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ISOMETRIES OF P-NUCLEAR TYPE OPERATORS

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Abstract. Let X be a Banach space X and let $C_p(\ell^{p^*},X) = \{T : \ell^{p^*} \to X : \|T\|_{C(p)} = \sup(\sum_{n=1}^{\infty} \|T\theta_n\|^p)^{\frac{1}{p}} < \infty\}$, where the supremum is taken over all p^*-orthonormal sequences in ℓ^{p^*}. The object of this paper is to study the isometries of $C_p(\ell^{p^*},X)$. We give full characterization of certain classes of onto isometries of $C_p(\ell^{p^*},X)$ for some Banach spaces X.

Keywords: Banach space; Isometries; P-nuclear operators.

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1. Introduction

Let X be a Banach space and T be a bounded linear operator on X. T is called an isometry if $\|Tx\| = \|x\|$ for all $x \in X$. The characterization of onto isometries on X has been an important topic in analysis. Isometries is a main tool to study the Geometry of Banach spaces like extreme points, smooth points and exposed points of the unit ball a Banach space. In [1], Kadison characterized the isometries of $L(H)$, the space of bounded linear operators on a Hilbert space H. The isomerties $C(I,X)$ were characterized by Lau [2]. The isometries of $\varphi-$nuclear operators on general Banach spaces were characterized by Khalil and Salih [3]. Isometries

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of $L(\ell^p)$ $1 \leq p \neq 2 < \infty$, was an open problem since 1951. Khalil and Saleh [4] gave a full characterization of such isometries. The isometries of the $p-$nuclear operators $N_p(\ell^p, X)$ were characterized by Yousef and Khalil [5].

In this paper we study the onto isometries of p-nuclear type operators, to be denoted by $C_p(\ell^{p^*}, X)$. These are Schatten type classes. We give full characterization of some classes of onto isometries of $C_p(\ell^{p^*}, X)$. We refer to [2], [3], [6] and [7] for the basic facts on tensor product of Banach spaces and functional analysis.

2. The Space $C_p(\ell^{p^*}, X)$

In this section, we introduce our space of p-nuclear type operators.

Definition 2.1. Let X be a Banach space, and (x_n) be a sequence in X. The sequence (x_n) is called p-orthogonal if $\|\sum \lambda_n x_n\| = (\sum |\lambda_n|^p \|x_n\|^p)^{\frac{1}{p}}$. It is called p-orthonormal if $\|x_n\| = 1$.

One can easily show that in ℓ^p-spaces, (x_n) is p-orthogonal if and only if the $x_n^{,s}$ have disjiont support.

Now, we introduce our space.

Definition 2.2. For a Banach space X, we set

$$C_p(\ell^{p^*}, X) = \{T : \ell^{p^*} \to X : \|T\|_{C(p)} = \sup(\sum_{n=1}^{\infty} \|T\theta_n\|^p)^{\frac{1}{p}} < \infty\},$$

where the supremum is taken over all p^*-orthonormal sequences in ℓ^{p^*}. One can easily see that $\left(C_p(\ell^{p^*}, X), \|.\|_{C(p)}\right)$ is a normed space.

Further, we have

Theorem 2.3. *If X is Banach space, then* $\left(C_p(\ell^{p^*}, X), \|.\|_{C(p)}\right)$ *is a Banach space.*

Proof. We claim that every absolutely convergent series is convergent. So let $T_n \in C_p(\ell^{p^*}, X)$ be a sequence such that $\sum_{n=1}^{\infty} \|T_n\|_{C(p)} < \infty$. We claim $\sum_{n=1}^{\infty} T_n \in C_p(\ell^{p^*}, X)$. Define $T : \ell^{p^*} \to X$ as

