beta-\alpha)}||\mathfrak{R}'||_{\infty}\int_{\alpha}^{\beta}(t-\alpha)(\beta-t)\phi^{(n+1)}(t)dt. \tag{22}$$

Since

$$\int_{\alpha}^{\beta}(t-\alpha)(\beta-t)\phi^{(n+1)}(t)dt = \int_{\alpha}^{\beta}[2t-(\alpha+\beta)]\phi^{(n)}(t)dt = (\beta-\alpha)\left[\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right] - 2\left(\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right).$$

Therefore, using identity (9) and the inequality (22), we deduce (21).

□

Now we intend to give the Ostrowski type inequalities related to generalizations of Popoviciu's inequality.

Theorem 3.5. *Suppose all the assumptions of Theorem 2.1 hold. Moreover, assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $|\phi^{(n)}|^p : [\alpha,\beta] \to \mathbb{R}$ be a R-integrable function for some $n \geq 2$. Then, we have*

$$\begin{aligned}(23)\quad & \Upsilon(\mathbf{x},\mathbf{p};\phi(x)) - \sum_{v=0}^{l} \frac{\phi^{(v+2)}(\alpha)}{v!} \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^v ds \\ & - \sum_{w=0}^{n-l-4} \left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\phi^{(l+3+w)}(\beta)}{(l+1+v)!(w-v)!} \right] \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^{l+1+v} ds \\ & \leq ||\phi^{(n)}||_p \left(\int_\alpha^\beta \left| \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))G_{n-2}(s,t)ds \right|^q dt \right)^{1/q}.\end{aligned}$$

The constant on the R.H.S. of (23) is sharp for $1 < p \leq \infty$ and the best possible for $p=1$.

Proof. Using identity (9) and applying H$\ddot{o}$lder's inequality, we obtain

$$\begin{aligned}& \Bigg| \Upsilon(\mathbf{x},\mathbf{p};\phi(x)) - \sum_{v=0}^{l} \frac{\phi^{(v+2)}(\alpha)}{v!} \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^v ds \\ & - \sum_{w=0}^{n-l-4} \left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\phi^{(l+3+w)}(\beta)}{(l+1+v)!(w-v)!} \right] \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^{l+1+v} ds \Bigg| \\ & = \left| \int_\alpha^\beta \mathfrak{R}(t)\phi^{(n)}(t)dt \right| \leq ||\phi^{(n)}||_p \left(\int_\alpha^\beta \left|\mathfrak{R}(t)\right|^q dt \right)^{1/q}.\end{aligned}$$

For the proof of the sharpness of the constant $\left(\int_\alpha^\beta \left|\mathfrak{R}(t)\right|^q dt \right)^{1/q}$, let us define the function ϕ for which the equality in (23) is obtained.

For $1 < p \leq \infty$ take ϕ to be such that

$$\phi^{(n)}(t) = \operatorname{sgn}\mathfrak{R}(t)|\mathfrak{R}(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $\phi^{(n)}(t) = \operatorname{sgn}\mathfrak{R}(t)$.

For $p = 1$, we prove that

$$(24)\qquad \left| \int_\alpha^\beta \mathfrak{R}(t)\phi^{(n)}(t)dt \right| \leq \max_{t\in[\alpha,\beta]} |\mathfrak{R}(t)| \left(\int_\alpha^\beta \phi^{(n)}(t)dt \right)$$

is the best possible inequality. Suppose that $|\mathfrak{R}(t)|$ attains its maximum at $t_0 \in [\alpha, \beta]$. To start with first we assume that $\mathfrak{R}(t_0) > 0$. For δ small enough we define $\phi_\delta(t)$ by

$$\phi_\delta(t) = \begin{cases} 0\,, & \alpha \le t \le t_0\,, \\ \frac{1}{\delta n!}(t-t_0)^n\,, & t_o \le t \le t_0+\delta\,, \\ \frac{1}{n!}(t-t_0)^{n-1}\,, & t_0+\delta \le t \le \beta\,. \end{cases}$$

