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EVOLUTION OF CURVES VIA THE VELOCITIES OF THE MOVING FRAME

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Abstract. The purpose of the present work is to construct new geometrical models for the motion of plane and space curves using an approach different from the one proposed by R. Mukherjee and R. Balakrishnan [1]. This approach is applied to a pair of coupled nonlinear partial differential equations (CNLPDEs) governing the time evolution of the curvature and torsion of the evolving curve. For each model, solutions for CNLPDEs have been displayed by numerical integration of Frenet−Seret equations..

Keywords: Curve evolution; Velocity; Moving frame; Governing equation.

2000 AMS Subject Classification: 47H17; 47H05; 47H09

1. Introduction

A lot of physical processes can be modeled in terms of the motion of curves, including the dynamics of vortex filaments in fluid dynamics [2], the growth of dendritic crystals in a plane [3], and more generally, the planar motion of interfaces [4]. The Subject of how space curves evolve in time is of great interest and has been investigated by many authors. Pioneering work is attributed to Hasimoto who showed in [2] the nonlinear Schrödinger equation describing the motion of an isolated non−stretching thin vortex filament. Lamb [5] used the Hasimoto transformation to connect other motions of curves to the mKdV and sine−Gordon equations. Nakayama, et al [6] obtained the sine−Gordon equation

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by considering a nonlocal motion. Also Nakayama and Wadati [7] presented a general formulation of evolving curves in two dimensions and its connection to mKdV hierarchy. Nassar, et al [8, 9, 10, 11] studied the evolution of plane curves, the motion of hyper surfaces and the evolution of space curves in R^n. R. Mukherjee and R. Balakrishnan [1] applied their method to the sine-Gordon equation and obtained links to five new classes of space curves, in addition to the two which were found by Lamb [5]. For each class, they displayed the rich variety of moving curves associated with the one−soliton, the breather, the two−soliton and the soliton−antisoliton solutions.

In this paper, we derive a pair of coupled nonlinear partial differential equations governing the time evolution of the curvature and torsion of the evolving curve. Exact solutions for these equations have been obtained. Also we have reconstructed the evolving curve from its curvatures. In addition, we shall present new geometrical models different from those in [1].

The article is organized as follows. In section 2, we introduce General curve evolution and derive CNLPDEs which formulates the problem directly in terms of the curvature and torsion. In section 3, we get exact solutions for CNLPDEs and display the moving curve for these solutions. In section 4, the heat equation for the curvature is derived in the case of plane curve and the moving curve associated with its solution is displayed.

2. CNLPDEs associated with space curve

2.1 General curve evolution and governing equations

In this section, we derive time−evolution equations that the intrinsic quantities of curves satisfy. Let us consider a curve embedded in three-dimensional space described in parametric form by a position vector $\mathbf{r} = \mathbf{r}(s)$, s being the usual arclength variable. The unit tangent vector $\mathbf{t} = \mathbf{r}_s$, the principal normal $\mathbf{n}$ and the binormal $\mathbf{b}$ form an orthonormal triad of unit vectors that satisfy the Frenet–Serret equations [12]

$$\begin{aligned} \mathbf{t_s} &= \kappa\mathbf{n}, \\ \mathbf{n_s} &= -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \mathbf{b_s} &= -\tau\mathbf{n} \end{aligned} \tag{1}$$

Here and hereafter, the subscripts denote partial derivatives. κ and τ are the curvature and torsion of the curve, $\kappa > 0$, whereas τ can carry a sign. There is a metric on the curve. that is

$$g(s,t) = \langle \mathbf{r_s}, \mathbf{r_s} \rangle \tag{2}$$

where $\langle , \rangle$ is the Euclidean scaler product. If this curve moves with time u, then all quantities in Eqs. (1) become functions of both s and u. The general temporal evolution in which the triad $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ remains orthonormal adopts the following form [13]

$$\begin{aligned} \mathbf{t_u} &= \alpha\mathbf{n} + \beta\mathbf{b}, \\ \mathbf{n_u} &= -\alpha\mathbf{t} + \gamma\mathbf{b}, \\ \mathbf{b_u} &= -\beta\mathbf{t} - \gamma\mathbf{n}. \end{aligned} \tag{3}$$

