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A NOTE ON WEYL SPECTRA OF UPPER-TRIANGULAR OPERATOR MATRICES

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Abstract. Let $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ be a 2×2 upper triangular operator matrix acting on the Hilbert space $\mathscr{H}\oplus\mathscr{K}$. In this paper, for given operators A and B, we give a new characterization of $\cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_w(M_C)$, where $\sigma_w(A)$ denote the Weyl spectrum of A.

Keywords: Weyl spectrum; Essential spectrum; Upper-triangular operator matrix.

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1. Introduction

Throughout this paper, let $\mathscr{H}$ and $\mathscr{K}$ be separable Hilbert spaces, $\mathscr{B}(\mathscr{K},\mathscr{H})$ denote the set of all bounded linear operators from $\mathscr{K}$ into $\mathscr{H}$ and abbreviate $\mathscr{B}(\mathscr{H},\mathscr{H})$ to $\mathscr{B}(\mathscr{H})$. If $A\in\mathscr{B}(\mathscr{H})$, write $N(A)$, $R(A)$, $n(A)$ and $d(A)$ for the null space, the range, the nullity and the defect of A, respectively. A is a semi-Fredholm operator if $R(A)$ is closed and $n(A)<\infty$ or $d(A)<\infty$, then define the semi-Fredholm index of A by $ind(A)=n(A)-d(A)$([3]). Suppose A is a semi-Fredholm operator, A is called an upper semi-Fredholm operator if $d(A)<\infty$ and A is called a lower semi-Fredholm operator if $n(A)<\infty$([3]). Moreover, A is called a Weyl operator

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if it is a Fredholm operator and its Fredholm index is zero. If $A \in \mathscr{B}(\mathscr{H})$, denote $\sigma(A)$, $\sigma_{ap}(A)$ and $\sigma_\delta(A)$ for the spectrum, the approximation point spectrum and the surjective spectrum of A, respectively. The Weyl spectrum $\sigma_w(A)$ and the Browder essential approximation point spectrum $\sigma_{ab}(A)$ of A are defined by $\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Weyl }\}$ and $\sigma_{ab}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not an upper semi-Fredholm operator with finite ascent }\}$, respectively Write iso$F$ for the set of all isolated points of $F \subset \mathbb{C}$. We denote by M_C an operator acting on $\mathscr{H} \oplus \mathscr{K}$ of the form,

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}, \tag{1}$$

where $C \in \mathscr{B}(\mathscr{K}, \mathscr{H})$. In the sequel, M_C has the form as (1).

Definition 1.1. For $T \in \mathscr{B}(\mathscr{H})$, define $\triangle_k(T)$ by

$$\triangle_k(T) = \{\lambda \in \sigma(T) : ind(T - \lambda) = k\},$$

where $k \in \mathbb{Z} \cup \{\pm\infty\}$. We call $\triangle_k(T)$ be the $k-$th component of $\sigma(T)$.

It is easy to know that $\triangle_k(T)$ is an open set for $k \neq 0$, but not necessary connected. For example, let

$$T = \begin{bmatrix} V & 0 \\ 0 & V + 2I \end{bmatrix}$$

be an operator on $\mathscr{B}(l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z}))$. Where V is unilateral shift operator on $\mathscr{B}(l^2(\mathbb{Z}))$. Then $\triangle_{-1}(V) = \mathbb{D}$ and $\triangle_{-1}(V + 2I) = \mathbb{D} + 2$. Thus $\triangle_{-1}(T) = \mathbb{D} \cup (\mathbb{D} + 2)$.

Definition 1.2. For $T \in \mathscr{B}(\mathscr{H})$, define $\triangle_k^0(T)$ by

$$\triangle_k^0(T) = \{\lambda \in \sigma(T) : ind(T - \lambda) = k \text{ and } n(T - \lambda) = 0 \text{ or } d(T - \lambda) = 0\},$$

where $k \in \mathbb{Z} \cup \{\pm\infty\}$. We call $\triangle_k^0(T)$ be the initial component of $\triangle_k(T)$.

