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GENERALIZED ORDER STATISTICS FROM $q-$ EXPONENTIAL TYPE- II DISTRIBUTION AND ITS CHARACTERIZATION

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Abstract. This article is concerned with $q-$ exponential type-II distribution. Recurrence relations for single and product moments of generalized order statistics have been derived from $q-$ exponential type-II distribution. Single and product moments of ordinary order statistics and upper k records cases have been discussed as a special case from generalized order statistics.

Keywords: generalized order statistics; order statistics; record values; single and product moments; recurrence relations; $q-$ exponential type-ii distribution and characterization.

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1. INTRODUCTION

The concept of generalized order statistics (gos) was introduced by Kamps (1995). Some types of ordered random variables such as: ordinary order statistics, upper k-records (upper record values when k = 1), sequential order statistics, ordering via truncated distributions, and censoring schemes can be discussed as special cases of the (gos).

Kamps introduced the model of generalized order statistics (gos) as follows:

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Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variable (rv) with the df $F(x)$ and the pdf $f(x)$. Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called (gos) if their joint pdf is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$ of $\Re^n$.

The joint density of the first r - gos is given by

$$f_{X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)}(x_1, x_2, \ldots, x_r)$$

$$= C_{r-1}\left(\prod_{i=1}^{r-1}[\bar{F}(x_i)]^{m_i} f(x_i)\right)[\bar{F}(x_r)]^{k+n-r+M_r-1} f(x_r) \tag{1.2}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$.

Then it is called generalized order statistics of a sample from distribution with df $F(x)$.

The pdf of r^{th} $m-gos$ is given by [Kamps, 1995]:

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}[F(x)] \tag{1.3}$$

and the joint pdf of $X(r, n, m, k)$ and $X(s, n, m, k)$, the r^{th} and s^{th} $m-gos$, $1 \leq r < s \leq n$, is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x, y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m g_m^{r-1}[F(x)]$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(x) f(y), \quad \alpha \leq x < y \leq \beta \tag{1.4}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -\log(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = \int_0^x (1-t)^m dt = h_m(x) - h_m(0), \quad x \in [0,1).$$

Choosing the parameters appropriately [Cramer, 2002], we get:

Table 1.1: Variants of the generalized order statistics

		$\gamma_n = k$	γ_r	m_r
i)	Sequential order statistics	α_n	$(n-r+1)\alpha_r$	$(\gamma_r - \gamma_{r+1} - 1)$
ii)	Ordinary order statistics	1	$n-r+1$	0
iii)	Record statistics	1	1	-1
iv)	Progressively type II censored order statistics	$R_n + 1$	$n-r+1+\sum_{j=r}^{n} R_j$	R_r
v)	Pfeifer's record statistics	β_n	β_r	$(\beta_r - \beta_{r+1} - 1)$

The $q-$ exponential distribution is a generalization of the exponential distribution. The main reason for introducing $q-$ exponential model is the switching property of the exponential form to corresponding binomial expansion. We refer the reader to Seetha and Thomas (2012) for a comprehensive study on the properties of $q-$ exponential distribution

$$\lim_{q\to 1}[1+(q-1)z]^{-\frac{1}{q-1}} = e^{-z}, \; 1 < q < 2$$

The main properties of the $q-$ exponential distribution as follows,

(1) Exponential distribution is a special case.

(2) It has equi- dispersed data via shape parameter.

(3) It allows for non- constant hazard rates.

A random variable X is said to have $q-$ exponential type-II distribution $(1 < q < 2)$ if its pdf is given by

$$f(x) = \upsilon(2-q)[1+(q-1)(\upsilon x)]^{-\frac{1}{(q-1)}} \ , \ x \geq 0 \tag{1.5}$$

and the corresponding df is

$$\overline{F}(x) = [1+(q-1)(\upsilon x)]^{\frac{q-2}{q-1}} \tag{1.6}$$

Therefore, in view of (1.5) and (1.6), we have

$$\overline{F}(x) = \frac{[1+(q-1)\ (\upsilon x)]}{\upsilon(2-q)} f(x) \tag{1.7}$$

Kamps (1998) investigated the importance of recurrence relations of order statistics in characterization. Recurrence relations for moments of order statistics and upper k-records were investigated, among others, by Joshi and Balakrishnan (1982), Khan *et al*. (1983a, 1983b), Grudzien and Szynal (1997), Pawlas and Szynal (1998, 1999) and Khan *et al*. (2015).