As is clear, the parameters α, β and γ (which are the velocities of the moving frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$) determine the motion of the curve. Here for an inextensible curves, the triad must satisfy the following compatibility conditions

$$\mathbf{t}_{us} = \mathbf{t}_{su}, \quad \mathbf{n}_{us} = \mathbf{n}_{su}, \quad \mathbf{b}_{us} = \mathbf{b}_{su}. \tag{4}$$

Inextensible curves mean that the flow defined by the equations Eqs.(3) preserves the class of curves in arc$-$length parametrization [14]

$$\begin{aligned} \frac{\partial}{\partial u}\|\frac{\partial \mathbf{r}}{\partial s}\|^2 &= \frac{\partial}{\partial u}\langle \mathbf{r}_s, \mathbf{r}_s \rangle \\ &= 2\langle \mathbf{r}_{us}, \mathbf{r}_s \rangle, \quad \text{where } \mathbf{r}_{us} = \mathbf{r}_{su} \\ &= 2\langle \mathbf{t}_u, \mathbf{r}_s \rangle, \quad \text{by using first Eq. in (1,3) we obtain} \\ &= 0. \end{aligned}$$

Apply the condition 4 to the systems(1),(3) yield

$$
\begin{aligned}
\kappa_u &= \alpha_s - \tau\beta, \\
\tau_u &= \gamma_s + \kappa\beta, \\
\beta_s &= \kappa\gamma - \tau\alpha.
\end{aligned} \tag{5}
$$

The temporal evolution of the curvature κ and the torsion τ of a curve may now be expressed in terms of the components of velocity $\{\alpha, \beta, \gamma\}$ which can be written as coupled nonlinear partial differential equations as follows,

$$
\begin{aligned}
&\kappa_u = \alpha_s - \beta\tau, \\
&\tau_u = (\frac{\beta_s + \tau\alpha}{\kappa})_s + \kappa\beta.
\end{aligned} \tag{6}
$$

Eqs.(6) is the main result of this paper. For a given $\{\alpha, \beta, \gamma\}$, the motion of the curve is determined from these equations. MATHEMATICA package software (computational software program used in scientific, engineering, mathematical fields and other areas of technical computing) was used for solving the Eqs.(6) which applies the tanh−and sech−methods [15]. The outline for given $\{\alpha, \beta, \gamma\}$ is that we get $\{\kappa, \tau\}$. In the next subsection we shall show how to recreate curves in space from their curvature and torsion via numerical integration of Frenet−Seret equations up to its position in space.

2.2 Reconstruction of a Curve from its Curvatures

One of the basic problems in geometry is to determine exactly the geometric quantities which distinguish one figure from another. For example, line segments are uniquely determined by their lengths, circles by their radii, triangles by side-angle-side, etc. It turns out that this problem can be solved in general for sufficiently smooth regular curves. We will see that a regular curve is uniquely determined by two scalar quantities, called curvature and torsion, as functions of the natural parameter, which follows from the next theorem.

2.2.1 Theorem (Fundamental existence and uniqueness theorem for space curves)

[16] Let $\kappa(s)$ and $\tau(s)$ be arbitrary continuous functions on $a \leq s \leq b$. then there exists, except for position in space , one and only one space curve C for which $\kappa(s)$ is the curvature, $\tau(s)$ is the torsion and s is a natural parameter along C.

The figures in the next section represent snapshots of the evolving space curve obtained by solving the Frenet−Serret equations (1) for a specified curvature and torsion using Mathematica [17]. Any moving space curve can be studied from two perspectives, namely the shape of the curve and the evolution of the curve. At every fixed time u, we clearly have a representation of the corresponding static space curve at that instant. The program [17] as it stands generates static space curves. It was extended slightly to generate the evolution of the space curves with time u.