Then for δ small enough

$$\left|\int_\alpha^\beta \mathfrak{R}(t)\phi^{(n)}(t)dt\right| = \left|\int_{t_0}^{t_0+\delta} \mathfrak{R}(t)\frac{1}{\delta}dt\right| = \frac{1}{\delta}\int_{t_0}^{t_0+\delta}\mathfrak{R}(t)dt.$$

Now from inequality (24), we have

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\mathfrak{R}(t)dt \le \mathfrak{R}(t_0)\int_{t_0}^{t_0+\delta}\frac{1}{\delta}dt = \mathfrak{R}(t_0).$$

Since

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\mathfrak{R}(t)dt = \mathfrak{R}(t_0),$$

the statement follows. The case when $\mathfrak{R}(t_0) < 0$, we define $\phi_\delta(t)$ by

$$\phi_\delta(t) = \begin{cases} \frac{1}{n!}(t-t_0-\delta)^{n-1}\,, & \alpha \le t \le t_0\,, \\ \frac{-1}{\delta n!}(t-t_0-\delta)^n\,, & t_o \le t \le t_0+\delta\,, \\ 0\,, & t_0+\delta \le t \le \beta\,, \end{cases}$$

and rest of the proof is the same as above.

□

4. Mean Value Theorems and $n-$ exponential convexity

We recall some definitions and basic results from [2], [6] and [12] which are required in sequel.

Definition 1. A function $\phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n}\xi_i\xi_j\,\phi\left(\frac{x_i+x_j}{2}\right) \ge 0,$$

hold for all choices $\xi_1,\ldots,\xi_n \in \mathbb{R}$ and all choices $x_1,\ldots,x_n \in I$. A function $\phi : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Definition 2. A function $\phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\phi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Proposition 4.1. *If $\phi : I \to \mathbb{R}$ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \le n$. Particularly,*

$$\det\left[\phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m} \ge 0$$

for all $m \in \mathbb{N}$, $m = 1,2,...,n$.

Remark 4.2. It is known that $\phi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2\phi(x) + 2\alpha\beta\phi\left(\frac{x+y}{2}\right) + \beta^2\phi(y) \ge 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Remark 4.3. *By the virtue of Theorem 2.2, we define the positive linear functional with respect to n-convex function ϕ as follows*

$$\begin{aligned}
(25)\quad \Gamma(\phi) := \Upsilon(\mathbf{x},\mathbf{p};\phi(x)) &- \sum_{v=0}^{l} \frac{\phi^{(v+2)}(\alpha)}{v!} \int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^{v} ds \\
&- \sum_{w=0}^{n-l-4}\left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\phi^{(l+3+w)}(\beta)}{(l+1+v)!(w-v)!}\right] \int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^{l+1+v} ds \ge 0.
\end{aligned}$$

Lagrange and Cauchy type mean value theorems related to defined functional is given in the following theorems.

Theorem 4.4. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi \in C^n[\alpha,\beta]$. If the inequality in (12) holds, then there exist $\xi \in [\alpha,\beta]$ such that*

$$\Gamma(\phi) = \phi^{(n)}(\xi)\Gamma(\varphi), \tag{26}$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\Gamma(\cdot)$ is defined by (25).

Proof. Similar to the proof of Theorem 4.1 in [7] (see also [3]). □

Theorem 4.5. *Let $\phi,\psi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi,\psi\in C^n[\alpha,\beta]$. If the inequality in (12) holds, then there exist $\xi\in[\alpha,\beta]$ such that*

$$\frac{\Gamma(\phi)}{\Gamma(\psi)}=\frac{\phi^{(n)}(\xi)}{\psi^{(n)}(\xi)}, \tag{27}$$

provided that the denominators are non-zero and $\Gamma(\cdot)$ is defined by (25).

Proof. Similar to the proof of Corollary 4.2 in [7] (see also [3]). □

Theorem 4.5 enables us to define Cauchy means, because if

$$\xi=\left(\frac{\phi^{(n)}}{\psi^{(n)}}\right)^{-1}\left(\frac{\Gamma(\phi)}{\Gamma(\psi)}\right),$$

which means that ξ is mean of α, β for given functions ϕ and ψ.

