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A HYBRID FIXED POINT THEOREM WITH PPF DEPENDENCE IN BANACH ALGEBRAS AND APPLICATIONS TO QUADRATIC FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, a fixed point theorem with PPF dependence in Banach algebras is proved and it is then applied to the hybrid differential equations of functional differential equations of delay and neutral type for proving the existence of PPF dependent solutions.

Keywords: Banach algebra; Hybrid fixed point theorem; PPF dependence; Quadratic functional differential equations; Existence theorem.

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1. Introduction

In the recent papers [1, 2], the authors proved some fixed point theorems for nonlinear operators in Banach spaces, wherein the domain and range of the operators is not same. The fixed point theorems of this kind are called PPF dependence fixed point theorems and are useful for proving the solutions of nonlinear functional differential and integral equations which may depend upon the past, present and future. The properties of a special Razumikhin or minimal

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or $\mathscr{D}$-class of functions are employed in the development of existence theory of PPF solutions for certain nonlinear equations in abstract spaces. The topic of PPF dependent theory is further developed in a series of papers [4, 5, 6, 7] for different classes of nonlinear operators and applications to different types of nonlinear functional differential equations.

Given a Banach space E with norm $\|\cdot\|_E$ and given a closed interval $I=[a,b]$ in $\mathbb{R}$, the set of real number, let $E_0 = C(I,E)$ be the Banach space of continuous E-valued continuous functions defined on I. We equip the space E_0 with the supremum norm $\|\cdot\|_{E_0}$ defined as

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E. \tag{1.1}$$

For a fixed point $c \in I$, the **Razumikhin** or **minimal** class $\mathscr{M}_c$ of functions is defined as

$$\mathscr{M}_c = \{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\}. \tag{1.2}$$

A Razumikhin class of functions $\mathscr{M}_c$ is said to be algebraically closed w.r.t. difference if $\phi - \xi \in \mathscr{M}_c$ whenever $\phi, \xi \in \mathscr{M}_c$. Similarly, $\mathscr{M}_c$ is topologically closed if it is closed in the topology of of E_0 generated by the norm $\|\cdot\|_{E_0}$. Similarly, other notions such as compactness and connectedness for $\mathscr{M}_c$ may be defined.

Let $T : E_0 \to E$. A point $\phi^* \in E_0$ is called a PPF dependent fixed point of T if $T\phi^* = \phi^*(c)$ for some $c \in I$ and any statement that guarantees the existence of PPF dependent fixed point is called a fixed point theorem with PPF dependence for the mapping T .

As mentioned in Bernfield *et al.* [1], the Razumikhin class of functions plays a significant role in proving the existence of PPF-fixed points with different domain and range of the operators. Very recently, generalizing a fixed point theorem of Bernfield *et al.* [1], the present author in Dhage [3] proved a fundamental fixed point theorems with PPF dependence for nonlinear contraction operators in Banach spaces.

Definition 1.1. A nonlinear operator $T : E_0 \to E$ is called a nonlinear contraction if there exists a upper semi-continuous function from the right $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|T\phi - T\xi\|_E \le \psi(\|\phi - \xi\|_{E_0}) \tag{1.3}$$

for all $\phi, \xi \in E_0$, where $\psi(r) < r, r > 0$. Similarly, T is called $\mathscr{B}$-contraction if there exists a nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous from right and satisfies (1.3).

Note that every contraction is $\mathscr{B}$-contraction and every $\mathscr{B}$-contraction is nonlinear contraction, however the converse may not be true. The details appears in a monographs of Browder [8] and the references given therein. The following fixed point theorem with PPF dependence is proved in Dhage [3].

Theorem 1.1 (Dhage [3]) *Suppose that $T : E_0 \to E$ is a nonlinear contraction. Then the following statements hold in E_0.*

(a) *If $\mathscr{M}_c$ is algebraically closed with respect to difference, then every sequence $\{\phi_n\}$ of successive iterates of T at each point $\phi_0 \in E_0$ converges to a PPF dependence fixed point of T.*

(b) *If $\mathscr{M}_c$ is topologically closed, then ϕ^* is the only fixed point of T in $\mathscr{M}_c$.*

In this paper, we prove a fixed point theorems with PPF dependence in a Banach algebra and apply it to hybrid differential equations of functional differential equations of delay and neutral type for proving the existence of PPF dependent solutions.

