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CHEBYSHEVIAN BASIS FUNCTION-TYPE BLOCK METHOD FOR THE SOLUTION OF FIRST-ORDER INITIAL VALUE PROBLEMS WITH OSCILLATING SOLUTIONS

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Abstract. In this paper, we develop a block method using Chebyshev polynomial basis function and use it to produce discrete methods which are simultaneously applied as numerical integrators by assembling them into a block method. The paper further investigates the properties of the block method and found it to be zero-stable, consistent and convergent. We also tested the efficiency of the method on some sampled oscillatory problems and found out that the method performed better than some existing ones with which we compared our results.

Keywords: Basis function; Block method; Chebyshev polynomial; Initial value problem, Oscillating.

2010 AMS Subject Classification: 65L05, 65D30.

1. Introduction

In this paper, we consider the approximate solution of first order Initial Value Problems (IVP-s) with oscillating solutions of the form,

(1) $$y' = f(x,y), \;\; y(x_0) = y_0, \; x \in [a,b],$$

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where $f:\Re\times\Re^m\to\Re^m,\quad y,y_0\in\Re^m$, f satisfies Lipchitz condition which guarantees the existence and uniqueness of solution of (1). Some of the set of challenging problems being encountered nowadays are the oscillatory differential equations. They are differential equations whose solutions are composed of smooth varying and of a 'nearly periodic' functions, i.e. they are oscillations whose wave form and period varies slowly with time (relative to the period), and where the solution is sought over a very large number of cycles (Stetter, 1994). Oscillatory problems have some of their Eigen values near the imaginary axis, and their solutions are oscillation processes with slowly varying amplitudes. The difficulty of solving such problems is explained by the necessity to ensure correct values of the amplitude and phase angle over many periods. To efficiently solve oscillatory problems, implicit methods are suitable (Skvortsov, 2011).

Oscillatory IVPs frequently arise in areas such as classical mechanics, celestial mechanics, quantum mechanics and biological sciences (Ngwane *et al.*, 2014). Several numerical methods based on power series polynomial basis functions have been developed for solving this class of important problems. See James *et al.* $(2013a, 2013b)$, Yakubu *et al.* 2013, Adesanya *et al.* 2014, among others. Other methods based on exponential fitting techniques which take advantage of the special properties of the solution that may be known in advance have been proposed (Fang *et al.* (2009), Jator *et al.* (2013), Ngwane *et al.* (2013)). The motivation governing the exponentially-fitted methods is inherent to the fact that if the frequency or a reasonable estimate of it is known in advance, these methods will be more advantageous than the polynomial based methods (Ngwane *et al.*, 2014). Recently, other authors developed a class of numerical inetgrators using a basis function that comprises of the combination of power series and exponential functions. These block integrators performed reasonably well on both oscillatory and stiff problems of the form (1). These authors include Sunday *et al.* $(2013a, 2013b)$, Sunday *et al.* $(2014a, 2014b)$, among others.

Definition 1.1 (Chebyshev Polynomial) The Chebyshev polynomial of the first kind $T_n(x)$ is a polynomial of degree n defined for $x\in[-1,1]$ by

$$T_n(x)=\cos(\arccos x),\ \ n=0,1,2,..., \tag{2}$$

where $-1 \le T_n \le 1$. By setting $x = \cos z$, we have

$$T_n = \cos nz \tag{3}$$

from which it is easy to deduce the expressions for the recursive relation of Chebyshev polynomials as, $T_0 = 1$, $T_1 = \cos z = x$, $T_2 = \cos 2z = 2\cos^2 z - 1 = 2x^2 - 1, \ldots$, $T_{n+1} = 2xT_n - T_{n-1}$, $n \ge 1$. The T_n is a polynomial of degree n with leading coefficient 2^{n-1} for $n \ge 1$.

Definition 1.2 (Oscillatory Differential Equation) A differential equation (1) is oscillatory if all the nontrivial solution of (1) have an infinite number of zeros on $x_0 \le x < \infty$.

Theorem 1.3. (Roots of Chebyshev Polynomials) *The roots of $T_n(x)$ of degree $n \ge 1$ has n simple zeros in $[-1,1]$ at $\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$, for each $k = 1,2,\ldots,n$. Moreover, $T_n(x)$ assumes its absolute extrema at $\bar{x}'_k = \cos\left(\frac{2k}{n}\right)$, with $T_n\left(\bar{x}'_k\right) = (-1)^k$, for each $k = 0,1,\ldots,n$.*

The Chebyshev polynomial possess the following properties.