Next we construct the non trivial examples of n-exponentially and exponentially convex functions from positive linear functional $\Gamma(\cdot)$. In the sequel I and J are intervals in $\mathbb{R}$.

Theorem 4.6. *Let $\Omega=\{\phi_t : t\in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $t\mapsto[x_0,\dots,x_n;\phi_t]$ is $n-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\dots,x_n\in I$. Then for the linear functional $\Gamma(\phi_t)$ as defined by (25), the following statements are valid:*

(i) The function $t\to\Gamma(\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\Gamma(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m\in\mathbb{N}, m\le n$, $t_1,..,t_m\in J$. Particularly,

$$\det[\Gamma(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m\ge 0 \textit{ for all } m\in\mathbb{N},\ m=1,2,...,n.$$

(ii) If the function $t\to\Gamma(\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Proof. (i) For $\xi_j\in\mathbb{R}$ and $t_j\in J$, $j=1,\dots,n$, we define the function

$$h(x)=\sum_{j,l=1}^{n}\xi_j\xi_l\phi_{\frac{t_j+t_l}{2}}(x).$$

Using the assumption that the function $t \mapsto [x_0, \ldots, x_n; \phi_t]$ is n-exponentially convex in the Jensen sense, we have

$$[x_0, \ldots, x_n, h] = \sum_{j,l=1}^{n} \xi_j \xi_l [x_0, \ldots, x_n; \phi_{\frac{t_j+t_l}{2}}] \geq 0,$$

which in turn implies that h is a n-convex function on J, therefore from Remark 4.3 we have $\Gamma(h) \geq 0$. The linearity of $\Gamma(\cdot)$ gives

$$\sum_{j,l=1}^{n} \xi_j \xi_l \Gamma(\phi_{\frac{t_j+t_l}{2}}) \geq 0.$$

We conclude that the function $t \mapsto \Gamma(\phi_t)$ is n-exponentially convex on J in the Jensen sense. The remaining part follows from Proposition 4.1.

(ii) If the function $t \to \Gamma(\phi_t)$ is continuous on J, then it is n-exponentially convex on J by definition. □

The following corollary is an immediate consequence of the above theorem

Corollary 4.7. *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0, \ldots, x_n; \phi_t]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Then for the linear functional $\Gamma(\phi_t)$ as defined by (25), the following statements hold:*

(i) The function $t \to \Gamma(\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[\Gamma(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1, .., t_m \in J$. Particularly,

$$\det[\Gamma(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^{m} \geq 0 \textit{ for all } m \in \mathbb{N},\ m = 1, 2, ..., n.$$

(ii) If the function $t \to \Gamma(\phi_t)$ is continuous on J, then it is exponentially convex on J.

Corollary 4.8. *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0, \ldots, x_n; \phi_t]$ is 2-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Let $\Gamma(\cdot)$ be linear functional defined by (25). Then the following statements hold:*

(i) If the function $t \mapsto \Gamma(\phi_t)$ is continuous on J, then it is 2-exponentially convex function on J. If $t \mapsto \Gamma(\phi_t)$ is additionally strictly positive, then it is also log-convex on J.

Furthermore, the following inequality holds true:

$$[\Gamma(\phi_s)]^{t-r} \leq [\Gamma(\phi_r)]^{t-s} [\Gamma(\phi_t)]^{s-r},$$

for every choice $r,s,t \in J$, such that $r < s < t$.

(ii) If the function $t \mapsto \Gamma(\phi_t)$ is strictly positive and differentiable on J, then for every $p,q,u,v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(\Gamma,\Omega) \leq \mu_{u,v}(\Gamma,\Omega), \tag{28}$$

where

$$\mu_{p,q}(\Gamma,\Omega) = \begin{cases} \left(\frac{\Gamma(\phi_p)}{\Gamma(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\Gamma(\phi_p)}{\Gamma(\phi_p)}\right), & p = q, \end{cases} \tag{29}$$

for $\phi_p, \phi_q \in \Omega$.