3. New geometrical models for the motion of space curve

In this section we consider some models of curve evolution specified by its local geometry. The nonlinear partial differential equations are generally difficult to solve and their exact solutions are difficult to obtain; therefore the models that we study depend on the viability of solving CNLPDEs. The set of five geometric parameters $\{\kappa, \tau, \alpha, \beta, \gamma\}$ appearing in the intrinsic Frenet-triad evolution equations (1) and (3) essentially describes a moving curve. Our strategy is to see under what conditions we can consistently find these functions such that (i) they satisfy all the compatibility conditions of (4), and (ii) they can be determined from κ and τ which satisfy the CNLPDEs Eqs. (6). In this study , we present five models of moving curves that satisfy the above requirements.

Before proceeding, we note that the basic Eqs.(1) and (3) when taken together, are invariant under the interchanges

$$\kappa \leftrightarrow \tau, \quad \alpha \leftrightarrow \gamma, \quad \beta \leftrightarrow -\beta \tag{7}$$

along with

$$\mathbf{b} \leftrightarrow \mathbf{t}, \quad \mathbf{n} \leftrightarrow -\mathbf{n} \tag{8}$$

Thus, if we can find a nontrivial set $\{\kappa, \tau, \alpha, \beta, \gamma\}$ that satisfy our requirements, another set can be found using the interchanges (7) but here we do not use such interchanges.

3.1 Model (1)

For a curve moving in space by the velocities

$$\{\alpha, \beta, \gamma\} = \{\kappa, \kappa_s, \frac{\kappa_{ss} + \tau\kappa}{\kappa}\}. \tag{9}$$

The evolution equations for the curvature and the torsion of the curve given from Eqs.(6) as follows,

$$\begin{aligned} \kappa_u &= \kappa_s - \kappa_s \tau, \\ \tau_u &= (\frac{\kappa_{ss} + \tau\kappa}{\kappa})_s + \kappa\kappa_s. \end{aligned} \tag{10}$$

The general solutions to this system are given by

$$\begin{aligned} \kappa_1(u, s) &= 2c_2 \text{sech}(c_1 u + c_2 s + c_3), \\ \tau_1(u, s) &= \frac{c_2 - c_1}{c_2}, \quad c_2 \neq 0, \end{aligned} \tag{11}$$

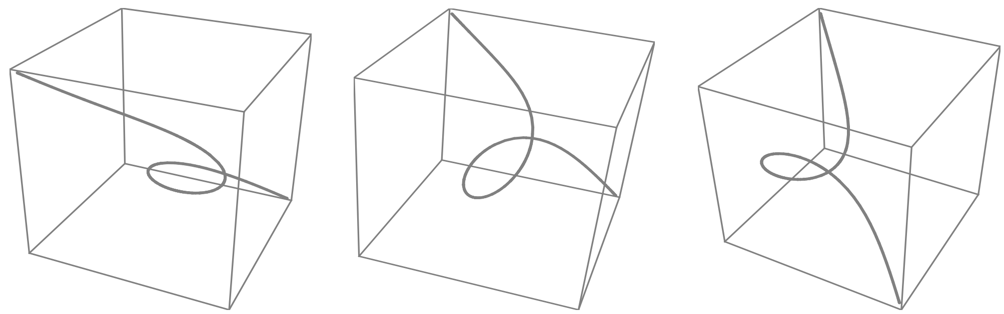
$$\begin{aligned} \kappa_2(u, s) &= c_4, \\ \tau_2(u, s) &= c_5 f(c_6 u + c_6 s + c_7), \end{aligned} \tag{12}$$

where $c_i, (i = 1, ..., 7)$ are arbitrary real constants and f is an arbitrary function of the parameter $y = (c_6 u + c_6 s + c_7)$. There are other solutions but we take the solution which has a geometric interpretation. As the first solution, it is easily verified that the set

(13)