Firstly, $|T_n(x)| \le 1,\ x \in [-1,1]$. Secondly, $T_n(x)$ is a polynomial of degree n. If n is even, $T_n(x)$ is an even polynomial and if n is odd, $T_n(x)$ is an odd polynomial. Thirdly, $T_n(x)$ assumes extreme values at $n+1$ points $x_k = \cos(k\pi/n)$, $k = 0,1,\ldots,n$ and extreme value at x_k is $(-1)^k$. Fourthly, $T_n(x)$ is orthogonal with respect to the weight function $W(x) = \frac{1}{\sqrt{1-x^2}}$ and

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx = \begin{pmatrix} 0,\ m \ne n \\ \frac{\pi}{2},\ m = n \ne 0 \\ \pi,\ m = n = 0 \end{pmatrix}.$$

In this paper, we derive a block method using the Chebyshev polynomial as our basis function. This method has the advantages of permitting easy change of step-size, does not require a starting value, it simultaneously generates more than one solution at a time, easy to program and less expensive in terms of the number of function evaluation per step.

2. Methodology: derivation of the block method

We consider the first five terms of the basis function (3) as our approximate solution. This is given by

$$y(x) = T_0(x) + T_1(x) + \sum_{n=1}^{3} \left(2xT_n(x) - T_{n-1}(x)\right). \tag{4}$$

We seek the solution of (1) on the partition $\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \ldots x_N = b$, of the integration interval $[a,b]$ with a constant step-size h , given by, $h = x_{n+1} - x_n, n = 0,1,2,\ldots,N$. Interpolating (4) at point $x_{n+s}, s = 0$ and collocating its first derivative at points $x_{n+r}, r = 0(1)3$ (where s and r are the number of interpolation and collocation points respectively), leads to the following system of equations,

(5) $$XA = U,$$

where

$$A = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4]^T,$$

$$U = [y_n \quad f_n \quad f_{n+1} \quad f_{n+2} \quad f_{n+3}]^T,$$

and

$$X = \begin{bmatrix} 1 & -2x_n & -6x_n^2 & 4x_n^3 & 8x_n^4 \\ 0 & -2 & -12x_n & 12x_n^2 & 32x_n^3 \\ 0 & -2 & -12x_{n+1} & 12x_{n+1}^2 & 32x_{n+1}^3 \\ 0 & -2 & -12x_{n+2} & 12x_{n+2}^2 & 32x_{n+2}^3 \\ 0 & -2 & -12x_{n+3} & 12x_{n+3}^2 & 32x_{n+3}^3 \end{bmatrix}.$$

Solving (5), for $a_j's, j = 0(1)4$ and substituting back into (4) gives a continuous linear multistep method of the form,

(6) $$y(x) = \alpha_0(x) y_n + h \sum_{j=0}^{3} \beta_j(x) f_{n+j},$$

where

(7) $$\begin{bmatrix} \alpha_0 = 1 \\ \beta_0 = -\frac{1}{24}(t^4 - 8t^3 + 22t^2 - 24t) \\ \beta_1 = \frac{1}{24}(3t^4 - 20t^3 + 36t^2) \\ \beta_2 = -\frac{1}{24}(3t^4 - 16t^3 + 18t^2) \\ \beta_3 = \frac{1}{24}(t^4 - 4t^3 + 4t^2) \end{bmatrix},$$

where $t = \frac{x - x_n}{h}$. Evaluating (6) at $t = 1(1)3$ gives a discrete block method of the form,

(8) $$A^{(0)} \mathbf{Y}_m = E \mathbf{y}_n + h d \mathbf{f}(\mathbf{y}_n) + h b \mathbf{F}(\mathbf{Y}_m),$$

where

$$\mathbf{Y}_m = [y_{n+1} \quad y_{n+2} \quad y_{n+3}]^T \ , \quad \mathbf{y}_n = [y_{n-2} \quad y_{n-1} \quad y_n]^T$$

$$\mathbf{F}(\mathbf{Y}_m) = [f_{n+1} \quad f_{n+2} \quad f_{n+3}]^T \quad , \ \mathbf{f}(\mathbf{y}_n) = [f_{n-2} \quad f_{n-1} \quad f_n]^T$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ , \ E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$d = \begin{bmatrix} 0 & 0 & \frac{9}{24} \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{9}{24} \end{bmatrix} \ , \ b = \begin{bmatrix} \frac{19}{24} & \frac{-5}{24} & \frac{1}{24} \\ \frac{4}{3} & \frac{1}{3} & 0 \\ \frac{27}{24} & \frac{27}{24} & \frac{9}{24} \end{bmatrix}$$

3. Analysis of basic properties of the block method

3.1. Order of the block method

Let the linear operator $L\{y(x);h\}$ associated with the block method (8) be defined as

$$L\{y(x);h\} = A^{(0)}\mathbf{Y}_m - E\mathbf{y}_n - h(d\mathbf{f}(\mathbf{y}_n) + b\mathbf{F}(\mathbf{Y}_m)) \tag{9}$$

expanding using Taylor series and comparing the coefficients of h gives,

$$L\{y(x);h\} = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + ... + c_p h^p y^p(x) + c_{p+1} h^{p+1} y^{p+1}(x) + ... \tag{10}$$

