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MULTIPLICATION OPERATORS ON ORLICZ-LORENTZ SEQUENCE SPACES

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Abstract. In this paper, we characterize the boundedness, compactness and closedness of the range of the multiplication operators on Orlicz-Lorentz sequence spaces.

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1. Introduction

Let (X,S,μ) be a $\sigma-$finite measure space and let f be a complex-valued measurable function defined on X. For $s \geq 0$ the distribution function μ_f of f is defined as

$$\mu_f(s) = \mu\left(\{x \in \mathbb{N} : \ |f(x)| > s\}\right)$$

and the non-incrasing rearrangement of f is defined as

$$f^*(t) = \inf\left\{s > 0 : \mu_f(s) \leq t\right\},\ t \geq 0.$$

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By a weight function w, we mean $w:(0,\infty)\to(0,\infty)$ is a non-increasing locally integrable function such that $\int_0^\infty w(t)dt=\infty$.

The Orlicz-Lorentz space $L_{\varphi,w}(\mu)$ is defined as

$$L_{\varphi,w}(\mu)=\left\{ f:X\to\mathbb{C}\text{ measurable}:\ \int_0^\infty \varphi\left(\alpha.f^*(t)\right)w(t)dt<\infty\text{ for some }\alpha>0\right\}.$$

Then $L_{\varphi,w}(\mu)$ is a Banach space with respect to the Luxemburg norm

$$\|f\|_{\varphi,w}=\inf\left\{\varepsilon>0:\int_0^\infty \varphi\left(\frac{|f^*(t)|}{\varepsilon}\right)w(t)dt\leq 1\right\},$$

where $\varphi:[0,\infty)\to[0,\infty)$ is a continuous convex function which satisfies the following conditions;

(i) $\varphi(x)=0$ if and only if $x=0$,

(ii) $\lim_{x\to\infty}\varphi(x)=\infty$.

Such a function φ is known as a Young's function. The Young's function φ is said to satisfy the $\triangle_2-$condition if for some $M>0$, $\varphi(2.x)\leq M.\varphi(x)$, $\forall x>0$.

For more details on Orlicz-Lorentz spaces one can refer $[8,11,12,14,16]$ and the references there in.

In this paper, we take $X=\mathbb{N}$, the set of natural numbers, $S=P(\mathbb{N})$ and μ is counting measure defined on $P(\mathbb{N})$, the family of all subsets of $\mathbb{N}$. A weight sequence $w=w(n)$ is a positive decreasing sequence such that $\lim_{n\to\infty}w(n)=0$ and $\lim_{n\to\infty}W(n)=\infty$, where $W(n)=\sum_{i=1}^{n}w(i)$ for every $n\in\mathbb{N}$ (see $[6],[9]$).

Let l^0 be the space of all sequences $a:\mathbb{N}\to\mathbb{R}$. We write for $n\in\mathbb{N}$, the distribution function μ_a of $a=\{a(n)\}_{i\geq1}$ can be written as

$$\mu_a(s)=\mu\left(\{n\in\mathbb{N}:\ |a(n)|>s\}\right),\ s\geq0.$$

The non-increasing rearrangemenet a^* of a is given as

$$a^*(t)=\inf\{s>0:\mu_a(s)\leq t\},\ t\geq0.$$

We can interpret the non-increasing rearrangement of a with $\mu_a(s) < \infty$, $s > 0$, as a sequence $\{a^*(n)\}$ if we define for $n-1 \leq t < n$,

$$a^*(n) = a^*(t) = \inf\{s > 0 : \mu_a(s) \leq n-1\}.$$

Then the sequence $a^* = \{a^*(i)\}$ is obtained by permutating $\{|a(n)|\}_{n\in S}$, where $S = \{n : a(n) \neq 0\}$, in the decreasing order with $a^*(n) = 0$ for $n > \mu(s)$ if $\mu(s) < \infty$.