$T(x)=\sum_{n=1}^{\infty}T_n(x)$. Clearly, T is bounded and $\|T\|\leq\|T\|_{C(p)}$. Further, we have

$$\begin{aligned}
\|T\|_{C(p)} &= \sup\left(\sum_{k=1}^{\infty}\|T(\theta_k)\|^p\right)^{\frac{1}{p}} \\
&= \sup\left(\sum_{k=1}^{\infty}\left\|\sum_{n=1}^{\infty}T_n(\theta_k)\right\|^p\right)^{\frac{1}{p}} \\
&\leq \sup\left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\|T_n(\theta_k)\|\right)^p\right)^{\frac{1}{p}} \\
&\leq \sup\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\|T_n(\theta_k)\|^p\right)^{\frac{1}{p}} \\
&\leq \sum_{n=1}^{\infty}\|T_n\|_{C(p)}<\infty.
\end{aligned}$$

Hence, we have $T\in C_p(\ell^{p^*},X)$. Remains to prove that $\sum_{n=1}^{\infty}T_n$ converge to T. So we claim $\|T-S_n\|_{C(p)}\to 0$.

$$\begin{aligned}
\|T-S_n\|_{C(p)} &= \sup\left(\sum_{k=1}^{\infty}\|T\theta_k-S_n\theta_k\|^p\right)^{\frac{1}{p}} \\
&= \sup\left(\sum_{k=1}^{\infty}\left\|\sum_{n+1}^{\infty}T_n\theta_k\right\|^p\right)^{\frac{1}{p}} \\
&\leq \sup\sum_{n+1}^{\infty}\left(\sum_{k=1}^{\infty}\|T_n\theta_k\|^p\right)^{\frac{1}{p}} \\
&\leq \sum_{n+1}^{\infty}\|T_n\|_{C(p)}.
\end{aligned}$$

But this goes to zero since it is the tail of a convergent series. Hence , $\left(C_p(\ell^{p^*},X),\|.\|_{C(p)}\right)$ is a Banach space.

Theorem 2.4. *Let X be a Banach space. Then the followings are equivalent:*

(i) $T\in C_p(\ell^{p^*},X)$

(ii) There exist $(\lambda_n)\in\ell^p$, *and* $g_n\in X$, *such that* $\|g_n\|=1$, *and* $T=\sum_{n=1}^{\infty}\lambda_n\delta_n\otimes g_n$.

Further, $\|T\|_{c(p)}=\|(\lambda_n)\|_p$.

Proof. First, we show $i\Rightarrow ii)$. Let $T\in C_p(\ell^{p^*},X)$ and (δ_n) be the natural basis in ℓ^{p^*}. Then

$$
\begin{aligned}
Tx &= T\,(\sum_{n=1}^{\infty} a_n\delta_n),\ \text{(where } x=(a_1,a_2,.......))\\
&= \sum_{n=1}^{\infty} a_n T\delta_n \text{ (since } T \text{ is bounded linear operator)}\\
&= \sum_{n=1}^{\infty} \lambda_n a_n g_n,\ \text{(where } g_n=\tfrac{T\delta_n}{\|T\delta_n\|} \text{ and } \lambda_n=\|T\delta_n\|)\\
&= \sum_{n=1}^{\infty} \lambda_n \langle \delta_n, x\rangle\, g_n\ .
\end{aligned}
$$

$(a_n=\langle\delta_n,x\rangle$, and $(\lambda_n)\in\ell^p$ since $T\in C_p(\ell^{p^*},X))$. Consequently, $T=\sum_{n=1}^{\infty}\lambda_n\delta_n\otimes g_n$. Remains to prove that $\|T\|_{c(p)}=\|(\lambda_n)\|_p$. Let $T\in C_p(\ell^{p^*},X)$, $1<p<\infty$, and $T=\sum_{n=1}^{\infty}\lambda_n\delta_n\otimes g_n$. Further, let (θ_k) be any p^* - orthonormal sequence in ℓ^{p^*}. Then $T\theta_k=\sum_{n=1}^{\infty}\lambda_n\langle\delta_n,\theta_k\rangle g_n$ and