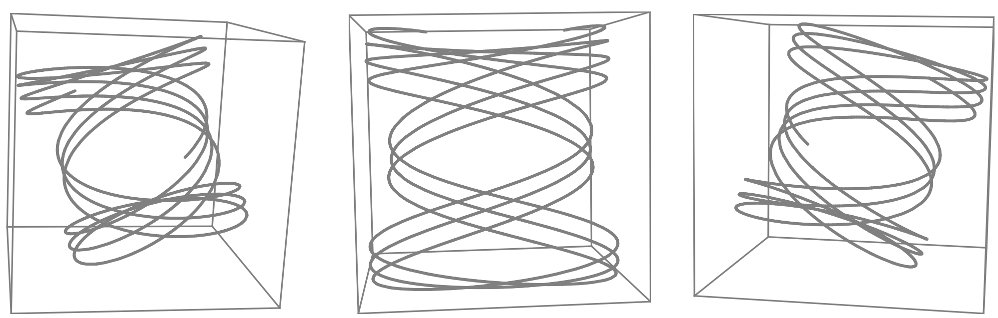
$$\{\kappa, \tau, \alpha, \beta, \gamma\} = \{2c_2 \text{sech}(y), 1 - \frac{c_1}{c_2}, 2c_2 \text{sech}(y), -2c_2^2 \text{sech}(y) \tanh(y), 1 - \frac{c_1}{c_2} + c_2^2 - 2c_2^2 \text{sech}^2(y)\}$$

satisfies the compatibility conditions Eqs. (5). If we take $c_1 = 0.4, c_2 = 0.5, c_3 = 0$ in Eq. (11), then $\kappa = \text{sech}(0.4u + 0.5s), \tau = 0.2$. we see that $\kappa \to 0$ as $s \to \pm\infty$. Thus, for large values of s, the curve straightens out at both ends as shown in Fig. 1. If we take $\kappa = 3, \tau = \sin(s + u)$ as a special case from Eq. (12), then we find $\tau \to \pm 1$ as $s \to \pm\infty$. Hence, the curve takes a helix form as in Fig. 2. The curve in this case repeats on intervals of length 2π because of the periodic function $\sin(s + u)$. Model (1) represents the motion of an inextensible curve of constant curvature and torsion.



(A) u=-1 (B) u=0 (C) u=1

FIGURE 1. Mod. (1) $\kappa = \operatorname{sech}(0.5s + 0.4u), \tau = 0.2$.



(A) u=-1 (B) u=0 (C) u=1

FIGURE 2. Mod. (1) $\kappa = 3, \tau = \sin(s + u)$.

In all the figures here and hereafter, we have used the total curve length of 20 ($-10 \leq s \leq 10$). In practice, the range of variation of u must remain much smaller than that of s so that the length of the curve suffices to display the complete geometric structure corresponding to the solution concerned.

3.2 Model (2)

We consider the case that the velocities are given by

$$\{\alpha, \beta, \gamma\} = \{0, \kappa, \frac{\kappa_s}{\kappa}\}. \tag{14}$$

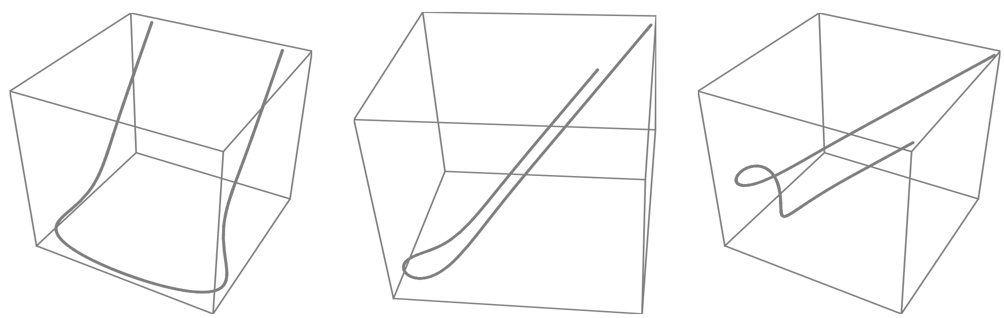
The evolution equations for the curvature and the torsion of the curve given from Eqs. (6) as follows,

$$\begin{aligned}\kappa_u &= -\kappa\tau,\\ \tau_u &= (\frac{\kappa_s}{\kappa})_s + \kappa^2.\end{aligned} \tag{15}$$

The general solution to this system is given by

$$\kappa(u,s) = \sqrt{c_1^2 + c_2^2}\mathrm{sech}(c_1 u + c_2 s + c_3), \quad \tau(u,s) = c_1 \tanh(c_1 u + c_2 s + c_3). \tag{16}$$

where c_1, c_2, c_3 are arbitrary real constants. If we take $c_1 = 1, c_2 = 1, c_3 = 0$ in Eq. (16), then $\kappa = \sqrt{2}\mathrm{sech}(u+s), \tau(u,s) = \tan(u+s)$. we see that $\kappa \to 0$ as $s \to \pm\infty$, $\tau \to \pm 1$ as $s \to \pm\infty$. Thus, for large values of s, the curve straightens out at both ends as shown in Fig. 3.