Proof. (i) This is an immediate consequence of Theorem 4.6 and Remark 4.2.

(ii) Since $p \mapsto \Gamma(\phi_t)$ is positive and continuous, by (i) we have that $t \mapsto \Gamma(\phi_t)$ is log-convex on J, that is, the function $t \mapsto \log\Gamma(\phi_t)$ is convex on J. Hence we get

$$\frac{\log\Gamma(\phi_p) - \log\Gamma(\phi_q)}{p-q} \leq \frac{\log\Gamma(\phi_u) - \log\Gamma(\phi_v)}{u-v}, \tag{30}$$

for $p \leq u, q \leq v, p \neq q, u \neq v$. So, we conclude that

$$\mu_{p,q}(\Gamma,\Omega) \leq \mu_{u,v}(\Gamma,\Omega).$$

Cases $p = q$ and $u = v$ follow from (30) as limit cases.

□

5. Applications to Cauchy means

In this section, we present some families of functions which fulfil the conditions of Theorem 4.6, Corollary 4.7 and Corollary 4.8. This enables us to construct a large families of functions which are exponentially convex. Explicit form of this functions is obtained after we calculate explicit action of functionals on a given family.

Example 5.1. *Let us consider a family of functions*

$$\Omega_1 = \{\phi_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$\phi_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, & t \neq 0, \\ \frac{x^n}{n!}, & t = 0. \end{cases}$$

Since $\frac{d^n\phi_t}{dx^n}(x) = e^{tx} > 0$*, the function* ϕ_t *is n-convex on* $\mathbb{R}$ *for every* $t \in \mathbb{R}$ *and* $t \mapsto \frac{d^n\phi_t}{dx^n}(x)$ *is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 4.6 we also have that* $t \mapsto [x_0,\ldots,x_n;\phi_t]$ *is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 4.7 we conclude that* $t \mapsto \Gamma(\phi_t)$ *is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping* $t \mapsto \phi_t$ *is not continuous for* $t = 0$*), so it is exponentially convex. For this family of functions,* $\mu_{t,q}(\Gamma,\Omega_1)$ *, from* (29)*, becomes*

$$\mu_{t,q}(\Gamma,\Omega_1) = \begin{cases} \left(\frac{\Gamma(\phi_t)}{\Gamma(\phi_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left(\frac{\Gamma(id\cdot\phi_t)}{\Gamma(\phi_t)} - \frac{n}{t}\right), & t = q \neq 0, \\ \exp\left(\frac{1}{n+1}\frac{\Gamma(id\cdot\phi_0)}{\Gamma(\phi_0)}\right), & t = q = 0, \end{cases}$$

where "id" is the identity function. By Corollary 4.8 $\mu_{t,q}(\Gamma,\Omega_1)$ *is a monotone function in parameters t and q.*

Since

$$\left(\frac{\frac{d^n f_t}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{t-q}}(\log x) = x,$$

using Theorem 4.5 it follows that:

$$M_{t,q}(\Gamma,\Omega_1) = \log\mu_{t,q}(\Gamma,\Omega_1),$$

satisfies

$$\alpha \leq M_{t,q}(\Gamma,\Omega_1) \leq \beta.$$

Hence $M_{t,q}(\Gamma,\Omega_1)$ *is a monotonic mean.*

Example 5.2. *Let us consider a family of functions*

$$\Omega_2 = \{g_t : (0,\infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$g_t(x) = \begin{cases} \frac{x^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{0,1,\dots,n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & t = j \in \{0,1,\dots,n-1\}. \end{cases}$$