$$
\begin{aligned}
\left(\sum_{k=1}^{\infty}\|T\theta_k\|^p\right)^{\frac{1}{p}} &= \left(\sum_{k=1}^{\infty}\left\|\sum_{n=1}^{\infty}\lambda_n\langle\delta_n,\theta_k\rangle g_n\right\|^p\right)^{\frac{1}{p}}\\
&\le \left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\lambda_n\langle|\delta_n|,|\theta_k|\rangle\right)^p\right)^{\frac{1}{p}}\\
&= \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\eta_k\lambda_n\langle|\delta_n|,|\theta_k|\rangle\right),
\end{aligned}
$$

where $\|(\eta_k)\|_{p^*}=1$ (By Hahn Banach Theorem and the fact that $(\ell^p)^*=\ell^{p^*}$). Now, if (e_n) is p - orthonormal in ℓ^p, then $(|e_n|)$ is p - orthonormal. Hence, $x=\sum_{n=1}^{\infty}|\lambda_n|\,|\delta_n|\in\ell^p$, $\|x\|_p=\|(\lambda_n)\|_p$, and $y=\sum_{k=1}^{\infty}|\eta_k|\,|\theta_k|\in\ell^{p^*}$, $\|y\|_{p^*}=1$. Now,

$$
\begin{aligned}
\left(\sum_{k=1}^{\infty}\|T\theta_k\|^p\right)^{\frac{1}{p}} &= \left|\sum_{n,k=1}^{\infty}|\lambda_n|\,|\eta_k|\,\langle|\delta_n|,|\theta_k|\rangle\right| = \left|\left\langle\sum_{n=1}^{\infty}|\lambda_n|\,|\delta_n|,\sum_{k=1}^{\infty}|\eta_k|\,|\theta_k|\right\rangle\right|\\
&= |\langle x,y\rangle| \le \|x\|_p\,\|y\|_{p^*} \le \left(\sum_{n=1}^{\infty}|\lambda_n|^p\right)^{\frac{1}{p}},\ \left(\|y\|_{p^*}=1\right).
\end{aligned}
$$

Hence, for any p - orthonormal sequence (θ_k), we have

$$
\left(\sum_{k=1}^{\infty}\|T\theta_k\|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{\infty}\|T\delta_n\|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty}|\lambda_n|^p\right)^{\frac{1}{p}}.
$$

So, $\sup\left(\sum_{k=1}^{\infty}\|T\theta_k\|^p\right)^{\frac{1}{p}}=\|(\lambda_n)\|_p$.

Next, we show $(ii) \Rightarrow (i)$. Let $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$, where $(\lambda_n) \in \ell^p$ and $\|g_n\| = 1$. Then as in case $(i \Rightarrow ii)$ we get

$$\|T\|_{C(p)} = \left(\sum_{n=1}^{\infty} |\lambda_n|^p \right)^{\frac{1}{p}} < \infty, \text{ and } T \in C_p(\ell^{p^*}, X).$$

This ends the proof of the Theorem.

Theorem 2.5. *Every operator $T \in C_p(\ell^{p^*}, X)$ has a representation for which the supremum is attained.*

Proof. From Theorem 2.4, we find the desired conclusion immediately.

3. The isometries

In this section, we study the isometric onto operators of $C_p(\ell^{p^*}, X)$.

Theorem 3.1. *Let A be an isometric onto operator of ℓ^{p^*}, and B be an isometric onto operator on X. Then the map defined by $F : C_p(\ell^{p^*}, X) \to C_p(\ell^{p^*}, X)$, $F(T) = BTA$ is an isometric onto operator of $C_p(\ell^{p^*}, X)$.*