In this paper, we are concerned with generalized order statistics from $q-$ exponential type-II distribution. Sections 2 and 3, presented the recurrence relations for single and product moments of generalized order statistics. Section 4, discussed the characterization result. Section 5, contains the numerical computations. Section 6, has the conclusion part.

2. RECURRENCE RELATIONS FOR SINGLE MOMENTS

Theorem 2.1: For the $q-$ exponential type-II distribution given (1.5) and $n \in N,\ m \in R\ , 2 \leq r \leq n$

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] =$$

$$\frac{j}{\gamma_r \upsilon(2-q)} \left\{E[X^{j-1}(r,n,m,k)] + \nu(q-1)\ E[X^j(r,n,m,k)]\right\} \tag{2.1}$$

Proof: From (1.3), we have

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [\overline{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))dx \tag{2.2}$$

Integrating by parts taking $[\overline{F}(x)]^{\gamma_r - 1} f(x)$ as the part to be integrated, we get

$$E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)] + \frac{jC_{r-1}}{\gamma_r (r-1)!} \int_0^\infty x^{j-1} [\overline{F}(x)]^{\gamma_r}\ g_m^{r-1}(F(x))dx$$

The constant of integration vanishes since the integral considered in (2.2) is a definite integral, on using (1.7), we obtain

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] = \frac{j}{\gamma_r\, \upsilon(2-q)}\left\{E[X^{j-1}(r,n,m,k)] + \nu\,(q-1)\; E[X^j(r,n,m,k)]\right\}$$

and hence the Theorem

Remark 2.1: Setting $m=0, k=1$ in the Theorem 2.1, we obtain the recurrence relations for the single moments of order statistics of the $q-$ exponential type-II distribution in the form

$$E[X^j{}_{r:n}] - E[X^j{}_{r-1:n}] = \frac{j}{\upsilon(2-q)(n-r+1)}\left\{E[X^{j-1}{}_{r:n}] + \nu\,(q-1)\; E[X^j{}_{r:n}]\right\}$$

Remark 2.2: Setting $m=-1, k=1$ in the Theorem 2.1, we get the recurrence relations for the single moments of upper $k-$ record of the $q-$ exponential type-II distribution in the form

$$E[X^j{}_{U(r)}]^k - E[X^j{}_{U(r-1)}]^k = \frac{j}{\upsilon(2-q)\,k}\left\{E[X^{j-1}{}_{U(r)}]^k + \nu\,(q-1)\; E[X^j{}_{U(r)}]^k\right\}$$

3. RECURRENCE RELATIONS FOR PRODUCT MOMENTS

Theorem 3.1: For the $q-$ exponential type-II distribution given (1.5) and $n \in N,\, m \in R$, $1 \le r \le s \le n-1$

$$E[X^i(r,n,m,k)\;X^j(s,n,m,k)] - E[X^i(r,n,m,k)\;X^j(s-1,n,m,k)] =$$

$$\frac{j}{\gamma_s\, \upsilon(2-q)}\left\{E[X^i(r,n,m,k)\;X^{j-1}(s,n,m,k)] + \upsilon(q-1)E[X^i(r,n,m,k)\;X^j(s,n,m,k)]\right\} \quad (3.1)$$

Proof: From (1.4), we have

$$E[X^i(r,n,m,k)\;X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^i\,[\bar F(x)]^m\, f(x)\, g_m^{r-1}(F(x))\; I(x)dx \quad (3.2)$$

where

$$I(x) = \int_x^\infty y^j\,[\bar F(x)]^{\gamma_s-1}\,[h_m(F(y) - h_m(F(x)]^{s-r-1}\, f(y)\,dy\,.$$

Solving the integral in $I(x)$ by parts and substituting the resulting expression in (3.2), we get

$$E[X^i(r,n,m,k)\;X^j(s,n,m,k)] - E[X^i(r,n,m,k)\;X^j(s-1,n,m,k)] =$$

$$\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^{j-1}[\bar F(x)]^m\, f(x)\, g_m^{r-1}(F(x)$$

$$\times [h_m(F(y) - h_m(F(x)]^{s-r-1}[F(y)]^{\gamma_s}\, dy\, dx$$

The constant of integration vanishes since the integral in $I(x)$ is definite integral. On using relation (1.7), we obtain

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] - E[X^i(r,n,m,k)\ X^j(s-1,n,m,k)] =$$

$$\frac{jC_{s-1}}{\gamma_s \upsilon(2-q)(r-1)!(s-r-1)!}\Big\{\int_0^\infty \int_x^\infty x^i y^{j-1}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)$$

$$\times [h_m(F(y) - h_m(F(x)]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y)\, dy\, dx$$

$$+\upsilon(q-1)\int_0^\infty \int_x^\infty x^i y^j [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)[h_m(F(y) - h_m(F(x)]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y)\, dy\, dx\Big\}$$