Definition 3.1. The linear operator L and the associated continuous linear multistep method (6) are said to be of order p if $c_0 = c_1 = c_2 = ... = c_p = 0$ and $c_{p+1} \neq 0$. The order is also defined as the largest positive real number that quantifies the rate of convergence of a numerical approximation of a differential equation to that of the exact solution. c_{p+1} is called the error constant (i.e. the accumulated error when the order of a method has been computed) and the local truncation error is given by

$$t_{n+k} = c_{p+1} h^{(p+1)} y^{(p+1)}(x_n) + \bigcirc(h^{p+2}). \tag{11}$$

For our method,

$$
(12)\qquad L\{y(x);h\} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} \\ -h \begin{bmatrix} \frac{9}{24} & \frac{19}{24} & -\frac{5}{24} & \frac{1}{24} \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0 \\ \frac{9}{24} & \frac{27}{24} & \frac{27}{24} & \frac{9}{24} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \end{bmatrix}.
$$

Expanding (12) in Taylor series gives,

$$
(13)\qquad \begin{bmatrix} \sum_{j-0}^{\infty} \frac{(h)^j}{j!} y_n^j - y_n - \frac{9h}{24} y_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \frac{19}{24}(1)^j - \frac{5}{24}(2)^j + \frac{1}{24}(3)^j \right\} \\ \sum_{j-0}^{\infty} \frac{(2h)^j}{j!} y_n^j - y_n - \frac{h}{3} y_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \frac{4}{3}(1)^j + \frac{1}{3}(2)^j + 0 \right\} \\ \sum_{j-0}^{\infty} \frac{(3h)^j}{j!} y_n^j - y_n - \frac{3h}{8} y_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \frac{9}{8}(1)^j + \frac{9}{8}(2)^j + \frac{3}{8}(3)^j \right\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

Hence, $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = 0,\ \bar{c}_5 = [-2.64(-02) \quad -1.11(-02) \quad 3.75(-02)]^T$. Therefore, the block method is of accurate fourth order.

3.2. Consistency

The block method (8) is consistent since it has order $p = 4 \geq 1$. It is important to note that consistency controls the magnitude of the local truncation error committed at each stage of the computation (Fatunla, 1988).

3.3. Zero stability

Definition 3.3.1. The block method (8) is said to be zero-stable, if the roots $z_s, s = 1, 2, ..., k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = \det(zA^{(0)} - E)$ satisfies $|z_s| \leq 1$ and every root satisfying $|z_s| = 1$ have multiplicity not exceeding the order of the differential equation. Moreover, as $h \to 0, \rho(z) = z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ and E (see Awoyemi *et al.* (2007) for details). The main consequence of zero-stability is to control the propagation of the error as the integration proceeds (Fatunla, 1988).

For our method,

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0 \tag{14}$$

$\rho(z) = z^2(z-1) = 0 \Longrightarrow z_1 = z_2 = 0, z_3 = 1$. Thus, the block method is zero-stable.

3.4. Convergence

The block method is convergent by consequence of Dahlquist theorem stated below.

Theorem 3.4.1 (Dahlquist, 1956) *The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.*

3.5. Region of absolute stability

Definition 3.5.1 (Yan, 2011) Region of absolute stability is a region in the complex z plane, where z=λh . It is defined as those values of $z$such that the numerical solutions of $y' = -\lambda y$ satisfy $y_j \to 0$ as $j \to \infty$ for any initial condition.

To determine the absolute stability region of the block method, we adopt the boundary locus method. This is achieved by substituting the test equation,

$$y' = -\lambda y \tag{15}$$

into the block formula (8). This gives

$$A^{(0)}\mathbf{Y}_m(w) = E\mathbf{y}_n(w) - h\lambda D\mathbf{y}_n(w) - h\lambda B\mathbf{Y}_m(w). \tag{16}$$

Thus,

$$\bar{h}(w) = -\left(\frac{A^{(0)}\mathbf{Y}_m(w) - E\mathbf{y}_n(w)}{D\mathbf{y}_n(w) + B\mathbf{Y}_m(w)} \right). \tag{17}$$

Since $\bar{h}$ is given by $\bar{h} = \lambda h$ and $w = e^{i\theta}$. Equation (17) is our characteristic or stability polynomial. For our method, equation (17) is given by

$$\bar{h}(w) = -h^3\left(\frac{1}{4}w^3 + \frac{1}{4}w^2\right) - h^2\left(\frac{11}{12}w^2 - \frac{11}{12}w^3\right) - h\left(\frac{3}{2}w^3 + \frac{3}{2}w^2\right) + w^3 - w^2. \tag{18}$$

This gives the stability region of the block method shown in fig.1 below.

