Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 4, 898-906
ISSN: 1927-5307

ON THE SPHERICAL CURVES AND BERTRAND CURVES IN MINKOWSKI-3 SPACE

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Abstract. Bertrand curves corresponding to the spherical curves in E^3 are given in [1] . In this paper, we apply the method of constructing Bertrand curves from the spherical curves to the curves in 3 dimensional Minkowski space. We also investigate the Bertrand curves corresponding to the spherical indicatrices of spacelike and timelike curves.

Keywords: Spherical Curves, Bertrand Curves, Spherical Indicatrices, Minkowski Space

2000 AMS Subject Classification: 53A35; 53B30

1. Introduction

In [9] and [4], the authors have characterized the spherical spacelike and timelike curves. In this work, firstly we give the Theorem about generating the Bertrand curves from the spherical curves in the three dimensional Minkowski spacetime. This is a generalization of the work of Izumiya and Takeuchi [1].We come up with the idea of considering the Sabban frame upon the casual characters of a curve's position vectors. In the next section, we define the spherical indicatrices and their types upon the casual character of the space

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Received February 27, 2012

curve. Finally, we calculate the Bertrand curves corresponding to the indicatrices by using the theorem given in previous section.

2. Preliminaries

Let E^3 be 3 dimensional Euclidean space, then the *pseudo scalar product* or *Lorentzian inner product* is defined by $\langle a,b\rangle_L = a_1b_1 + a_2b_2 - a_3b_3$ where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are two vectors in E^3. $(E^3, \langle,\rangle_L)$ is called a 3 dimensional *pseudo Euclidean space* or *Minkowski* 3 *space*. An arbitrary vector $a = (a_1, a_2, a_3)$ in L^3 can have one of three Lorentzian causal characters; it is *spacelike* if $\langle a,a\rangle_L > 0$ or $a = 0$, *timelike* if $\langle a,a\rangle_L < 0$ and *null*(*lightlike*) if $\langle a,a\rangle_L = 0$ and $a \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in L^3 is locally *spacelike*, *timelike* or *null*(*lightlike*), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. Recall that the pseudo-norm of an arbitrary vector $a \in L^3$ is given by $|a| = \sqrt{|\langle a,a\rangle_L|}$ and the velocity υ of the curve is given by $\upsilon = \|\alpha'(s)\|$. Therefore, α is a unit speed curve if and only if $\langle\alpha'(s), \alpha'(s)\rangle_L = \pm 1$. For any $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in L^3$, the pseudo-vector product of a and b is defined as follows

$$a \times b = \begin{vmatrix} i & j & -k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Denote by $\{t(\sigma), n(\sigma), b(\sigma)\}$ the moving Frenet frame along the nonlightlike curve α parameterized by a pseudo-arclength parameter σ. Let $t(\sigma) = \alpha'(\sigma)$, $n(\sigma) = \dfrac{\alpha''(\sigma)}{|\alpha''(\sigma)|}$ and $b(\sigma) = t(\sigma) \times n(\sigma)$ be the tangent, the principal normal and the binormal vector of the curve α respectively. When α is a timelike curve, i.e. when t is a timelike vector, the Frenet formulae read;

$$t' = \kappa n, n' = \kappa t + \tau b, b' = -\tau n$$

When α is a spacelike curve, there are three possibilities depending on the causal character of t'. If t' is spacelike, then the Frenet formulae read;

$$t' = \kappa n, n' = -\kappa t + \tau b, b' = \tau n$$

If t' is timelike, then the Frenet formulae read;

$$t' = \kappa n, n' = \kappa t + \tau b, b' = \tau n$$

If t' is lightlike, then the Frenet formulae read;

$$t' = n, n' = \tau n, b' = -t - \tau b$$

The functions κ and τ are called the *curvature* and the *torsion* of α respectively [7]. When the curve α is not parametrized by the arc-length, the corresponding formulae of the curvature and the torsion are;

$$\begin{aligned} \kappa^2(t) &= \delta \frac{|\alpha'(t)|^2 |\alpha'(t)|^2 - \langle \alpha'(t), \alpha'(t) \rangle^2}{|\alpha'(t)|^6} \\ \tau(t) &= -\delta \frac{\det(\alpha'(t), \alpha'(t), \alpha'(t))}{\kappa^2(t) |\alpha'(t)|^6} \end{aligned}$$

where δ is 1 or -1 depending on $\alpha'(t)$ is a spacelike or timelike vector, respectively [6]. For the timelike curves see [7].