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] - E[X^i(r,n,m,k)\ X^j(s-1,n,m,k)] =$$

$$\frac{j}{\gamma_s\, \upsilon(2-q)}\Big\{E[X^i(r,n,m,k)\ X^{j-1}(s,n,m,k)] + \upsilon(q-1) E[X^i(r,n,m,k)\ X^j(s,n,m,k)]\Big\}$$

and hence the Theorem

Remark 3.1: Setting $m = 0, k = 1$ in the Theorem 3.1, we obtain the recurrence relations for the product moments of order statistics of the $q-$ exponential type-II distribution in the form

$$E[X_{r,s:n}^{i\ j}] - E[X_{r,s-1:n}^{i\ j}] = \frac{j}{\upsilon(2-q)(n-s+1)}\Big\{E[X_{r,s:n}^{i\ j-1}] + \nu(q-1)\ E[X_{r,s:n}^{i\ j}]\Big\}$$

Remark 3.2: Setting $m = -1, k = 1$ in the Theorem 3.1, we get the recurrence relations for the product moments of upper $k-$ record of the $q-$ exponential type-II distribution in the form

$$E[X^i{}_{U(r)}\ X^j{}_{U(s)}]^k - E[X^i{}_{U(r)}\ X^j{}_{U(s-1)}]^k = \frac{j}{\upsilon(2-q)k}\Big\{E[X^i{}_{U(r)} X^{j-1}{}_{U(s)}]^k + \nu(q-1)\ E[X^i{}_{U(r)}\ X^j{}_{U(s)}]^k\Big\}$$

4. CHARACTERIZATION

Theorem 4.1: Let X be a non-negative random variable having absolutely continuous distribution $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$, for all $x > 0$

$$E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)] + \frac{j}{\gamma_r\, \upsilon(2-q)} E[X^{j-1}(r,n,m,k)] + \frac{j(q-1)}{\gamma_r(2-q)}\ E[X^j(r,n,m,k)] \qquad (4.1)$$

if and only if

$$\bar{F}(x) = [1 + (q-1)(\upsilon x)]^{\frac{q-2}{q-1}}$$

Proof: The necessary part follows immediately from equation (2.1). On the other hand if the recurrence relation in equation (4.1) is satisfied, then on using equation (1.3), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))dx = \frac{(r-1)\,C_{r-1}}{\gamma_r\,(r-1)!}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r + m} f(x)\, g_m^{r-2}(F(x))dx$$

$$+\frac{j\,C_{r-1}}{\gamma_r\,\upsilon(2-q)(r-1)!}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))\,dx$$

$$+\frac{j\,C_{r-1}\,(q-1)}{\gamma_r\;\;(r-1)!(2-q)}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))\,dx \qquad (4.2)$$

Integrating the first integral on the right hand side of equation (4.2), by parts, we get

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))dx = -\frac{jC_{r-1}}{\gamma_r (r-1)!}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_r} f(x)\, g_m^{r-1}(F(x))dx$$

$$+\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))dx$$

$$+\frac{j\,C_{r-1}}{\gamma_r\,\upsilon(2-q)(r-1)!}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))\,dx$$

$$+\frac{j\,C_{r-1}\,(q-1)}{\gamma_r\;\;(r-1)!(2-q)}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))\,dx\,.$$

Which reduces to

$$\frac{jC_{r-1}}{\gamma_r (r-1)!}\int_0^\beta x^{j-1}[\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}(F(x))\left[\bar{F}(x) - \frac{1}{\upsilon(2-q)} f(x) - x\frac{(q-1)}{(2-q)} f(x)\right]dx = 0 \qquad (4.3)$$

Now applying a generalization of the Muntz- Szasaz Theorem (Hawang and Lin, 1984) to equation (4.3), we get

$$\frac{f(x)}{\bar{F}(x)} = \frac{\upsilon(2-q)}{[1 + (q-1)(\upsilon x)]}$$

Which proves that

$$\bar{F}(x) = [1 + (q-1)(\upsilon x)]^{\frac{q-2}{q-1}}$$

5. NUMERICAL COMPUTATIONS

Table1: Four moments of order statistics from the $q-$ exponential type-II distribution