Since $\frac{d^n g_t}{dx^n}(x) = x^{t-n} > 0$, *the function* g_t *is* $n-$*convex for* $x > 0$ *and* $t \mapsto \frac{d^n g_t}{dx^n}(x)$ *is exponentially convex by definition. Arguing as in Example 5.1 we get that the mappings* $t \mapsto \Gamma(g_t)$ *is exponentially convex. Hence, for this family of functions* $\mu_{p,q}(\Gamma,\Omega_2)$ *, from* (29)*, are equal to*

$$\mu_{t,q}(\Gamma,\Omega_2) = \begin{cases} \left(\frac{\Gamma(g_t)}{\Gamma(g_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Gamma(g_0 g_t)}{\Gamma(g_t)} + \sum\limits_{k=0}^{n-1} \frac{1}{k-t}\right), & t = q \notin \{0,1,\dots,n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Gamma(g_0 g_t)}{2\Gamma(g_t)} + \sum\limits_{\substack{k=0 \\ k \neq t}}^{n-1} \frac{1}{k-t}\right), & t = q \in \{0,1,\dots,n-1\}. \end{cases}$$

Again, using Theorem 4.5 we conclude that

$$\alpha \leq \left(\frac{\Gamma(g_t)}{\Gamma(g_q)}\right)^{\frac{1}{t-q}} \leq \beta. \tag{31}$$

Hence $\mu_{t,q}(\Gamma,\Omega_2)$ *is a mean and its monotonicity is followed by (28).*

Example 5.3. *Let*

$$\Omega_3 = \{\zeta_t : (0,\infty) \to \mathbb{R} : t \in (0,\infty)\}$$

be a family of functions defined by

$$\zeta_t(x) = \begin{cases} \frac{t^{-x}}{(-\log t)^n}, & t \neq 1; \\ \frac{x^n}{(n)!}, & t = 1. \end{cases}$$

Since $\frac{d^n \zeta_t}{dx^n}(x) = t^{-x}$ *is the Laplace transform of a non-negative function (see* [16]*) it is exponentially convex. Obviously* ζ_t *are n-convex functions for every* $t > 0$.

For this family of functions, $\mu_{t,q}(\Gamma,\Omega_3)$ *, in this case for* $[\alpha,\beta]\subset\mathbb{R}^+$*, from (29) becomes*

$$\mu_{t,q}(\Gamma,\Omega_3) = \begin{cases} \left(\frac{\Gamma(\zeta_t)}{\Gamma(\zeta_q)}\right)^{\frac{1}{t-q}}, & t\neq q; \\ exp\left(-\frac{\Gamma(id.\zeta_t)}{t\Gamma(\zeta_t)}-\frac{n}{t\log t}\right), & t=q\neq 1; \\ exp\left(-\frac{1}{n+1}\frac{\Gamma(id.\zeta_1)}{\Gamma(\zeta_1)}\right), & t=q=1, \end{cases}$$

where id is the identity function. By Corollary 4.8 $\mu_{p,q}(\Gamma,\Omega_3)$ *is a monotone function in parameters t and q.*

Using Theorem 4.5 it follows that

$$M_{t,q}(\Gamma,\Omega_3) = -L(t,q)log\mu_{t,q}(\Gamma,\Omega_3),$$

satisfy

$$\alpha\leq M_{t,q}(\Gamma,\Omega_3)\leq\beta.$$

This shows that $M_{t,q}(\Gamma,\Omega_3)$ *is a mean. Because of the inequality (28), this mean is monotonic. Furthermore,* $L(t,q)$ *is logarithmic mean defined by*

$$L(t,q) = \begin{cases} \frac{t-q}{\log t-\log q}, & t\neq q; \\ \\ t, & t=q. \end{cases}$$

Example 5.4. *Let*

$$\Omega_4 = \{\gamma_t : (0,\infty)\to\mathbb{R} : t\in(0,\infty)\}$$

be a family of functions defined by

$$\gamma_t(x) = \frac{e^{-x\sqrt{t}}}{(-\sqrt{t})^n}.$$

Since $\frac{d^n\gamma_t}{dx^n}(x) = e^{-x\sqrt{t}}$ *is the Laplace transform of a non-negative function (see* [16]*) it is exponentially convex. Obviously* γ_t *are n-convex function for every* $t>0$*.*