A hyperbola and a pseudo sphere are defined by respectively;

$$\begin{aligned} H_1^2(r)_m &= \left\{ a = (a_1, a_2, a_3) \in \mathbb{R}^3 \mid \langle a - m, a - m \rangle_L = -r^2 \right\} \\ S_1^2(r)_m &= \left\{ a = (a_1, a_2, a_3) \in \mathbb{R}^3 \mid \langle a - m, a - m \rangle_L = r^2 \right\} \end{aligned}$$

where $m = (m_1, m_2, m_3)$ and $r \in \mathbb{R}^+$. $H_1^2(r) := H_1^2(r)_m - \{m\}$,$S_1^2(r) := S_1^2(r)_m - \{m\}$ are respectively *a hyperbola* and *a pseudo sphere* with radius r at the center m. Denote that $S_0^2 = S_1^2(1)_0$ and $H_0^2 = H_1^2(1)_0$ [3].

3. Spherical Curves and Bertrand Curves in $\mathbf{E}_1^3$

In [9] and [4], the autors have characterized the spherical spacelike and timelike curves. In this section, we give the Theorem about generating the Bertrand curves from the spherical curves in E_1^3.

Theorem 3.1. *There are no null curves lying on the Lorentzian sphere in E_1^3 [4].*

Let γ be a unit speed nonlightlike curve. If σ is the arc lenght parameter of γ, then the unit tangent vector of γ is $t(\sigma) = \dot{\gamma}(\sigma)$ where $\dot{\gamma} = \dfrac{d\gamma}{d\sigma}$. We define a vector $s(\sigma) = \gamma(\sigma) \times t(\sigma)$, then we have a pseudo-orthonormal frame $\{\gamma(\sigma), t(\sigma), s(\sigma)\}$ along γ . If γ is a spacelike curve, then the vector s is timelike where γ is in S_1^2 and the vector s is spacelike where γ is in H_1^2. Similarly, if the curve γ is timelike, then the vector s is spacelike.

Theorem 3.2. *Let γ be a unit speed spherical curve in E_1^3, then the spherical Frenet formulas of γ are*

i. *If γ is a spacelike curve in H_1^2*

$$\begin{aligned}\dot{\gamma}(\sigma) &= t(\sigma)\\ \dot{t}(\sigma) &= \gamma(\sigma) + \kappa_g(\sigma)\, s(\sigma)\\ \dot{s}(\sigma) &= -\kappa_g(\sigma)\, t(\sigma)\end{aligned}$$

ii. *If γ is a spacelike curve in S_1^2*

$$\begin{aligned}\dot{\gamma}(\sigma) &= t(\sigma)\\ \dot{t}(\sigma) &= -\gamma(\sigma) - \kappa_g(\sigma)\, s(\sigma)\\ \dot{s}(\sigma) &= -\kappa_g(\sigma)\, t(\sigma)\end{aligned}$$

iii. *If γ is a timelike curve*

$$\begin{aligned}\dot{\gamma}(\sigma) &= t(\sigma)\\ \dot{t}(\sigma) &= \gamma(\sigma) + \kappa_g(\sigma)\, s(\sigma)\\ \dot{s}(\sigma) &= -\kappa_g(\sigma)\, t(\sigma)\end{aligned}$$

where κ_g is the geodesic curvature of γ.

Theorem 3.3. *Let γ be a unit speed spherical curve in E_1^3, then*

$$\widetilde{\gamma}(\sigma) = a\left[\int_{\sigma_0}^{\sigma} \gamma(\upsilon)\,d\upsilon + \varepsilon \coth\theta \int_{\sigma_0}^{\sigma} s(\upsilon)\,d\upsilon\right] + c \tag{1}$$

is a Bertrand curve where a and θ are constant numbers, c is constant vector and $\varepsilon = \pm 1$. We take $\varepsilon = 1$ when the curve γ is spacelike and $\varepsilon = -1$ when the curve γ is timelike. Moreover, all the Bertrand curves with nonlightlike pirincipal normal can be constructed by this method.