Proof. Let $x \in \ell^{p^*}$ and let$T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$ be an element in $C_p(\ell^{p^*}, X)$. Since B is an isometry, we have

$$\begin{aligned} F(T)x &= BTAx = \sum_{n=1}^{\infty} \lambda_n \langle \delta_n, Ax \rangle Bg_n \\ &= \sum_{n=1}^{\infty} \lambda_n \langle A^* \delta_n, x \rangle Bg_n = \sum_{n=1}^{\infty} \lambda_n \langle A^* \delta_n, x \rangle \overset{\wedge}{g_n}, \end{aligned}$$

where $\left\| \overset{\wedge}{g_n} \right\| = 1$. Further, Since A^* is an isometric onto operator on ℓ^p, then $A^* \delta_n = \delta_{\varphi(n)}$, where $\varphi : \mathbb{N} \to \mathbb{N}$ is $(1-1)$ and onto map on the set of natural numbers. Thus $F(T) = \sum_{n=1}^{\infty} \lambda_n \delta_{\varphi(n)} \otimes \overset{\wedge}{g_n}$ $= \hat{T}$, say. Now, $\left\| \hat{T} \right\| = \left(\sum_{n=1}^{\infty} |\lambda_n|^p \right)^{\frac{1}{p}}$, and F is an isometry by Theorem 3.3.4.

To show that F is onto, let $S = \sum_{n=1}^{\infty} a_n \delta_n \otimes g_n \in C_p(\ell^{p^*}, X)$, and $\hat{S} = \sum_{n=1}^{\infty} a_n \delta_{\varphi^{-1}(n)} \otimes B^{-1} g_n$. Clearly $F(\hat{S}) = \sum_{n=1}^{\infty} a_n \delta_n \otimes g_n = S$. Then F is onto.This completes the proof.

Definition 3.2. A basic atom in $C_p(\ell^{p^*}, X)$ is an operator of the form $\delta_k \otimes h$ for some $k \in N$ and $h \in X$.

Theorem 3.3. *Let $F : C_p(\ell^{p^*},X) \to C_p(\ell^{p^*},X)$. If F preserves basic atoms and if F preserves rank, then the following are equivalent:*

(i) F is an isometric onto operator.

(ii) There exist two isometric onto operators: $A : \ell^p \to \ell^p$ and $B : X \to X$, and a sequence (a_n), $|a_n| = 1$ for all n with $F(\sum \delta_n \otimes x_n) = \sum A(\delta_n) \otimes a_n B x_n$.

Proof. First, we show $(i) \Rightarrow (ii)$. Let F be an isometric onto operator. We divide the proof into steps:

Step (i) . Let $X_1 = \delta_1 \otimes X = \{\delta_1 \otimes x : x \in X\}$.

Then $F(X_1) = \{\delta_k \otimes y : y \in X$, for fixed k , $\forall y \in X\}$.

Claim. Let $x_1, x_2 \in X$. If possible assume $F(\delta_1 \otimes x_1) = \delta_{k_1} \otimes \hat{x}_1$and $F(\delta_1 \otimes x_2) = \delta_{k_2} \otimes \hat{x}_2$, and $\delta_{k_1} \neq \delta_{k_2}$. Then, $\delta_1 \otimes x_1 + \delta_1 \otimes x_2 = \delta_1 \otimes (x_1 + x_2)$ is a basic atom. Since F preserves basic atoms then $F(\delta_1 \otimes x_1 + \delta_1 \otimes x_2) = \delta_j \otimes y$ for some $y \in X$ and $j \in N$. Hence, $\delta_j \otimes y = \delta_{k_1} \otimes \hat{x}_1 + \delta_{k_2} \otimes \hat{x}_2$, which is a contradiction since $\delta_{k_1} \neq \delta_{k_2}$. So $\delta_{k_1} \otimes \hat{x}_1 + \delta_{k_2} \otimes \hat{x}_2$ is not a basic atom. So $F(\delta_1 \otimes x) = \delta_k \otimes y$, for fixed $k \in N$. Similarly for $\delta_2, \delta_3, \ldots$.

Step (ii) . Define $A : \ell^p \to \ell^p$, $A\delta_1 = \delta_k$, where $F(\delta_1 \otimes X) = \delta_k \otimes X$. Similarly for $\delta_2, \delta_3, \ldots$. Then A is an isometric onto operator since it permutes the basis (δ_k) and F is onto. So A can be recognized as: $A\delta_n = \delta_{\varphi(n)}$, where φ is a permutation on the set of natural numbers, N.