2. Hybrid fixed point theory with PPF dependence

Throughout subsequent part of this paper, unless otherwise specified, let E denote a Banach algebra with norm $\|\cdot\|_E$. Then $E_0 = C(I,E)$ becomes a Banach algebra with respect to the norm (1.1) and the multiplication " $\cdot$ " defined by

$$(x \cdot y)(t) = x(t) \cdot y(t) = x(t)y(t)$$

for all $t \in I$, whenever $x, y \in E_0$. When there is no confusion, we simply write xy instead of $x \cdot y$.

Definition 2.1. A mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a **dominating function** or, in short, **$\mathscr{D}$-function** if it is upper semi- continuous and nondecreasing function satisfying $\psi(r) = 0 \iff r = 0$. A mapping $Q : E_0 \to E$ is called **strong $\mathscr{D}$-Lipschitz** if there is a $\mathscr{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\|Q\phi - Q\xi\|_E \leq \psi(\|\phi(c) - \xi(c)\|_E) \tag{2.1}$$

for all $\phi, \xi \in E$. If $\psi(r) = kr$, $k > 0$, then Q is called **strong Lipschitz** with the Lipschitz constant k. In particular, if $k < 1$, then Q is called a **strong contraction** on X with the contraction constant k. Further, if $\psi(r) < r$ for $r > 0$, then Q is called **strong nonlinear $\mathscr{D}$-contraction** and the function ψ is called $\mathscr{D}$-function of Q on X.

There do exist $\mathscr{D}$-functions and the commonly used $\mathscr{D}$-functions are

$$\begin{aligned}
&\psi(r) = k\,r, \ \text{ for some constant } k > 0,\\
&\psi(r) = \frac{L\,r}{K + r}, \ \text{ for some constants } L > 0, K > 0 \text{ with } L \le K,\\
&\psi(r) = \tan^{-1} r,\\
&\psi(r) = \log(1 + r),\\
&\psi(r) = e^r - 1.
\end{aligned}$$

The above $\mathscr{D}$-functions have been widely used in the theory of nonlinear differential and integral equations for proving the existence results via fixed point methods.

Another notion that we need in the sequel is the following definition.

Definition 2.2. An operator Q on a Banach space E into itself is called compact if $Q(E)$ is a relatively compact subset of E. Q is called totally bounded if for any bounded subset S of E, $Q(S)$ is a relatively compact subset of E. If Q is continuous and totally bounded, then it is called completely continuous on E.

Our main hybrid fixed point theorem with PPF dependence is the following result.

Theorem 2.1 *Let $A : E_0 \to E$ and $B : E \to E$ be two operators such that*

(a) *A is bounded and strong $\mathscr{D}$-Lipschitz with the $\mathscr{D}$-function ψ, and*

(b) *B is continuous and compact, and*

(c) *$M\psi(r) < r$ if $r > 0$, where $M = \|B(E)\| = \sup\{\|Bx\| : x \in E\}$.*

Further, if the Razumikhin class of functions $\mathscr{R}_c$ is topologically and algebraically closed with respect to difference, then for a given $c \in [a,b]$ the operator equation

$$A\phi\,B\phi(c) = \phi(c) \tag{2.2}$$

has a PPF dependent solution.

Proof. Let $\xi \in E_0$ be fixed and let $c \in [a,b]$ be given. Define an operator $T_{\xi(c)} : E_0 \to E$ by

$$T_{\xi(c)}(\phi) = A\phi\, B\xi(c). \tag{2.3}$$

Clearly, $T_{\xi(c)}$ is a nonlinear $\mathscr{D}$-contraction on E_0. To see this, let $\phi_1, \phi_2 \in E_0$. Then,

$$\begin{aligned}\|T_{\xi(c)}(\phi_1) - T_{\xi(c)}(\phi_2)\|_E &= \|A\phi_1 - A\phi_2\|_E\, \|B\xi(c)\|_E \\ &\leq \|B(E)\|_E\, \psi(\|\phi_1(c) - \phi_2(c)\|_E) \\ &\leq M\, \psi(\|\phi_1(c) - \phi_2(c)\|_E).\end{aligned}$$