(A) u=-1 (B) u=0 (C) u=1

FIGURE 3. Mod. (2) $\kappa = \sqrt{2}\mathrm{sech}(s+u), \tau = \tanh(u+s)$.

3.3 Model (3)

Consider a curve moving in space such that the velocity of the frame is

$$\{\alpha, \beta, \gamma\} = \{\tau\kappa, 0, \tau^2\}.$$

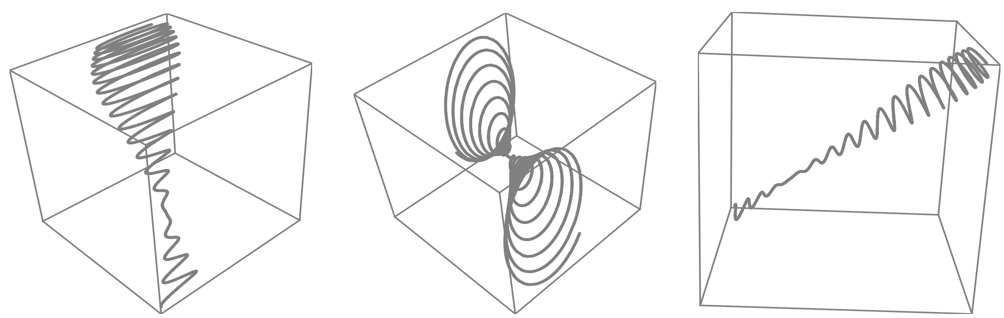
The evolution equations for the curvature and the torsion of the curve given from Eqs.(6) as follows,

$$\begin{aligned}\kappa_u &= \kappa_s\tau + \kappa\tau_s,\\ \tau_u &= 2\tau\tau_s.\end{aligned} \tag{17}$$

The general solution to this system is given by

$$\kappa(u,s) = c_1 f(u + s/c_2 + c_3), \quad \tau(u,s) = c_2, \tag{18}$$

where c_1, c_2, c_3 are arbitrary real constants and f is an arbitrary function of $(u+s/c_2+c_3)$. If we take $\kappa = \sinh(u + s/5), \tau = 5$ as a special case from Eq. (18), we see that $\kappa \to \pm\infty$, as $s \to \pm\infty$ as we find in Fig.(4), and we note that Fig. (5) is similar to Fig. (3) in the work of Tomasz Lipniacki [18]



(A) u=-1 (B) u=0 (C) u=1

FIGURE 4. Mod. (3) $\kappa = \sinh(u + s/5), \tau = 5$.

3.4 Model (4)

For a curve moving in space by the velocities

$$\{\alpha, \beta, \gamma\} = \{\tau\kappa, \kappa_s, \frac{\kappa\tau^2 + \kappa_{ss}}{\kappa}\}, \tag{19}$$

and from Eqs. (6) we have the system

$$\begin{aligned} \kappa_u &= \kappa\tau_s, \\ \tau_u &= (\frac{\kappa\tau^2 + \kappa_{ss}}{\kappa})_s + \kappa\kappa_s. \end{aligned} \tag{20}$$