Proof. If γ is a spacelike curve in H_1^2, then we have

$$\begin{aligned}
\dot{\widetilde{\gamma}}(\sigma) &= a(\gamma(\sigma) + \coth\theta s(\sigma)) \\
\ddot{\widetilde{\gamma}}(\sigma) &= a(1 - \coth\theta\kappa_g(\sigma))t(\sigma) \\
\dddot{\widetilde{\gamma}}(\sigma) &= -a\coth\theta\dot{\kappa}_g(\sigma)t(\sigma) + a(1 - \coth\theta\kappa_g(\sigma))(\gamma(\sigma) + \kappa_g(\sigma)s(\sigma))
\end{aligned}$$

$\widetilde{\gamma}$ is a spacelike curve, hence the curvature and torsion of $\tilde{\gamma}$ are

$$\begin{aligned}
\kappa(\sigma) &= \varepsilon\frac{\sinh^2\theta\,(1 - \coth\theta\kappa_g(\sigma))}{a} \\
\tau(\sigma) &= -\frac{\sinh^2\theta\,(\kappa_g(\sigma) - \coth\theta)}{a}
\end{aligned}$$

where $\varepsilon = \pm 1$. Since

$$-a(\varepsilon\kappa(\sigma) - \coth\theta\tau(\sigma)) = 1 \tag{2}$$

$\widetilde{\gamma}$ is a Bertrand curve. If γ is a spacelike curve in S_1^2 or a timelike one, then the proof is similar to the mentioned above.

For the converse, let $\widetilde{\gamma}$ be a Bertrand curve with nonlightlike principal normal and $\{T, N, B\}$ be its Frenet frame. If $\widetilde{\gamma}$ is a spacelike curve, we have two cases upon the character of T' :

If T' is a spacelike vector, we define a curve

$$\gamma(\sigma) = \varepsilon(\sinh\theta T(\sigma) + \cosh\theta B(\sigma))$$

In this case, γ is a spacelike curve. Using the equation (2), we calculate $\acute{\gamma}(\sigma) = \dfrac{\varepsilon \sinh \theta N(\sigma)}{a}$. When σ_1 is the arc lenght parameter of γ, we have $\dfrac{d\sigma_1}{d\sigma} = \dfrac{\varepsilon \sinh \theta}{a}$.

$$\begin{aligned} a\gamma(\sigma)\frac{d\sigma_1}{d\sigma} &= \sinh\theta\left(\sinh\theta T(\sigma) + \cosh\theta B(\sigma)\right) \qquad (3) \\ a\coth\theta s(\sigma)\frac{d\sigma_1}{d\sigma} &= a\coth\theta\gamma(\sigma)\times\frac{d\gamma}{d\sigma_1}\frac{d\sigma_1}{d\sigma} \\ &= -\cosh\theta\left(\cosh\theta T(\sigma) + \sinh\theta B(\sigma)\right) \qquad (4) \end{aligned}$$

If we substitute the equations (3) and (4) in (1) , we have

$$a\int_0^{\sigma_1} \gamma(v)\,dv + a\coth\theta\int_0^{\sigma_1} s(v)\,dv = -\int_{\sigma_0}^{\sigma} T(v)\,dv = -\widetilde{\gamma}(\sigma) + c$$

Here we note that the curve $\tilde{\gamma}$ is negatively oriented. If T' is a timelike vector or $\tilde{\gamma}$ is a timelike curve, then the proof is similar to the mentioned above.

4. Bertrand Curves Corresponding to The Spherical Indicatrices in $\mathbf{E}_1^3$

We define the spherical indicatrices of spacelike and timelike curves and then we investigate the Bertrand curves corresponding to them.

Definition 4.1. Let $\gamma : I \longrightarrow E_1^3$ be a unit speed curve, σ be its arc lenght parameter and $\kappa(\sigma) \neq 0$. The curve γ_t given by $\gamma_t(s) = t(s)$ is called the tangent indicatrix of γ. If the curve γ is timelike or spacelike, then γ_t is on the sphere H_1^2 or S_1^2 respectively.

Theorem 4.1. *The Bertrand curve corresponding to the tangent indicatrix of a curve in E_1^3 is*

$$\tilde{\gamma}(\sigma_t) = a\left[\gamma(\sigma_t) + \varepsilon\coth\theta\int_{\sigma_{t_0}}^{\sigma_t} b(v)\,dv\right] + c$$

where σ_t is the arc lenght parameter of γ_t. If γ_t is a spacelike or timelike curve, then we take $\varepsilon = 1$ or $\varepsilon = -1$ respectively.