For this family of functions, $\mu_{t,q}(\Gamma,\Omega_4)$*, in this case for* $[\alpha,\beta]\subset\mathbb{R}^+$*, from (29) becomes*

$$\mu_{t,q}(\Gamma,\Omega_4) = \begin{cases} \left(\frac{\Gamma(\gamma_t)}{\Gamma(\gamma_q)}\right)^{\frac{1}{t-q}}, & t\neq q; \\ exp\left(-\frac{\Gamma(id.\gamma_t)}{2\sqrt{t}\Gamma(\gamma_t)}-\frac{n}{2t}\right), & t=q; \end{cases} \quad i=1,2.$$

By Corollary 4.8, it is a monotone function in parameters t and q.

Using Theorem 4.5 it follows that

$$M_{t,q}(\Gamma,\Omega_4) = -\left(\sqrt{t}+\sqrt{q}\right) ln\mu_{t,q}(\Gamma,\Omega_4),$$

satisfy

$$\alpha \leq M_{t,q}(\Gamma,\Omega_4) \leq \beta.$$

This shows that $M_{t,q}(\Gamma,\Omega_4)$ *is a mean. Because of the above inequality (28), this mean is monotonic.*

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Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] R. P. Agarwal and P. J. Y. Wong, Error Inequalities in Polynomial Interpolation and their Applications, Kluwer Academic Publishers, Dordrecht, (1993).

[2] S. N. Bernstein, Sur les fonctions absolument monotones, Acta Math. 52 (1929), 1–66.

[3] S. I. Butt and J. Pečarić, Generalized Hermite-Hadamard's Inequality, Proc. A. Razmadze Math. Inst. 163 (2013), 9-27.

[4] P. Cerone, S. S. Dragomir, Some new Ostrowski-type bounds for the Čebyšev functional and applications, J. Math. Inequal. 8(1) (2014), 159-170.

[5] L. Horváth, K.A. Khan and J. Pečarić, Combinatorial Improvements of Jensen's Inequality / Classical and New Refinements of Jensen's Inequality with Applications, Monographs in inequalities 8, Element, Zagreb. (2014), pp. 229.

[6] J. Jakšetic and J. Pečarić, Exponential Convexity Method, J. Convex. Anal. 20(1) (2013), 181-197.

[7] J. Jakšetić, J. Pečarić, A. Perušić, Steffensen inequality, higher order convexity and exponential convexity, Rend. Circ. Mat. Palermo. 63(1) (2014), 109-127.

[8] K. A. Khan, J. Pečarić and I. Perić, Differences of weighted mixed symmetric means and related results, Journal of Inequalities and Applications. 2010 (2010), Article ID 289730, 16 pages.

[9] K. A. Khan, J. Pečarić and I. Perić, Generalization of Popoviciu Type Inequalities for Symmetric Means Generated by Convex Functions, J. Math. Comput. Sci. 4(6) (2014), 1091-1113.

[10] C. P. Niculescu, The Integral Version of Popoviciu's Inequality, J. Math. Inequal. 3(3) (2009), 323-328.

[11] C. P. Niculescu and F. Popovici, A Refinement of Popovicius Inequality, Bull. Soc. Sci. Math. Roum. 49 (97) (2006), 285-290.

[12] J. Pečarić and J. Perić, Improvement of the Giaccardi and the Petrović Inequality and Related Stolarsky Type Means, An. Univ. Craiova Ser. Mat. Inform. 39(1) (2012), 65-75.

[13] J. Pečarić, F. Proschan and Y. L. Tong, Convex functions, Partial Orderings and Statistical Applications, Academic Press, New York, (1992).

[14] T. Popoviciu, Sur certaines inegalites qui caracterisent les fonctions convexes, Analele Ştiinţifice Univ. Al. I. Cuza, Iasi, Sectia Mat., 11 (1965), 155-164.

[15] P. M. Vasić and Lj. R. Stanković, Some inequalities for convex functions, Math. Balkanica. 6(44) (1976), 281-288.

[16] D. V. Widder, Completely convex function and Lidstone series, Trans. Am. Math. Soc. 51 (1942), 387-398.