The Orlicz-Lorentz sequence space $\ell_{\varphi,w}(\mathbb{N})$ (or $\ell_{\varphi,w}$) is defined as

$$\ell_{\varphi,w}(\mathbb{N}) = \left\{ a \in l^0 : \sum_{n=1}^{\infty} \varphi(\alpha.a^*(n))\, w(n) < \infty, \text{ for some } \alpha > 0 \right\}.$$

The space $\ell_{\varphi,w}$ equipped with the Luxemburg norm

$$\|a\|_{\varphi,w} = \inf\left\{ \varepsilon > 0 : \sum_{n=1}^{\infty} \varphi\left(\frac{a^*(n)}{\varepsilon}\right) w(n) \ \leq 1 \right\}$$

is a Banach space. In [8], [9], a description of the duals, isomorphic ℓ^p-subspaces of Orlicz-Lorentz sequence spaces is given and in [6] geometric properties of Orlicz-Lorentz sequence spaces are discussed.

If $\varphi(u) = u^p,\ 1 \leq p < \infty$, then $d(w,p) := \ell_{\varphi,w}$ is a Lorentz sequence space. If $w(n) = 1$ for every $n \in \mathbb{N}$, then $\ell_\varphi := \ell_{\varphi,w}$ is an Orlicz sequence spaces (see [12], [13]).

Singh and Komal [18] initiated the study of composition operators on sequence space. Recently, Komal and Gupta [10], stuied multiplication operators on Orlicz spaces and Arora, Datt and Verma [2], [3] studied multiplication and composition operators on Orlicz-Lorentz space. Multiplication operators are studied in various function and sequence spaces [1, 2, 3, 4, 5, 10, 15, 17].

Let $u = \{u(n)\}$ be a complex sequence. We define a linear transformation M_u on the Orlicz-Lorentz sequence spaces $\ell_{\varphi,w}$, into the linear spaces of au complex sequences by

$$M_u(a) = u.a = \{u(n).a(n)\},$$

where $a = \{a(n)\}$. If M_u is bounded with range in $\ell_{\varphi,w}$, then it is called a multiplication operator on $\ell_{\varphi,w}$. By $B(\ell_{\varphi,w})$ we mean the algebra of all bounded linear operators on $\ell_{\varphi,w}$.

2. Characterizations

In this section boundedness, invertibility, range and compactness of the multiplication operator M_u on the space $\ell_{\varphi,w}$, induced by a sequence $u = \{u(n)\}$ are characterized.

Theorem 1. *The multiplication transformation $M_u : \ell_{\varphi,w} \to \ell_{\varphi,w}$ is bounded if and only if u is bounded.*

Proof. Assume that u is bounded, then $|u(n)| \leq K$ for all $n \in \mathbb{N}$ and some $K > 0$. Hence for any $a = \{a(n)\}$ in $\ell_{\varphi,w}$, $ua = \{u(n).a(n)\}$ satisfies $|u(n).a(n)| \leq K|a(n)|$. For the nonnegative rearrangement of $M_u f$, one can find the distribution function of $M_u a = u.a$ as

$$\begin{aligned} \mu_{M_u a}(s) &= \mu\{n \in \mathbb{N} : (M_u(a))(n) > s\} \\ &= \mu\{n \in \mathbb{N} : |u(n).a(n)| > s\} \\ &\leq \mu\{n \in \mathbb{N} : K.|a(n)| > s\} \end{aligned}$$

$$= \mu_{K.a}(s). \tag{2.1}$$

Hence for each $t \geq 0$, by (2.1) we get

$$\{s > 0 : \mu_{K.a}(s) \leq t\} \subseteq \{s > 0 : \mu_{M_u a}(s) \leq t\}$$

and we find $(M_u(a))^*(n) \leq K.a^*(n)$ for each $n \in \mathbb{N}$, and so we obtain

$$\sum_{n=1}^{\infty} \varphi\left(\frac{(M_u(a))^*(n)}{K.\|a\|_{\varphi,w}}\right).w(n) \leq \sum_{n=1}^{\infty} \varphi\left(\frac{K.a^*(n)}{K.\|a\|_{\varphi,w}}\right) \leq 1.$$

Hence for $a \in \ell_{\varphi,w}$,

$$\|M_u a\|_{\varphi,w} \leq K.\|a\|_{\varphi,w}.$$

Thus M_u is bounded on $\ell_{\varphi,w}$.

