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# MINIMIZING A FUNCTION USING POSYNOMIAL APPROXIMATION

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**Abstract.** This paper discusses the use of Bernstein polynomial approximation to minimize a function of single variable on an interval. Bernstein operator is used to prove that a single dimensional minimization problem is equivalent to a Posynomial Programming problem. Original problem is solved using the dual of this approximate problem.

**Keywords:** Bernstein Polynomial, Posynomial Programming, Primal-Dual relation.

**2000 AMS Subject Classification:** 90C30, 90C56, 90C25.

## 1. Introduction

There are several reasons for discussing one dimensional optimization, since some of the theoretical and numerical aspects of unconstrained optimization in $R^n$ can be conveniently illustrated in one dimension and some of the iterative methods for $n-$dimensional problems include steps in which extreme points are sought along certain directions in $R^n$, and these steps are equivalent to one dimensional optimization. Most of the existing search methods in the literature of numerical optimization to find the solution of an unconstrained optimization problem are mostly divided in two categories; gradient based methods and gradient free methods. But in practical situations many optimization problems exist, which are not differentiable. So gradient based search methods fail to solve these type problems. The important

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references for single dimensional gradient free search methods include Golden Section method due to Wilde[7], Fibonacci search method due to Kiefer [5], Newton's method, Steepest Descent method etc. The third category is direct method, which approximates the function whose minimum is sought by quadratic or cubic degree polynomials. In this paper we develop a solution method for $\min_{x\in[a,b]} f(x),\ f \in \mathcal{C}[a,b]$, which belongs to third category. We convert the original problem to an approximate problem using a linear transformation known as Bernstein operator. Our approximating polynomial is not necessarily of second or third degree. Rather, we get more approximating solution in case degree of the polynomial is large. In addition to this, we will solve the dual of the approximate problem in stead of solving the approximate problem directly as in direct method. The Bernstein operator $B_n$ is a linear operator from $\mathcal{C}[0,1]$ to the subspace $P_n$ of polynomials of degree $n$, and it is defined for all positive integral values of $n$. The Bernstein polynomial approximation of degree $n$ for $f \in \mathcal{C}[0,1]$, which is named after its creator, S.N.Bernstein [2] in 1912, is specified by the expression

$$(B_n f)(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} f(k/n),\ 0 \leq x \leq 1.$$

The sequence of Bernstein polynomials has following important properties. (For detail discussions one may refer[6]).

**Property 1.1** *Given a function $f \in \mathcal{C}[a,b]$ and any $\epsilon > 0$, there exists an integer $N$ such that $\mid f(x) - (B_n f)(x) \mid < \epsilon$, $0 \leq x \leq 1$, for all $n \geq N$ and $\parallel f - B_n f \parallel_{max} \leq \frac{3}{2}\omega(\frac{1}{\sqrt{n}})$, where $\omega(\delta) = sup_{|x_1 - x_2| \leq \delta} \mid f(x_1) - f(x_2) \mid$, $\delta > 0, x_1, x_2 \in [a,b]$.*

**Property 1.2** *Bernstein polynomial of single variable has shape preserving property. For example, the Bernstein polynomial of a convex function in $R$ is itself convex.*

**Property 1.3** *If $f$ is convex on $[0,1]$, then $(B_n f)(x) \geq f(x)$ for all $n \geq 1$.*

**Property 1.4** *Each term of the Bernstein polynomial is positive if $f$ is a positive function, which is not true for Lagrange polynomial, Legendre polynomial etc.*

**Property 1.5** *Bernstein polynomial just depends upon value of $f$ at $n+1$ discrete points, does not use differentiability assumption like Taylor polynomial which is useful only for infinitely differentiable function.*

These interesting behavior of Bernstein operator motivated us to develop a new method to solve a minimization problem of a function of single variable. First, using Bernstein Polynomial approximation, we transform the original minimization problem to an approximating problem, which may be treated as a Posynomial Programming problem in $R^2$ over a line segment and prove that solution of the transformed problem and the original problem are approximately same. Dual of the approximate problem has linear constraints. Solution of this system of linear equations becomes the approximating solution of the original problem. In fact there exist several type of approximating polynomials for a continuous function. But Bernstein polynomial has shape preserving property, which other polynomials do not have. This is the most important factor to prove that solution of the original problem and the approximating problem are lying in a neighborhood.

## 2. Methodology

**Theorem 2.1** *Let $\hat{x}$ be a local optimal solution of the optimization problem* $\min f(x)$, *where* $f \in \mathcal{C}[0,1]$ *and* $(B_n f)(x)$ *be the* $n^{th}$ *degree Bernstein polynomial approximation of* $f$. *Then there exists* $\hat{\hat{x}}$ *in the closure of neighborhood of* $\hat{x}$ *such that for* $\epsilon > 0$ *there exists a large number* $N$ *satisfying* $(B_n f)(\hat{\hat{x}}) - f(\hat{x}) < \epsilon \ \forall n \geq N$ *and* $\hat{\hat{x}}$ *is a local minimum solution of min*$(B_n f)(x)$. *Moreover if* $f$ *is a convex function then* $\hat{x}$ *and* $\hat{\hat{x}}$ *are global minimum solutions.*

**Proof.** Since $\hat{x}$ is the local minimum point of $f(x)$, so $f$ is convex in a neighborhood $N(\hat{x})$. By Property 1.2, $B_n f$ is also convex in this neighborhood. Hence $B_n f$ attains its minimum at some point say $\hat{\hat{x}}$ in the closure of this neighborhood.