Clearly, $\triangle_0^0(T) = \emptyset$ and $\triangle_k^0(T) \subset \triangle_k(T)$ for each k. For A and B in $\mathscr{B}(\mathscr{H})$, denote

$$\begin{cases} U_{-\infty} &= \emptyset, \\ U_\infty &= \triangle_{-\infty}(A) \cap \triangle_\infty(B), \\ U_k &= \triangle_{-k}(A) \cap \triangle_k(B),\ k \in \mathbb{Z} \setminus \{0\}, \\ U_0 &= (\triangle_0(A) \cap \triangle_0(B)) \cup (\triangle_0(A) \cap \rho(B)) \cup (\triangle_0(B) \cap \rho(A)). \end{cases}$$

$$U_k^0 = \begin{cases} \triangle_{-k}^0(A) \cap \triangle_k^0(B), & k \geq 0, \\ \emptyset, & k < 0. \end{cases}$$

and $\nabla_k = U_k \setminus U_k^0$ for $k \in \mathbb{Z} \cup \{\pm\infty\}$. It is easy to know that $U_k^0 \subset U_k$ for each k.

Spectra of upper triangular operator matrices have been studied in operator theory for many years and many interesting results have been obtained, see[1-2], [4-11]. In particular, given $A \in \mathscr{B}(\mathscr{H})$ and $B \in \mathscr{B}(\mathscr{K})$, the set $\cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_\tau(M_C)$ were discussed in some works, where $\sigma_\tau(M_C)$ can be equal to the spectrum, the Weyl spectrum, the essential spectrum of M_C. For example, in [6], H. Du and J. Pan have proved that,

$$\bigcap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma(M_C) = \sigma_{ap}(A) \cup \sigma_\delta(B) \cup \{\lambda \in \mathbb{C} : n(B-\lambda) \neq d(A-\lambda)\}. \tag{2}$$

D. S. Djordjević [4] has obtained that

$$\bigcap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_e(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup W(A,B), \tag{3}$$

where $W(A,B) = \{\lambda \in \mathrm{C} : dim R(A-\lambda)^\perp \neq dim N(B-\lambda)$ and one of them is infinite$\}$. Meanwhile, D. S. Djordjević has also obtained that

$$\bigcap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_w(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup W_0(A,B), \tag{4}$$

where $W_0(A,B) = \{\lambda \in \mathrm{C} : N(A-\lambda) \oplus N(B-\lambda)$ is not ismorphic to$X/\overline{R(A-\lambda)} \oplus Y/\overline{R(B-\lambda)}\}$.

In the present paper, we investigate the similar questions. Studying in detail the structure of spectra of concerning operators, we give another characterization of $\bigcap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_w(M_C)$. For $A \in \mathscr{B}(\mathscr{H})$ and $B \in \mathscr{B}(\mathscr{K})$,

$$\cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_w(M_C) = (\cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_e(M_C)) \cup (\cup_{-k}^{\infty}((\triangle_{-k}(A) \cup \triangle_k(B)) \setminus U_k)).$$

Comparing the formula (4), our result is more clear in the structure of the spectrum.

2. Main results

To prove the main results, we begin with some lemmas.

Lemma 2.1. [11] *Given $A \in \mathscr{B}(\mathscr{H})$, $B \in \mathscr{B}(\mathscr{K})$, then each U_k^0 is an open set and*

$$(\sigma(A) \cup \sigma(B)) \setminus \bigcap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma(M_C) = \bigcup_{k=0}^{\infty} U_k^0,$$

where M_C, U_k^0 are defined as in Section 1.