Conversely, suppose M_u is a bounded operators. Then there exists $K > 0$ such that

$$\|M_u(a)\|_{\varphi,w} \leq K.\|a\|_{\varphi,w}$$

for all $a \in \ell_{\varphi,w}$. If the sequence $u = \{u(n)\}$ is not bounded then for every positive integer k, there exists n_k such that $|u(n_k)| > k$. It is not hard to see $\chi_{\{n_k\}} \in \ell_{\varphi,w}$ satisfies

$$\left\|\chi_{\{n_k\}}\right\|_{\varphi,w} = \frac{1}{\varphi^{-1}\left(\frac{1}{c}\right)},$$

where $c = \sum_{m=1}^{\mu(\{n_k\})} w(m)$. Also we have

$$\left(u.\chi_{\{n_k\}}\right)^*(m) \geq k.\chi^*_{\{n_k\}}(m).$$

This give us

$$\begin{aligned}\left\|M_u.\chi_{\{n_k\}}\right\|_{\varphi,w} &\geq \inf\left\{\varepsilon > 0 : \sum_{m=1}^{\infty} \varphi\left(\frac{k.\chi^*_{\{n_k\}}(m)}{\varepsilon}\right).w(m) \leq 1\right\} \\ &= k.\left\|\chi_{\{n_k\}}\right\|_{\varphi,w}.\end{aligned}$$

This contradicts the boundedness of M_u. Hence u must be a bounded sequence.

Theorem 2. *Let $M_u \in B\left(\ell_{\varphi,w}\right)$. Then M_u is invertible if and only if there is $\delta > 0$ such that $|u(n)| \geq \delta$ for all $n \in \mathbb{N}$.*

Proof. If M_u is invertible then we find $\delta > 0$ such that

$$\|M_u a\|_{\varphi,w} \geq \delta. \|a\|_{\varphi,w}$$

for all $a \in \ell_{\varphi,w}$. In particular, for $e_n = \{e_n(m)\}$ this gives $|u(n)| \geq \delta$.

Conversely, if $|u(n)| \geq \delta$ for all $n \in \mathbb{N}$ and some $\delta > 0$, then define another sequence $v(n) = \frac{1}{u(n)}$. Clearly in view of Theorem 2.1, M_v is bounded on $\ell_{\varphi,w}$ and $M_v = M_u^{-1}$.

Theorem 3. *Let $M_u \in B\left(\ell_{\varphi,w}\right)$. Then M_u has closed range if and only if for some $\delta > 0$,*

$$|u(n)| \geq \delta \qquad \text{for all } n \in S, \tag{2.2}$$

where $S = \{n \in \mathbb{N} : \ u(n) \neq 0\}$.

Proof. If $|u(n)| \geq \delta$ for all $n \in S$, then for $x \in \ell_{\varphi,w}(\mathbb{N})$, where $x = \{x(k)\}_{k\geq 1}$,

$$(u.x.\chi_S)^*(k) \geq \delta. (x.\chi_S)^*(k)$$

and so we get

$$\|M_u.x.\chi_S\|_{\varphi,w} \geq \delta. \|x.\chi_S\|_{\varphi,w}. \tag{2.3}$$

Let $x \in cl\,(ranM_u)$. Then there exists a sequence $\{x_n\} \subset \ell_{\varphi,w}(\mathbb{N})$, where $x_n = \{x_n(k)\}$ such that $M_u x_n \to x$ as $n \to \infty$. Then we have

$$\|M_u x_n - M_u x_m\|_{\varphi,w} \to 0$$

as $n,m \to \infty$.

Let define $(x_n \chi_S)$ such that $x_n \chi_S = \begin{cases} x_n(k) & \text{, if } k \in S \\ 0 & \text{,otherwise} \end{cases}$. Then $M_u x_n \chi_S = M_u x_n$ and therefore it follows (2.2), that

$$\|x_n \chi_S - x_m \chi_S\|_{\varphi,w} \leq \frac{1}{\delta} \|M_u x_n - M_u x_m\|_{\varphi,w} \to 0$$

as $n,m \to \infty$. Thus $\{x_n \chi_S\}$ is a Cauchy sequence in $\ell_{\varphi,w}(\mathbb{N})$ and in view of completness of $\ell_{\varphi,w}(\mathbb{N})$, there exists $y \in \ell_{\varphi,w}(\mathbb{N})$ such that $x_n \chi_S \to y$ as $n \to \infty$. In others words $M_u x_n \to M_u y$. Hence $x = M_u y$ so that M_u has a closed range. Suppose that M_u has closed range. Therefore there exists a $\delta > 0$ such that $\|M_u a\|_{\varphi,w} \geq \delta . \|a\|_{\varphi,w}$ for all $a \in \ell_{\varphi,w}(S)$, where

$$\ell_{\varphi,w}(S) = \left\{a = a(n) \in \ell_{\varphi,w}(\mathbb{N}) : \ a(n) = 0 \text{ for } n \in \mathbb{N} - S\right\} = \left\{a \chi_S : \ a \in \ell_{\varphi,w}\right\}.$$