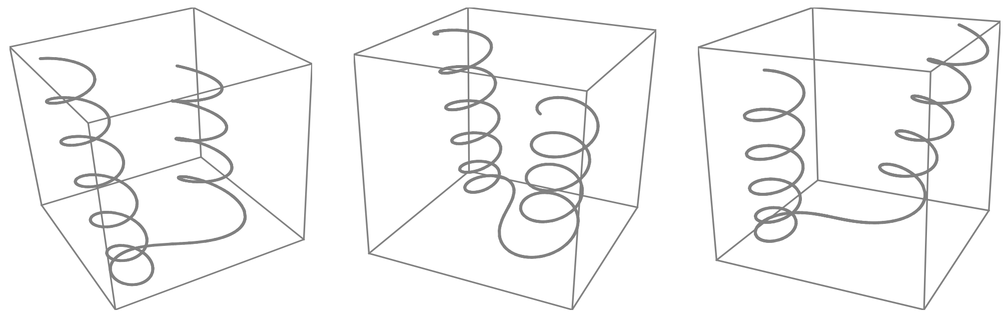
The general solution to this system is given by

(21)

$$\kappa(u,s) = -c_5 \mathrm{sech}(c_1 u + c_2 s + c_3) + c_5 \tanh(c_1 u + c_2 s + c_3)$$

$$\tau(u,s) = \frac{-c_1}{c_2}\mathrm{sech}^2(c_1 u + c_2 s + c_3) - \frac{c_1}{c_2}\tanh(c_1 u + c_2 s + c_3)\mathrm{sech}(c_2 s + c_1 u + c_3) + c_4,$$

where c_1, c_2, c_3, c_4 are arbitrary real constants. If we take $c_1 = 1, c_2 = 1, c_3 = 0, c_4 = 1, c_5 = 3$ in Eq. (21), then for $s \to \infty, \kappa \to -3, \tau \to 1$. Thus for a large value of s the curve collapses into helix at the end. On the other hand, for $s \to \infty, \kappa \to 3, \tau \to 1$, thus for a large value of s the curve collapses into helix at the end as shown in Fig. 5.



(A) u=-1 (B) u=0 (C) u=1

FIGURE 5. space curves corresponding to model(4).

3.5 Model (5)

For a curve moving in space by the velocities,

$$\{\alpha, \beta, \gamma\} = \{\tau, 0, \frac{\tau^2}{\kappa}\}. \tag{22}$$

The evolution equations for curvature and torsion are given from Eqs. (6) as follows,

$$\begin{aligned} \kappa_u &= \tau_s, \\ \tau_u &= (\frac{\tau^2}{\kappa})_s. \end{aligned} \tag{23}$$

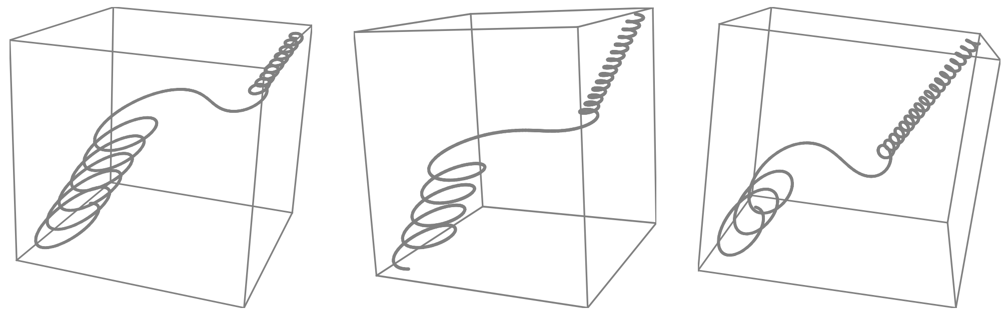
The general solution to this system is given by

(24)

$$\kappa(u,s) = \frac{c_2}{c_1}[c_7 \tanh^3(c_1 u + c_2 s + c_3) + c_6 \tanh^2(c_1 u + c_2 s + c_3) + c_5 \tanh(c_1 u + c_2 s + c_3) + c_4],$$

$$\tau(u,s) = c_7 \tanh^3(c_1 u + c_2 s + c_3) + c_6 \tanh^2(c_1 u + c_2 s + c_3) + c_5 \tanh(c_1 u + c_2 s + c_3) + c_4.$$

where $c_i, (i = 1, ..., 7)$ are arbitrary real constants. If we take $c_1 = 1, c_2 = 3, c_3 = 0, c_4 = 0, c_5 = 1, c_6 = 1, c_7 = 1$ in Eq. (24) then for $s \to \infty, \kappa \to 9, \tau \to 3$, thus for a large value of s the curve collapses into helix at the end. On the other hand, $s \to -\infty, \kappa \to -3, \tau \to -1$ thus, for a large value of s the curve collapses into helix at the end as shown in Fig. 6.