Lemma 2.2.([8]) *Let $A \in \mathscr{B}(\mathscr{H})$, $B \in \mathscr{B}(\mathscr{K})$ and $C \in \mathscr{B}(\mathscr{K},\mathscr{H})$. If two of the three operators are Fredholm, then the other is also Fredholm. Moreover, $ind(M_C) = ind(A) + ind(B)$.*

Lemma 2.3. *For $A \in \mathscr{B}(\mathscr{H})$ and $B \in \mathscr{B}(\mathscr{K})$, if $\lambda \in \sigma(A) \cup \sigma(B)$, then there exists a $k \in \mathbb{Z} \cup \{\infty\}$ such that $\lambda \in U_k$ if and only if there exists an operator $C \in \mathscr{B}(\mathscr{K},\mathscr{H})$ such that $M_C - \lambda$ is a Weyl operator.*

Proof. Suppose that there exists a $k \in \mathbb{Z} \cup \{\infty\}$ such that $\lambda \in U_k$. If $k \in \mathbb{Z}$, the result is clear. If $k = \infty$, that is to say, $\lambda \in \triangle_{-\infty}(A) \cap \triangle_{\infty}(B)$, then $n(A-\lambda) < \infty$, $d(B-\lambda) < \infty$ and $d(A-\lambda) = n(B-\lambda) = \infty$. Suppose that $n(A-\lambda) = m_1$, $d(B-\lambda) = m_2$. Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be orthonormal basis of $N(B-\lambda)$ and $R(A-\lambda)^{\perp}$, respectively. Define an operator C by

$$\begin{cases} Cf_{m_1+j} = g_{m_2+j}, & 1 \le j \le \infty, \\ Cf = 0, & f \perp \vee\{f_i\}_{i=m_1+1}^{\infty}. \end{cases}$$

Thus $n(M_C - \lambda) = d(M_C - \lambda) = m_1 + m_2$. So $ind(M_C - \lambda) = 0$, $M_C - \lambda$ is Weyl.

On the contrary, suppose that there exists an operator $C \in B(K,H)$ such that $M_C - \lambda$ is a Weyl operator. Thus $A - \lambda$ is lower semi-Fredholm and $B - \lambda$ is upper semi-Fredholm. From Lemma 2.2, if one of $A - \lambda$ and $B - \lambda$ is Fredholm, then there exists a $k \in \mathbb{Z}$ such that $\lambda \in U_k$. If $ind(A-\lambda) = -\infty$, then $ind(B-\lambda) = \infty$. Therefore the result holds. The proof is finished.

Theorem 2.4. *For given $A \in \mathscr{B}(\mathscr{H})$, $B \in \mathscr{B}(\mathscr{K})$, then*

$$\begin{aligned} \cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_w(M_C) &= (\cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma(M_C)) \setminus (\cup_{-k}^{\infty} U_k) \\ &= (\cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_e(M_C)) \cup (\cup_{-k}^{\infty}((\triangle_{-k}(A) \cup \triangle_k(B)) \setminus U_k)). \end{aligned}$$

Proof. For convenience, we divided the proof into two steps.

Step 1. We prove $\cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_w(M_C) = \cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma(M_C) \setminus (\cup_{-k}^{\infty} U_k)$. From Lemma 1, it is sufficient to prove that $\cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_w(M_C) = \cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma(M_C) \setminus (\cup_{-k}^{\infty} (U_k \setminus U_k^0))$.

Since for any $C \in \mathscr{B}(\mathscr{K},\mathscr{H})$, we have $\sigma_w(M_C) \subset \sigma(M_C)$, thus

$$\cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_w(M_C) \subset \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma(M_C).$$

For any $\lambda \in \cup_{-k}^{\infty}(U_k \setminus U_k^0)$, then there exists a $k \in \mathbb{Z}\cup\{\infty\}\setminus\{0\}$ such that $\lambda \in U_k \setminus U_k^0$. If $k<\infty$, then $M_C-\lambda$ is Weyl for any $C \in \mathscr{B}(\mathscr{K},\mathscr{H})$. If $k=\infty$, from Lemma 2.3, there exists an operator $C \in \mathscr{B}(\mathscr{K},\mathscr{H})$ such that $M_C-\lambda$ is Weyl. So $\lambda \notin \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_w(M_C)$. Thus

$$\cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_w(M_C) \subset \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma(M_C)\setminus(\cup_{-k}^{\infty}(U_k\setminus U_k^0)).$$