Since $f$ is continuous in $[0,1]$, for $\epsilon_1 > 0$, there exists $\delta > 0$ such that $\mid f(\hat{\hat{x}}) - f(\hat{x}) \mid < \epsilon_1$, whenever $\parallel \hat{\hat{x}} - \hat{x} \parallel < \delta$.

From Property 1.1, for $\epsilon_2 > 0$, there exists an integer $N$ such that $\mid f(\hat{\hat{x}}) - (B_n f)(\hat{\hat{x}}) \mid < \epsilon_2$, for all $n \geq N$.

Hence

$$\mid (B_n f)(\hat{\hat{x}}) - f(\hat{x}) \mid \leq \mid (B_n f)(\hat{\hat{x}}) - f(\hat{\hat{x}}) \mid + \mid f(\hat{\hat{x}}) - f(\hat{x}) \mid < \epsilon_1 + \epsilon_2 = \epsilon \ (say),$$

for all $n \geq N$, whenever $\parallel \hat{\hat{x}} - \hat{x} \parallel < \delta$.

From Property 1.3, $(B_n f)(\hat{\hat{x}}) \geq f(\hat{\hat{x}})$ for all $n \geq 1$. Also $f(\hat{\hat{x}}) \geq f(\hat{x})$ since $\hat{x}$ is the local minimum of $f$. Hence $(B_n f)(\hat{\hat{x}}) - f(\hat{x}) \geq 0$. Proof of the theorem follows for $N(\hat{x}) = (\hat{x} - \delta', \hat{x} + \delta'),\ \delta' < \delta$.

Consider the optimization problem,

$$(P:) \ \min_{x \in [a,b]} f(x), \ f : [a,b] \to (0, \ \infty) \ is \ continuous.$$

Take $\frac{x-a}{b-a} = t$. Then $\min_{x \in [a,b]} f(x) = \min_{t \in [0,1]} g(t)$, where $g(t) = f(a + t(b-a))$. By Theorem 2.1, $\min_{t \in [0,1]} g(t) \approx \min_{t \in [0,1]} (B_n g)(t)$ for large $n$, where $B_n g$ is the $n^{th}$ degree Bernstein approximating polynomial of $g$. So the optimization problem $(P)$ is equivalent to approximating optimization problem $(BP)$,

$$(BP): \ \min_{u+v=1} g(u) = \sum_{k=0}^{n} C_k u^k v^{n-k} g(k/n), \ where \ C_k = \frac{n!}{k!(n-k)!}$$

In $(BP)$, $0 < u, v < 1$ should be considered since $g$ is a strictly positive function. Here $u = t$ and $v = 1-t$. By Property 1.4, coefficients in the objective function of $(BP)$ are greater than zero. Hence $(BP)$ can be treated as a Posynomial Programming problem ([3],[4]), which can be solved by converting to its dual $(DBP)$, as given below.

$$(DBP): \ G(\alpha) = \max \prod_{k=0}^{n} \left(\frac{C_k g(k/n) \sum_{k=0}^{n} \alpha_{0k}}{\alpha_{0k}}\right)^{\alpha_{0k}} \alpha_{11}^{-\alpha_{11}} \alpha_{12}^{-\alpha_{12}} \alpha_{21}^{\alpha_{21}} \alpha_{22}^{\alpha_{22}}$$

$$subject\ to\ \sum_{k=0}^{n} \alpha_{0k} = 1,$$

$$\sum_{k=0}^{n} k\alpha_{0k} + \alpha_{11} - \alpha_{21} = 0,$$

$$\sum_{k=0}^{n} (n-k)\alpha_{0k} + \alpha_{12} - \alpha_{22} = 0,$$

$$\alpha = (\alpha_{0k}, \alpha_{11}\alpha_{12}, \alpha_{21}\alpha_{22}) > 0.$$

If $(DBP)$ has zero degree of difficulty, then the constraints of $(DBP)$, which is a system of linear equations, will yield an unique solution $\hat{\alpha} = (\hat{\alpha}_{0k}, \hat{\alpha}_{11}\hat{\alpha}_{12}, \hat{\alpha_{21}}, \hat{\alpha_{22}})$ from which the original objective function can be found as

$$\hat{g} = g(\hat{u}) = G(\hat{\alpha}) = \prod_{k=0}^{n} \left(\frac{C_k g(k/n) \sum_{k=0}^{n} \hat{\alpha}_{0k}}{\hat{\alpha}_{0k}}\right)^{\hat{\alpha}_{0k}} \hat{\alpha}_{11}^{-\hat{\alpha}_{11}} \hat{\alpha}_{12}^{-\hat{\alpha}_{12}} \hat{\alpha}_{21}^{\hat{\alpha}_{21}} \hat{\alpha}_{22}^{\hat{\alpha}_{22}}$$

Since $g(t) > 0$, so $\hat{g} > 0$. Once $\hat{g}$ is known, solution of $(BP)$ $\hat{u}$ and $\hat{v}$ may be determined from the primal dual relation between $(BP)$ and $(BDP)$. But degree of difficulty of $(DBP)$ may not be always zero since the approximation is more accurate for large $n$. In case of positive degree of difficulty, we may follow the algorithm given by Alejandre, Allueva and Gonzalez[1].

## 3. Conclusion

We may observe that Theorem 2.1 holds for any approximating polynomial of $f \in \mathcal{C}[a, b]$, which has shape preserving property. But in that case, the approximate problem may not be expressed in terms of a posynomial. Since Bernstein Polynomial has binomial terms, so the approximate problem is a standard Posynomial Programming problem. Our result is true for the strictly positive functions.

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