This shows that $T_{\xi(c)}$ is a strong nonlinear $\mathscr{D}$-contraction and hence nonlinear $\mathscr{D}$-contraction on E_0. By Theorem 2.3), there is a unique PPF dependence fixed point $\phi^* \in E_0$ such that

$$T_{\xi(c)}(\phi^*) = \phi^*(c) \quad \text{ or } \quad A\phi^* B\xi(c) = \phi^*(c).$$

Next, we define a mapping $Q : E \to E$ by

$$Q\xi(c) = \phi^*(c) = A\phi^* B\xi(c). \tag{2.4}$$

It then follows that

$$\begin{aligned}\|Q\xi_1(c) - Q\xi_2(c)\|_E &= \|A\phi_1^* B\xi_1(c) - A\phi_2^* B\xi_2(c)\|_E \\ &\leq \|A\phi_1^* - A\phi_2^*\|_E\, \|B\xi_1\|_E + \|A\phi_2^*\|_E\, \|B\xi_1(c) - B\xi_2(c)\|_E \qquad (2.5) \\ &\leq M\, \psi(\|\phi_1^*(c) - \phi_2^*(c)\|_E) + k\, \|B\xi_1(c) - B\xi_2(c)\|_E,\end{aligned}$$

where k is a bound of A on E_0. Since B is compact, if $\{B\xi_n(c)\}$ is any sequence in E, then $\{B\xi_n(c)\}$ has a convergent subsequence. Without loss of generality, call it the same sequence. Hence, $\{B\xi_n(c)\}$ is a Cauchy sequence. From inequality (2.5), we obtain

$$\|Q\xi_m(c) - Q\xi_n(c)\|_E \leq M\, \psi(\|\phi_m^*(c) - \phi_n^*(c)\|_E) + k\, \|B\xi_m(c) - B\xi_n(c)\|_E.$$

Taking the limit superior in above inequality yields

$$\begin{aligned}&\limsup_{m,n\to\infty} \|Q\xi_m(c) - Q\xi_n(c)\|_E \\ &\qquad \leq M \limsup_{m,n\to\infty} \psi(\|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E) + k \limsup_{m,n\to\infty} \|B\xi_m(c) - B\xi_n(c)\|_E \\ &\qquad \leq M\, \psi\left(\limsup_{m,n\to\infty} \|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E\right).\end{aligned}$$

Hence,

$$\lim_{m,n\to\infty} \|Q\xi_m(c) - Q\xi_n(c)\|_E = 0.$$

As a result, $\{Q\xi_n(c)\}$ is a Cauchy sequence. Since E is complete, $\{Q\xi_n(c)\}$ has a convergent subsequence. Now a direct application of Schauder fixed point principle yields that there is a point $\xi \in E_0$ such that $Q\xi^*(c) = \xi^*(c)$. Consequently $A\xi^* B\xi^*(c) = \xi^*(c)$. This completes the proof.

Rremark 2.1. If we consider Theorem 2.1 in a closed, convex and bounded subset of the Banach space E, then condition of the boundedness of the operator A is not required because in that case the boundedness of A follows immediately from the strong Lipschitz condition.

3. Quadratic functional differential equations

In this section, we apply the abstract result of the previous section to functional differential equations for proving the existence of solutions under a weaker Lipschitz condition. Given a closed interval $I_0 = [-r, 0]$ in $\mathbb{R}$ for some real number $r > 0$, let $\mathscr{C}$ denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathscr{C}$ with supremum norm $\|\cdot\|_{\mathscr{C}}$ defined by

$$\|\phi\|_{\mathscr{C}} = \sup_{\theta\in I_0} |\phi(\theta)|. \tag{3.1}$$

It is clear that $\mathscr{C}$ is a Banach space with this norm called the history space of the problem under consideration.

For each $t \in I = [0, T]$ and and given a function $x \in C(J, \mathbb{R})$, define a function $t \to x_t \in \mathscr{C}$ by

$$x_t(\theta) = x(t+\theta),\ \theta \in I_0, \tag{3.2}$$

where the argument θ represents the delay in the argument of solutions and $t + \theta \in J = [-r, T]$.

Now we are equipped with the necessary details to study the nonlinear problems of functional differential equations.

3.1. Functional differential equation of delay type

Given a function $\phi \in \mathscr{C}$, consider the perturbed or a hybrid differential equation of functional differential equations of delay type (in short HDE),

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] &= g(t,x_t),\ t \in I, \\ x_0 &= \phi \end{aligned}\right\} \tag{3.3}$$

where $f : I \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $g : I \times \mathscr{C} \to \mathbb{R}$ are continuous functions.