On the other hand, if $\lambda \in \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma(M_C)$ and $\lambda \notin \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_w(M_C)$, then $M_C-\lambda$ is weyl and not invertible. Then $ind(M_C-\lambda)=0$, $A-\lambda$ is left semi-Freholm operator and $B-\lambda$ is right semi-Freholm operator. Thus there exist k_1, $k_2 \in \mathbb{Z}\cup\{\infty\}$ such that $\lambda \in \triangle_{-k_1}(A)$ and $\lambda \in \triangle_{k_2}(B)$. From Lemma 2.3, $k_1=k_2$. Let $k=k_1=k_2$. So $\lambda \in U_k\setminus U_k^0$ by Lemma 2.1. Hence $\lambda \notin \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma(M_C)\setminus(\cup_{-k}^{\infty}(U_k\setminus U_k^0))$. Therefore

$$\cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma(M_C)\setminus(\cup_{-k}^{\infty}(U_k\setminus U_k^0)) \subset \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_w(M_C).$$

So

$$\cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_w(M_C) = \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma(M_C)\setminus(\cup_{-k}^{\infty}(U_k\setminus U_k^0)).$$

Step 2. We prove

$$\cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_w(M_C) = (\cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_e(M_C))\cup_{-k}^{\infty}((\triangle_{-k}(A)\cup\triangle_k(B))\setminus U_k).$$

Since for any $C \in \mathscr{B}(\mathscr{K},\mathscr{H})$, we have $\sigma_e(M_C)\subset\sigma_w(M_C)\subset\sigma(M_C)$, thus

$$\cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_e(M_C) \subset \cap_{C\in\mathscr{B}(K,H)}\sigma_w(M_C) \subset \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma(M_C).$$

Moreover, $\cup_{-k}^{\infty}((\triangle_{-k}(A)\cup\triangle_k(B))\setminus U_k) \subset \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_w(M_C)$, thus

$$(\cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_e(M_C))\cup_{-k}^{\infty}((\triangle_{-k}(A)\cup\triangle_k(B))\setminus U_k) \subset \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_w(M_C). \tag{5}$$

If $\lambda \notin (\cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_e(M_C))\cup_{-k}^{\infty}((\triangle_{-k}(A)\cup\triangle_k(B))\setminus U_k)$, then

$$\lambda \notin \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_e(M_C) \text{ and } \lambda \notin \cup_{-k}^{\infty}((\triangle_{-k}(A)\cup\triangle_k(B))\setminus U_k).$$

If $\lambda \notin \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma_e(M_C)$ and $\lambda \in \cap_{C\in\mathscr{B}(\mathscr{K},\mathscr{H})}\sigma(M_C)$, then there exists an operator $C \in \mathscr{B}(\mathscr{K},\mathscr{H})$ such that $M_C-\lambda$ is a Fredhlom operator, so $\lambda \in \cup_{-k}^{\infty}((\triangle_{-k}(A)\cup\triangle_k(B))$. And also $\lambda \notin \cup_{-k}^{\infty}((\triangle_{-k}(A) \cup \triangle_k(B)) \setminus U_k)$, thus there exists an integer k such that $\lambda \in U_k$. So there exists an operator C_0 such that $M_{C_0} - \lambda$ is Weyl. Therefore $\lambda \notin \cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_w(M_C)$. Hence

$$\cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_w(M_C) \subset (\cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_e(M_C)) \cup_{-k}^{\infty} ((\triangle_{-k}(A) \cup \triangle_k(B)) \setminus U_k). \tag{6}$$

Combining the formula (5) with the formula (6), then

$$\cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_w(M_C) = (\cap_{C \in \mathscr{B}(\mathscr{K},\mathscr{H})} \sigma_e(M_C)) \cup_{-k}^{\infty} ((\triangle_{-k}(A) \cup \triangle_k(B)) \setminus U_k).$$

The proof is completed.

Conflict of Interests

The author declares that there is no conflict of interests.

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