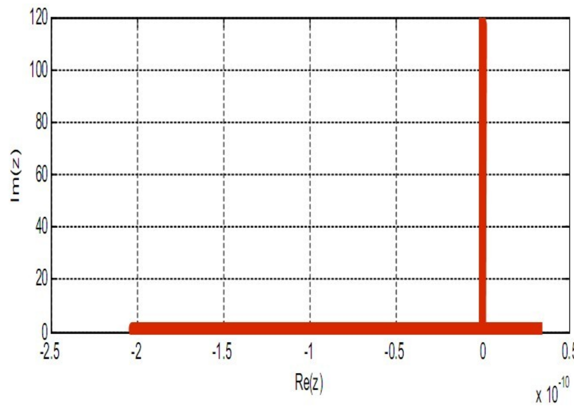

Figure 1

Lambert (1973) showed that the stability region for L-stable methods must encroach into the positive half of the complex plane. Thus, the block method developed is said to be L-stable by virtue of fig. 1.

4. Numerical experiments

We shall apply the block method developed on two sampled oscillatory ODEs and compare our results with those of some existing methods. We shall use the following notations in the tables below;

ERR- Exact Solution-Computed Solution

EAU- Error in Adeboye and Umar (2013)

ESJ- Error in Sunday *et al.* (2013)

Problem 4.1. Consider the oscillatory ODE

$$y' = -\sin x - 200(y - \cos x),\ y(0) = 0 \tag{19}$$

with the exact solution

$$y(x) = \cos x - e^{-200x}. \tag{20}$$

Problem 4.2. Consider the Prothero-Robinson oscillatory ODE

$$y' = Ly + \cos x - L\sin x,\ L = -1,\ \ y(0) = 0, \tag{21}$$

which has the exact solution

$$y(x) = \sin x \tag{22}$$

Table 1: Showing the result for oscillatory problem 1.

x	Exact Solution	Computed Solution	ERR	ESJ	Time/s
0.0010	0.1812687469220599	0.1812687472945497	$3.724898e-010$	$6.581226e-006$	0.0163
0.0020	0.3296779539650273	0.3296779544867199	$5.216926e-010$	$2.937887e-006$	0.0178
0.0030	0.4511838639093485	0.4511838645880447	$6.786962e-010$	$9.396094e-006$	0.0195
0.0040	0.5506630358934450	0.5506630366535412	$7.600962e-010$	$1.130466e-005$	0.0213
0.0050	0.6321080588545993	0.6321080595958538	$7.412545e-010$	$7.910709e-006$	0.0229
0.0060	0.6987877881417979	0.6987877888867506	$7.449528e-010$	$1.031328e-005$	0.0247
0.0070	0.7533785361584351	0.7533785368805425	$7.221074e-010$	$1.042596e-005$	0.0265
0.0080	0.7980714821760109	0.7980714828324983	$6.564874e-010$	$7.798045e-006$	0.0281
0.0090	0.8346606120517877	0.8346606126650458	$6.132581e-010$	$8.490002e-006$	0.0297
0.0100	0.8646147171800526	0.8646147177437180	$5.636654e-010$	$8.038839e-006$	0.0315

Table 2: Showing the result for oscillatory problem 2

x	Exact Solution	Computed Solution	ERR	EAU	Time/s
0.1000	0.0998334166468281	0.0998334166602505	$1.342236e-011$	$2.0e-11$	0.1618
0.2000	0.1986693307950612	0.1986693308165252	$2.146397e-011$	$3.0e-11$	0.1633
0.3000	0.2955202066613396	0.2955202066936986	$3.235895e-011$	$1.0e-10$	0.1647
0.4000	0.3894183423086506	0.3894183423505273	$4.187661e-011$	$2.0e-10$	0.1664
0.5000	0.4794255386042032	0.4794255386505803	$4.637712e-011$	$1.0e-10$	0.1679
0.6000	0.5646424733950356	0.5646424734484032	$5.336764e-011$	$2.0e-10$	0.1693
0.7000	0.6442176872376914	0.6442176872966270	$5.893563e-011$	$1.0e-10$	0.1713
0.8000	0.7173560908995231	0.7173560909597445	$6.022138e-011$	$2.0e-10$	0.1728
0.9000	0.7833269096274838	0.7833269096908259	$6.334211e-011$	$3.0e-10$	0.1743
1.0000	0.8414709848078968	0.8414709848729562	$6.505940e-011$	$3.0e-10$	0.1760

5. Conclusion

A block method for the solution of first-order ODEs with oscillating solutions have been developed using the Chebyshev polynomial as a basis function.The block method developed was found to be zero-stable, consistent and convergent. The method was also found to perform better than some existing ones with which we compared our results with.

Conflict of Interests

The authors declare that there is no conflict of interests.

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