By a solution x of the HDE (3.3) we mean a function $x \in C(J,\mathbb{R})$ that satisfies

(i) the function $t \mapsto \dfrac{x}{f(t,x)}$ is continuous in I for each $x \in \mathbb{R}$, and
(ii) x satisfies the equations in (3.3) on J,

where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

The hybrid differential equation (3.3) is not new to the theory of nonlinear functional differential equations and the details of the different types of nonlinear differential equations appears in Dhage [4]. The existence theorems for the HDE (3.3) are generally proved using the hybrid fixed point theorems of Dhage [4], however, the novelty of the present paper lies in the nice applicability of our Theorem 2.1 for proving the existence of solutions with PPF dependence for the HDE (3.3) on J.

We consider the following hypotheses in what follows.

(H_0) The function $f(t,x)$ is increasing in x for each $t \in I$.

(H_1) There exist real numbers $L > 0$ and $K > 0$ such that

$$|g(t,x) - g(t,y)| \le \frac{L|x(0) - y(0)|}{K + |x(0) - y(0)|}$$

for all $t \in I$ and $x, y \in \mathscr{C}$.

(H_2) The function f is uniformly continuous and there exists a real number $M_f > 0$ such that

$$|f(t,x)| \le M_f$$

for all $t \in I$ and $x \in \mathbb{R}$.

Remark 3.1. If $L < K$ in hypothesis (H_1), then it reduces to the usual Lipschitz condition of g, namely,

$$|g(t,x) - g(t,y)| \le (L/K)|x(0) - y(0)|$$

for all $t \in I$ and $x, y \in \mathscr{C}$.

Theorem 3.1. *Assume that the hypotheses(H_0), (H_1) and (H_2) hold. Furthermore, if*

$$LT\ \max\{M_f, 1\} \le K,$$

then the HDE (3.3) has a PPF dependent solution defined on J.

Proof. Set $E = C(J, \mathbb{R})$. Then E is a Banach algebra with respect to the usual supremum norm $\|\cdot\|_E$ and the multiplication " $\cdot$ " defined by

$$\|x\|_E = \sup_{t\in J} |x(t)|$$

and

$$(x \cdot y)(t) = x(t) \cdot y(t) = x(t)y(t)$$

for all $t \in I$, whenever $x, y \in E$. Define a set of functions

$$\widehat{E} = \big\{\hat{x} = (x_t)_{t\in I} : x_t \in \mathscr{C}, x \in C(I, \mathbb{R}) \text{ and } x_0 = \phi\big\}. \tag{3.4}$$

Define a norm $\|\hat{x}\|_{\widehat{E}}$ in $\widehat{E}$ by

$$\|\hat{x}\|_{\widehat{E}} = \sup_{t\in I} \|x_t\|_{\mathscr{C}} \tag{3.5}$$

Clearly, $\hat{x} \in C(I_0, \mathbb{R}) = \mathscr{C}$. Next we show that $\widehat{E}$ is a Banach space. Consider a Cauchy sequence $\{\hat{x}_n\}$ in $\hat{E}$. Then, $\{(x_t^n)_{t\in I}\}$ is a Cauchy sequence in $\mathscr{C}$ for each $t \in I$. This further implies that $\{x_t^m(s)\}$ is a Cauchy sequence in in $\mathbb{R}$ for each $s \in [-r, 0]$. Then $\{x_t^m(s)\}$ converges to $x_t(s)$ for each $t \in I_0$. Since $\{x_t^n\}$ is a sequence of uniformly continuous functions for a fixed $t \in I$, $x_t(s)$ is also continuous in $s \in [-r, 0]$. Hence the sequence $\{\hat{x}_n\}$ converges to $\hat{x} \in \hat{E}$. As a result, $\widehat{E}$ is complete.