(A) u=-1 (B) u=0 (C) u=1

FIGURE 6. space curves corresponding to model(5).

4. Reduction of time-evolution equations from 3D to 2D

In this section, we reduce the time–evolution equations that we derived in section **??** to lower dimensional cases. In order to reduce the time–evolution equations (6) to those that describe time evolution of a curve in a tow–dimensional Euclidean space, we set

$$\tau = \beta = \gamma = 0. \tag{25}$$

Then, (6) yields

$$\kappa_u = \alpha_s. \tag{26}$$

For a given α, Eq. (26) determines the motion of curves in tow–dimensional Euclidean space. The authors in [6] considered a curve moving in the plane under the velocity

$$\mathbf{r}_t = U\mathbf{n} + W\mathbf{t}, \tag{27}$$

thus the dynamical equation for the curvature becomes

$$\kappa_t = U_{ss} + \kappa^2 U + \kappa_s W, \tag{28}$$

where (U, W) are the normal and tangent projections of the velocity. For a given (U, W), the motion of of the curve is determined from equation (28).

4.1 Geometrical models for the motion of plane curve

4.1.1 Model (1)

If we take $\alpha = \kappa_s$, then the dynamical equation for the curvature becomes

$$\kappa_u = \kappa_{ss}, \tag{29}$$

which is precisely the Heat or diffusion equation and has a general solution in the following form

$$\kappa(u, s) = \exp(-u/c_1^2)(c_2 \cos(s/c_3) + c_3 \sin(s/c_3)), \tag{30}$$

where c_1, c_2, c_3 are arbitrary real constants. If we take $c_1 = c_2 = c_3 = 1$,then $\kappa(u, s) = \exp(-u)(\cos(s) + \sin(s))$,we see that $\kappa \to 0$ as $u \to \infty$. Thus the curve converts to straight line. On the other hand, $\kappa \to \infty$ as $u \to -\infty$. Thus more loop generates as in Fig.7

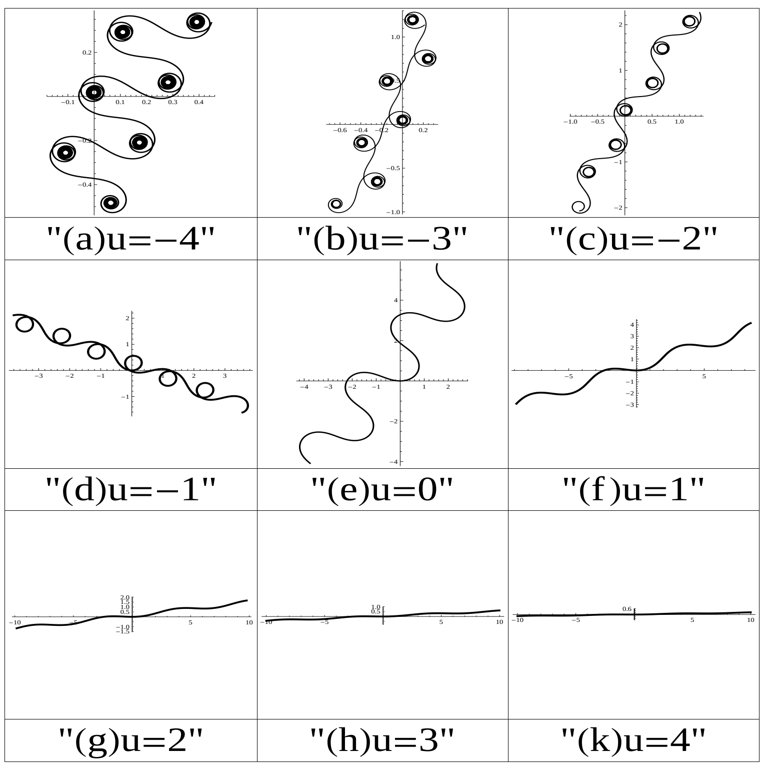


FIGURE 7. $\kappa(u, s) = \exp(-u)(\cos(s) + \sin(s))$.