Proof. The tangent vector of γ_t is $t_t = \dfrac{dt}{d\sigma_t}$. If the curve γ is timelike, γ_t is a spacelike curve and if the curve γ is spacelike, γ_t can be a spacelike or timelike curve. Wheter the curve γ is spacelike or timelike, we have $\dfrac{d\sigma_t}{d\sigma} = \kappa(\sigma)$. Now we define $s_t = t \times t_t$, then the Bertrand curve corresponding to γ_t is

$$\begin{aligned}\tilde{\gamma}(\sigma_t) &= a\int_{\sigma_{t_0}}^{\sigma_t} t(\upsilon)\,d\upsilon + a\varepsilon\coth\theta\int_{\sigma_{t_0}}^{\sigma_t} s_t(\upsilon)\,d\upsilon + c \\ &= a\gamma(\sigma_t) + a\varepsilon\coth\theta\int_{\sigma_{t_0}}^{\sigma_t} b(\upsilon)\,d\upsilon + c\end{aligned}$$

Definition 4.2. Let $\gamma : I \longrightarrow E_1^3$ be a unit speed curve, σ be its arc lenght parameter and $\kappa(\sigma) \neq 0$. The curve γ_n given by $\gamma_n(s) = n(s)$ is called the principal normal indicatrix of γ. If the vector n is spacelike or timelike, then γ_t is on the sphere S_1^2 or H_1^2 respectively.

Theorem 4.2. *The Bertrand curve corresponding to the principal normal indicatrix of a curve in E_1^3 is*

$$\tilde{\gamma}(\sigma_n) = a\left[\int_{\sigma_{n_0}}^{\sigma_n} n(\upsilon)\,d\upsilon + \varepsilon\coth\theta\int_{\sigma_{n_0}}^{\sigma_n} d(\upsilon)\,d\upsilon\right] + c$$

where σ_n is the arc lenght parameter of γ_n. If γ_n is a spacelike or timelike curve, then we take $\varepsilon = 1$ or $\varepsilon = -1$ respectively.

Proof. The tangent vector of γ_n is $t_n = \dfrac{dn}{d\sigma_n}$. Wheter the curve γ is timelike or spacelike, the curve γ_n can be spacelike or timelike. In both cases, we have $\dfrac{d\sigma_n}{d\sigma} = \|w(\sigma)\|$ where w is the darboux vector of γ. Now we define $s_n = n \times t_n$, then the Bertrand curve corresponding to γ_n is

$$\begin{aligned}\tilde{\gamma}(\sigma_n) &= a\int_{\sigma_{n_0}}^{\sigma_n} n(\upsilon)\,d\upsilon + a\varepsilon\coth\theta\int_{\sigma_{n_0}}^{\sigma_n} s_n(\upsilon)\,d\upsilon + c \\ &= a\int_{\sigma_{n_0}}^{\sigma_n} n(\upsilon)\,d\upsilon + a\varepsilon\coth\theta\int_{\sigma_{n_0}}^{\sigma_n} d(\upsilon)\,d\upsilon + c\end{aligned}$$

Definition 4.3. Let $\gamma : I \longrightarrow E_1^3$ be a unit speed curve, σ be its arc lenght parameter and $\kappa(\sigma) \neq 0$. The curve γ_b given by $\gamma_b(s) = b(s)$ is called the binormal indicatrix of γ. If the curve γ is timelike or spacelike, in both cases, the curve γ_b is on the sphere S_1^2.

Theorem 4.3. *The Bertrand curve corresponding to the binormal indicatrix of a curve in E_1^3 is*

$$\tilde{\gamma}(\sigma_b) = a\left[\int_{\sigma_{b_0}}^{\sigma_b} b(\upsilon)\, d\upsilon + \varepsilon \coth\theta \int_{\sigma_{b_0}}^{\sigma_b} t(\upsilon)\, d\upsilon\right] + c$$

where σ_b is the arc lenght parameter of γ_b. If the curve γ is timelike, then we take $\varepsilon = 1$ where γ_b is spacelike and $\varepsilon = -1$ where γ_b is timelike. If the curve γ is spacelike, then the sign of ε reverses.

Proof. The tangent vector of γ_b is $t_b = \dfrac{db}{d\sigma_b}$. Wheter the curve γ is timelike or spacelike, γ_b can be a spacelike or timelike curve and we have $\dfrac{d\sigma_b}{d\sigma} = \tau(\sigma)$. Now we define $s_b = b \times t_b$, then the Bertrand curve corresponding to γ_b is

$$\begin{aligned}
\tilde{\gamma}(\sigma_b) &= a\int_{\sigma_{b_0}}^{\sigma_b} b(\upsilon)\, d\upsilon + a\varepsilon \coth\theta \int_{\sigma_{b_0}}^{\sigma_b} s_b(\upsilon)\, d\upsilon + c \\
&= a\int_{\sigma_{b_0}}^{\sigma_b} b(\upsilon)\, d\upsilon + a\varepsilon \coth\theta \int_{\sigma_{b_0}}^{\sigma_b} t(\upsilon)\, d\upsilon + c
\end{aligned}$$

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