Now the HDE (3.3) is equivalent to the nonlinear hybrid integral equation (in short HIE)

$$x(t) = \begin{cases} \big[f(t, x(t))\big] \left(\dfrac{\phi(0)}{f(0, \phi(0))} + \displaystyle\int_0^t g(s, x_s)\, ds\right), & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.6}$$

Consider two operators $A : \widehat{E} \to C(J,\mathbb{R})$ and $B : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by and

$$A\hat{x} = A(x_t)_{t\in I} = \begin{cases} \dfrac{\phi(0)}{f(0,\phi(0))} + \displaystyle\int_0^t g(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.7}$$

and

$$Bx(t) = \begin{cases} f(t,x(t)), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0. \end{cases} \tag{3.8}$$

Then the HIE (3.5) is equivalent to the operator equation

$$A\hat{x}B\hat{x}(0) = \hat{x}(0). \tag{3.9}$$

We shall show that the operators A and B satisfy all the condition of Theorem 2.1. First we show that A is a bounded operator on $\widehat{E}$ into E. Now for any $\hat{x} \in \widehat{E}$, one has

$$\begin{aligned} \|A\hat{x}\|_E &\le \|A0\|_E + \|A(x_t)_{t\in I} - A0\|_E \\ &\le \|A0\|_E + \left| \int_0^t g(s,x_s)\,ds - \int_0^t g(s,0)\,ds \right| \\ &\le \|A0\|_E + \int_0^t \frac{L|x_t(0)-0|}{K+|x_t(0)-0|}\,ds \\ &\le \|A0\| + \int_0^t \frac{L\|\hat{x}(0)\|_{\mathscr{C}}}{K+\|\hat{x}\|_{\mathscr{C}}}\,ds \\ &\le \|A0\| + LT \end{aligned}$$

which shows that A is a bounded operator on $\widehat{E}$ with bound $\|A0\| + LT$.

Next, we that A is a strong $\mathscr{B}$-Lipschitz on $\widehat{E}$. Then,

$$\begin{aligned} \|A\hat{x} - A\hat{y}\|_E &= \|A(x_t)_{t\in I} - A(y_t)_{t\in I}\| \\ &\le \left| \int_0^t g(s,x_s)\,ds - \int_0^t g(s,y_s)\,ds \right| \\ &\le \int_0^t \frac{L|x_t(0)-y_t(0)|}{K+|x_t(0)-y_t(0)|}\,ds \\ &\le \int_0^t \frac{L\|\hat{x}(0)-\hat{y}(0)\|_{\mathscr{C}}}{K+\|\hat{x}(0)-\hat{y}(0)\|_{\mathscr{C}}}\,ds \\ &= \psi(\|\hat{x}(0)-\hat{y}(0)\|_E) \end{aligned}$$

for all $\hat{x},\hat{y} \in \widehat{E}$, where $\psi(r) = \dfrac{LTr}{K+r}$. Hence, A is a strong $\mathscr{D}$-Lipschitz on $\widehat{E}$ with $\mathscr{D}$-function ψ.

Next, we show that B is compact and continuous operator on $C(J,\mathbb{R})$. Let $\{x_n\}$ be a sequence in $C(J,\mathbb{R})$ such that $x_n \to x$ as $n \to \infty$. Then by continuity of f,,

$$\lim_{n\to\infty} Bx_n(t) = \lim_{n\to\infty} f(s,x_n(s))\,ds = f(s,x(s))\,ds = Bx(t)$$

for all $t \in I$. Similarly, if $t \in I_0$, then $\lim\limits_{n\to\infty} Bx_n(t) = 1 = Bx(t)$. This shows that $\{Bx_n(t)\}$ converges to Bx point-wise on J. But $\{Bx_n(t)\}$ is a sequence of uniformly continuous functions on J, so $Bx_n \to Bx$ uniformly. Hence, B is a continuous operator on E into itself.

Secondly, we show that B is compact. To, finish, it is enough to show that $B(E)$ is uniformly bounded and equi-continuous set in E. Let $x \in E$ be arbitrary. Then,

$$|Bx(t)| \le |f(s,x(s))| \le M_f$$

for all $t \in J$, and $|Bx(t)| \le 1$ for all $t \in I_0$. From this it follows that

$$|Bx(t)| \le \max\{M_f\,,\,1\} = M^*$$

for all $t \in J$, whence B is uniformly bounded on E.