4.1.2 Model (2)

If we take $\alpha = \kappa$, then the evolution equation of the curvature of the evolving curve is

$$\kappa_u = \kappa_s, \tag{31}$$

which has a general solution in the form

$$\kappa(u,s) = \phi(u+s), \tag{32}$$

where $\phi(u+s)$ is an arbitrary function, so we can take the solutions of equation (31) in the forms

$$\begin{aligned} \kappa_1(u,s) &= s+u, \\ \kappa_2(u,s) &= \sin(s+u). \end{aligned}$$

The geometric visualization of the curves corresponding to these solutions are given in Fig. (8) and Fig. (9), respectively

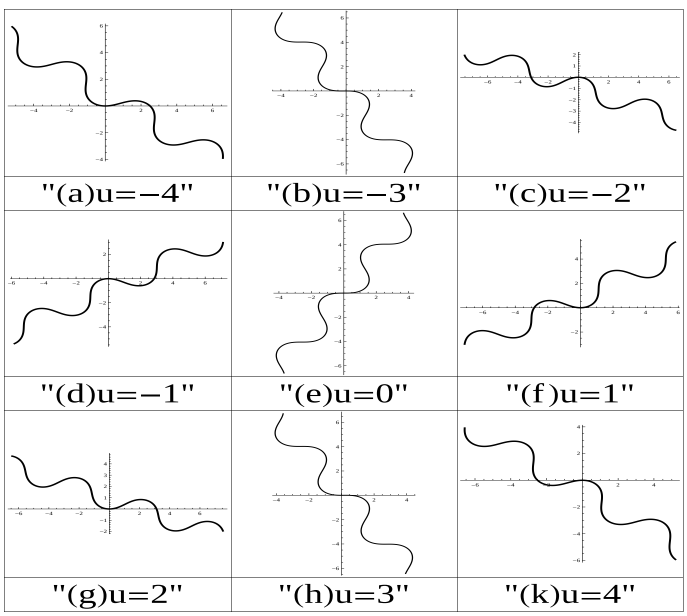


FIGURE 8. $\kappa = \sin(s+u)$.

The curve in th case $\kappa = \sin(s+u)$ repeats on intervals of length 2π because of the periodic function $\sin(s+u)$.

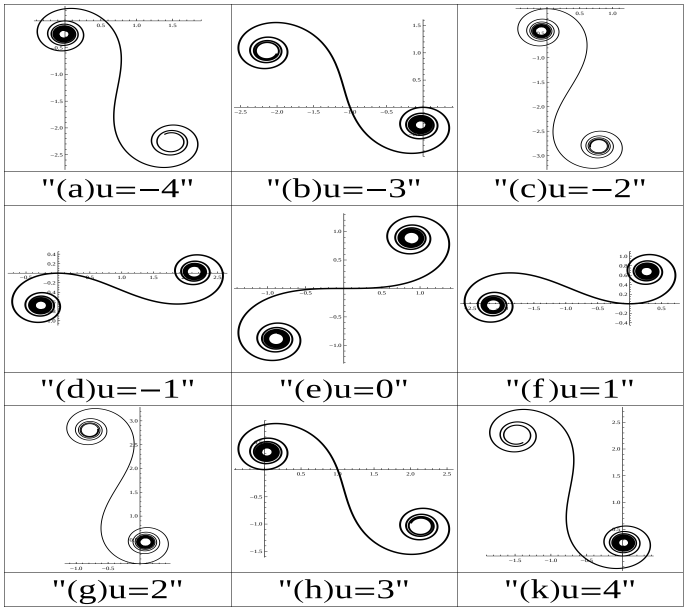


FIGURE 9. $\kappa = s + u$.

Conclusion

In this paper we have presented geometrical models for the motion of space and plane curves other than mkdv, sin−Gordan equations which were known before and different from those in [1]. For each geometrical model, we display the rich variety of moving curves associated with the CNLPDEs and Heat equation solutions via numerical integration of Frenet−Seret equations.

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