If the condition (2.2) does not hold, then for each $k \in \mathbb{N}$ we can find $n_k \in S$ such that $|u(n_k)| < \frac{1}{k}$. It is seen that $\chi_{\{n_k\}} \in \ell_{\varphi,w}(S)$ satisfies

$$\left\|\chi_{\{n_k\}}\right\|_{\varphi,w} = \frac{1}{\varphi^{-1}\left(\frac{1}{c}\right)},$$

where $c = \sum\limits_{m=1}^{\mu(\{n_k\})} w(m)$. Also we have

$$\left(u.\chi_{\{n_k\}}\right)^*(m) \leq \frac{1}{k}.\left(\chi_{\{n_k\}}\right)^*(m).$$

Hence

$$\begin{aligned}
\left\|M_u.\chi_{\{n_k\}}\right\|_{\varphi,w} &= \inf\left\{\varepsilon > 0 : \sum_{m=1}^{\infty} \varphi\left(\frac{\left(u.\chi_{\{n_k\}}\right)^*(m)}{\varepsilon}\right).w(m) \leq 1\right\} \\
&< \inf\left\{\varepsilon > 0 : \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{k}\left(\chi_{\{n_k\}}\right)^*(m)}{\varepsilon}\right).w(m) \leq 1\right\} \\
&= \frac{1}{k}.\left\|\chi_{\{n_k\}}\right\|_{\varphi,w},
\end{aligned}$$

which is a contradiction. This completes the proof.

Theorem 4. *$M_u \in B\left(\ell_{\varphi,w}\right)$ is compact if and only if $\ell_{\varphi,w}(U_\delta)$ is finite dimensional for each $\delta > 0$, where $\ell_{\varphi,w}(U_\delta) = \left\{a.\chi_{U_\delta} : \ a \in \ell_{\varphi,w}\right\}$ and $U_\delta = \{n \in \mathbb{N} : \ |u(n)| \geq \delta\}$.*

Proof. Assume that $M_u \in B\left(\ell_{\varphi,w}\right)$ is compact. Then $M_u\Big|_{\ell_{\varphi,w}(U_\delta)}$ is also a compact operator and

$$\left\|M_u.\chi_{U_\delta}.a\right\|_{\varphi,w} \leq \delta \left\|\chi_{U_\delta}.a\right\|_{\varphi,w}$$

for each $a \in \ell_{\varphi,w}$. Since $M_u\Big|_{\ell_{\varphi,w}(U_\delta)}$ is a compact and invertible, we get that $\ell_{\varphi,w}(U_\delta)$ is finite dimensional for each $\delta > 0$.

Conversely, suppose that $\ell_{\varphi,w}(U_\delta)$ is finite dimensional for each $\delta > 0$. For each $n \in \mathbb{N}$, if we define $u_n = \{u_n(m)\}$ such that

$$u_n(m) = \begin{cases} u(m) & \text{, if } m \in U_{\frac{1}{n}} \\ 0 & \text{,otherwise} \end{cases}$$

, then it is easy to see that all operators M_{u_n} are compact. Also, for each $a \in \ell_{\varphi,w}$ and for all $s \geq 0$, we have

$$\{m \in \mathbb{N} : |(u_n - u)(m).a(m)| \geq s\} \subset \{m \in \mathbb{N} : |a(m)| \geq s\}$$

and so

$$((u_n - u).a)^*(m) \leq \frac{1}{n}.a^*(m).$$

Therefore

$$\|(M_{u_n} - M_u).a\|_{\varphi,w} \leq \frac{1}{n}.\|a\|_{\varphi,w}$$

and so M_u is compact operator.

Conflict of Interests

The authors declare that there is no conflict of interests.

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