To show equi-continuity, let $t,\tau \in I$. Then, from uniformly continuity of f it follows that

$$|Bx(t) - Bx(\tau)| \le |f(t,x(t)) - f(\tau,x(\tau))| < \varepsilon$$

for all $x \in C(J,\mathbb{R})$. If $\tau \in I_0$ and $t \in I$, then $\tau \to 0$ and $t \to 0$ whenever, $|\tau - t| \to 0$. Whence it follows that

$$|Bx(t) - Bx(\tau)| \le |Bx(\tau) - Bx(0)| + |Bx(t) - Bx(0)| \to 0 \text{ as } t \to \tau$$

for all $x \in C(J,\mathbb{R})$. From this, it follows that $B(E)$ is an equi-continuous set in E. Now an application of Arzella-Ascoli theorem yields that B is a compact operator on E into itself. Finally,

$$M\psi(r) = \frac{LT\max\{M_f,1\}\,r}{K+r} < r$$

for all $r > 0$ and so, all the conditions of Theorem 2.1 are satisfied. Again, here the Razumikhin class $\mathscr{R}_0$, $0 \in [-r,T]$ is $C([0,T],\mathbb{R})$ which is topologically and algebraically closed with respect to difference. Hence, an application of Theorem 2.1 yields that the integral equation (3.5) has a solution on J. This further implies that the HDE (3.3) has a PPF dependent solution defined on J. This completes the proof.

3.2. Functional differential equation of neutral type

Given a function $\phi \in \mathscr{C}$, consider the perturbed or a hybrid functional differential equation of neutral type (in short HDE)

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x_t)}\right] &= g(t,x(t)) \\ x_0 &= \phi \end{aligned}\right\} \tag{3.10}$$

for all $t \in I$, where $f : I \times \mathscr{C} \to \mathbb{R} \setminus \{0\}$ and $g : I \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a solution x of the FDE (3.10) we mean a function $x \in C(J,\mathbb{R})$ that satisfies

(i) the function $t \mapsto \dfrac{x(t)}{f(t,x_t)}$ is continuous in I, and

(ii) x satisfies the equations in (3.10) on J,

where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

We consider the following hypotheses in what follows.

(H_3) There exist real numbers $L > 0$ and $K > 0$ such that

$$|f(t,x) - f(t,y)| \le \frac{L|x(0)-y(0)|}{K + |x(0)-y(0)|}$$

for all $x, y \in \mathscr{C}$.

(H_4) There exists a real number $M_g > 0$ such that

$$|g(t,x)| \le M_g$$

for all $t \in I$ and $x \in \mathbb{R}$.

Theorem 3.2. *Assume that hypotheses (H_0), (H_3) and (H_4) hold. Further if* $L\left[\|\phi\|_{\mathscr{C}} + M_g T\right] \le K$, *then the HDE (3.10) has a PPF dependent solution defined on J.*

Proof. Set $E = C(J,\mathbb{R})$. Then E is a Banach algebra with respect to the norm and the multiplication defined as in Theorem 3.1. Define a set of functions $\widehat{E}$ by (3.4) which is equipped with the norm $\|\hat{x}\|_{\widehat{E}}$ defined by (3.5). Clearly, $\hat{x} \in C(I_0,\mathbb{R}) = \mathscr{C}$. It can be shown as in Theorem 3.1 that $\widehat{E}$ is a Banach space.

Now the HDE (3.10) is equivalent to the nonlinear hybrid integral equation (in short HIE)

$$x(t) = \begin{cases} \big[f(t,x_t)\big]\left(\phi(0)+\int_0^t g(s,x(s))\,ds\right), & \text{if } t\in I,\\ \phi(t), & \text{if } t\in I_0. \end{cases} \tag{3.11}$$

Consider two operators $A:\widehat{E}\to C(J,\mathbb{R})$ and $B:C(J,\mathbb{R})\to C(J,\mathbb{R})$ defined by

$$A\hat{x} = A(x_t)_{t\in I} = \begin{cases} f(t,x_t), & \text{if } t\in I,\\ 1, & \text{if } t\in I_0, \end{cases} \tag{3.12}$$

and

$$Bx(t) == \begin{cases} \phi(0)+\int_0^t g(s,x(s))\,ds, & \text{if } t\in I,\\ \phi(t), & \text{if } t\in I_0. \end{cases} \tag{3.13}$$

Then the HIE (3.11) is equivalent to the operator equation

$$A\hat{x}B\hat{x}(0) = \hat{x}(0). \tag{3.14}$$

We shall show that the operators A and B satisfy all the condition of Theorem 2.1. First we show that A is bounded on $\widehat{E}$.

$$\begin{aligned} \|A\hat{x}\| &\le \|A0\| + \|A(x_t)_{t\in I} - A0\| \\ &\le \sup_{t\in J}|f(t,0)| + |f(s,x_t)-f(s,0)| \\ &\le F_0 + \frac{L|x_t(0)-0|}{K+|x_t(0)-0|} \\ &\le F_0 + \frac{L\|\hat{x}(0)\|_{\mathscr{C}}}{K+\|\hat{x}(0)\|_{\mathscr{C}}} = F_0 + L, \end{aligned}$$

for all $\hat{x}\in\widehat{E}$, where $F_0=\sup_{t\in I}|f(t,0)|$. Hence, A is bounded on $\widehat{E}$ with bound F_0+L.