n	r	$\nu = 0.5, j = 1$		$\nu = 1.5, j = 1$	
		$q = 1.2$	$q = 1.3$	$q = 1.2$	$q = 1.3$
1	1	3.333333	5.00000	1.111111	1.666667
2	1	1.428571	1.818182	0.4761905	0.606060
	2	5.238095	8.181818	1.746032	2.727273
3	1	0.9090909	1.111111	0.3030303	0.370370
	2	2.467532	3.232323	0.8225108	1.077441
	3	6.623377	10.65657	2.207792	3.552189
4	1	0.6666667	0.80000	0.2222222	0.266666
	2	1.636364	2.044444	0.5454545	0.681481
	3	3.298701	4.420202	1.099567	1.473401
	4	7.731602	12.73535	2.577201	4.245118
5	1	0.5263158	0.625	0.1754386	0.208333
	2	1.22807	1.5000	0.4093567	0.50000
	3	2.248804	2.861111	0.7496013	0.953703
	4	3.998633	5.459596	1.332878	1.819865
	5	8.664844	14.55429	2.888281	4.851431
n	r	$\nu = 0.5, j = 2$		$\nu = 1.5, j = 2$	
		$q = 1.2$	$q = 1.3$	$q = 1.2$	$q = 1.3$
1	1	33.33333	200	1.111111	22.22222
2	1	4.761905	9.090909	0.4761905	1.010101
	2	61.90476	390.9091	1.746032	43.43434
3	1	1.818182	2.962963	0.3030303	0.3292181
	2	10.64935	21.3468	0.8225108	2.371867
	3	87.53247	575.6902	2.207792	63.96558
4	1	0.952381	1.454545	0.2222222	0.1616162
	2	4.415584	7.488215	0.5454545	0.8320239
	3	16.88312	35.20539	1.099567	3.91171
	4	111.0823	755.8518	2.577201	83.98354
5	1	0.5847953	0.862069	0.1754386	0.0957854
	2	2.422723	3.824451	0.4093567	0.424939
	3	7.404876	12.98386	0.7496013	1.442651
	4	23.20194	50.01974	1.332878	5.557749
	5	133.0523	932.3099	2.888281	103.59

6. CONCLUSION

This paper deals with the generalized order statistics from the $q-$ exponential type-II distribution. Recurrence relations between the single and product moments are derived. Characterizations of the $q-$ exponential type-II distribution based on the recurrence relations are discussed. Special cases are also deduced.

Conflict of Interests

The authors declare that there is no conflict of interests.

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REFERENCES

[1] Cramer, E. Contributions to Generalized Order Statistics. Ph.D.Thesis, University of Oldenburg, (2002).

[2] Grudzien, Z., Szynal, D. Characterization of continuous distributions via moments of the kth record values with random indices. J. Appl. Statist. Sci., 5 (1997), 259–266.

[3] Hwang, J. S. and Lin, G.D.: On a generalized moments problem II. Proc. Amer. Math. Soc., 91 (1984), 577-580.

[4] Joshi, P. C. and Balakrishnan, N.: Recurrence relations and identities for the product moments of order statistics. Sankhyā, Ser. B, 44 (1982), 39-49.

[5] Kamps, U. A Concept of Generalized Order Statistics. Stuttgart: Teubner. (1995).

[6] Kamps, U. Characterizations of distributions by recurrence relations and identities for moments of order statistics. In: Balakrishnan, N., Rao, C. R., eds. Handbook of Statistics, 16, Order Statistics: Theory and Methods. Amsterdam: North-Holland. (1998).

[7] Khan, A. H., Parvez, S. and Yaqub, M.: Recurrence relations between product moments of order statistics. J. Statist. Plann. Inference, 8 (1983), 175-183.

[8] Khan., M. I., and Khan, M. A. R., and Khan, M. A.: Generalized order statistics from q-exponential type-I distribution and its characterization. Int. J. Comp. Theo. Stat., 2 (2015) (To appear).

[9] Khan, A. H., Yaqub, M. and Parvez, S.: Recurrence relations between moments of order statistics. Naval Res. Logist. Quart., 30 (1983b), 419-441. Corrections (1985), 32, 693.

[10] Pawlas, P., Szynal, D. Relations for single and product moments of kth record values from exponential and Gumbel distributions. J. Appl. Statist. Sci., 7 (1998), 53–61.

[11] Pawlas, P., Szynal, D. Recurrence relations for single and product moments of kth record values from Pareto, generalized Pareto and Burr distributions. Commun. Statist. Theory Methods, 28(1999), 1699–1709.

[12] V. L. Seetha and Thomas, C. On $q-$ exponential count models. J. Appl. Statist. Sci., 20 (2012), 149-157.