Next, we show that a strong $\mathscr{B}$-Lipschitz on $\widehat{E}$. Then,

$$\begin{aligned} \|A\hat{x}-A\hat{y}\|_E &= \|A(x_t)_{t\in I} - A(y_t)_{t\in I}\| \\ &\le |f(s,x_t)-f(s,y_t)| \\ &\le \frac{L|x_t(0)-y_t(0)|}{K+|x_t(0)-y_t(0)|} \\ &\le \frac{L\|\hat{x}(0)-\hat{y}(0)\|_{\mathscr{C}}}{K+\|\hat{x}(0)-\hat{y}(0)\|_{\mathscr{C}}} \end{aligned}$$

$$= \psi(\|\hat{x}(0) - \hat{y}(0)\|_E)$$

for all $\hat{x}, \hat{y} \in \widehat{E}$, where $\psi(r) = \dfrac{Lr}{K+r}$. Hence, A is a strong $\mathscr{D}$-Lipschitz on $\widehat{E}$ with $\mathscr{D}$-function ψ.

Next, we show that B is compact and continuous operator on $C(J,\mathbb{R})$. Let $\{x_n\}$ be a sequence in $C(J,\mathbb{R})$ such that $x_n \to x$ as $n \to \infty$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} Bx_n(t) &= \phi(0) + \lim_{n\to\infty} \int_0^t g(s, x_n(s))\, ds \\ &= \phi(0) + \int_0^t \lim_{n\to\infty} g(s, x_n(s))\, ds = Bx(t)\end{aligned}$$

for all $t \in I$. Similarly, if $t \in I_0$, then $\lim_{n\to\infty} Bx_n(t) = \phi(t) = Bx(t)$. This shows that $\{Bx_n(t)\}$ converges to Bx point-wise on J. But $\{Bx_n(t)\}$ is a sequence of uniformly continuous functions on J, So $Bx_n \to Bx$ uniformly. Hence, B is a continuous operator on E into itself.

Secondly, we show that B is compact. To, finish, it is enough to show that $B(E)$ is uniformly bounded and equi-continuous set in E. Let $x \in E$ be arbitrary. Then,

$$|Bx(t)| \le |\phi(0)| + \int_0^t |g(s, x(s))|\, ds \le \|\phi\|_{\mathscr{C}} + M_g T$$

for all $t \in J$ which shows that $B(E)$ is uniformly bounded set in E. To show equi-continuity, let $t, \tau \in I$. Then,

$$|Bx(t) - Bx(\tau)| \le \left| \int_\tau^t |g(s, x(s))|\, ds \right| \le M_g\, |t - \tau|.$$

If $\tau \in I_0$ and $t \in I$, then $\tau \to 0$ and $t \to 0$ whenever, $|\tau - t| \to 0$. Whence it follows that

$$|Bx(t) - Bx(\tau)| \le |Bx(\tau) - Bx(0)| + |Bx(t) - Bx(0)| \le M_g\, |t - \tau|.$$

From the above inequalities it follows that $B(E)$ is an equi-continuous set in E. Now an application of Arzelá-Ascoli theorem yields that B is a compact operator on E into itself. Finally,

$$M\psi(r) = \frac{L\left[\|\phi\|_{\mathscr{C}} + M_g T\right] r}{K + r} < r$$

for all $r > 0$. Again, Again, here the Razumikhin class $\mathscr{R}_0$, $0 \in [-r, T]$ is $C([0,T],\mathbb{R})$ which is topologically and algebraically closed with respect to difference and so, all the conditions of Theorem 2.1 are satisfied. Hence, an application of Theorem 2.1 yields that the integral equation

(3.11) has a solution on J. This further implies that the HDE (3.10) has a PPF dependent solution defined on J. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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