Step (iii) . $F(\delta_1 \otimes x) = \delta_{\varphi(1)} \otimes x_1$, and $F(\delta_2 \otimes x) = \delta_{\varphi(2)} \otimes x_1$.

Claim. If possible assume $F(\delta_1 \otimes x) = \delta_{\varphi(1)} \otimes y$ and $F(\delta_2 \otimes x) = \delta_{\varphi(2)} \otimes z$, $y \neq z$. Now, $\delta_1 \otimes x + \delta_2 \otimes x = (\delta_1 + \delta_2) \otimes x$, which is a $1-$rank operator. But $F((\delta_1 + \delta_2) \otimes x) = \delta_{\varphi(1)} \otimes y + \delta_{\varphi(2)} \otimes z$. Now, since $\|y\| = \|z\| = \|x\|$, (since F is an isometry) then either y, z are independent or $y = \pm z$. If y, z are independent, then $F((\delta_1 + \delta_2) \otimes x) = \delta_{\varphi(1)} \otimes y + \delta_{\varphi(2)} \otimes z$ is a two rank operator which is a contradiction, since F preserves rank. Hence, $F(\delta_1 \otimes x) = \delta_{\varphi(1)} \otimes a_1 y$,and $F(\delta_2 \otimes x) = \delta_{\varphi(2)} \otimes a_2 y$, with $|a_i| = 1$.

In a similar way one can prove $\{F(\delta_k \otimes x) : k \in N\} = \{\delta_{\varphi(k)} \otimes a_k y : \ |a_k| = 1, k \in N\}$.

Step (iv) . Define $B : X \to X$, $B(x) = y$, where $F(\delta_k \otimes x) = a_k \delta_{\varphi(k)} \otimes y$, and $|a_k| = 1$. Then B is well-defined linear maps.To prove the linearity of B, let $x_1, x_2 \in X$, and $\beta \in \mathbb{R}$. Then, $F(\delta_k \otimes (\beta x_1 + x_2)) = F(\beta \delta_k \otimes x_1 + \delta_k \otimes x_2) = \beta F(\delta_k \otimes x_1) + F(\delta_k \otimes x_2) = \beta a_k \delta_{\varphi(k)} \otimes \hat{x}_1 + a_k \delta_{\varphi(k)} \otimes \hat{x}_2 = a_k \delta_{\varphi(k)} \otimes \left(\beta \hat{x}_1 + \hat{x}_2\right) = a_k \delta_{\varphi(k)} \otimes (\beta B(x_1) + B(x_2))$,, where $|a_k| = 1$. That is

$B(\beta x_1 + x_2) = \beta B(x_1) + B(x_2)$. Since F is an isometric operator, we have $\|x\| = \|\delta_k \otimes x\| = \|F(\delta_k \otimes x)\| = \left\|a_k \delta_{\varphi(k)} \otimes y\right\| = \|y\| = \|B(x)\|$. Hence B is an isometry. Finally, let $y \in X$. Then $\delta_k \otimes y = F(\delta_{\varphi^{-1}(k)} \otimes x)$, since F is onto. Therefore, $y = B(x)$ for some $x \in X$. Thus B is an isometric onto operator.

Now, we want to show that $F(T) = \sum\limits_{n=1}^{\infty} A(\delta_n) \otimes a_n B x_n$. Indeed, we have $F(\sum\limits_{n=1}^{\infty} \delta_n \otimes x_n) = \sum\limits_{n=1}^{\infty} F(\delta_n \otimes x_n) = \sum\limits_{n=1}^{\infty} \delta_{\varphi(n)} \otimes a_n y_n = \sum\limits_{n=1}^{\infty} A(\delta_n) \otimes a_n B x_n$.

Now, we are in a position to show $(ii) \Rightarrow (i)$. Let $T = \sum\limits_{n=1}^{\infty} \delta_n \otimes x_n$ be an element in $C_p(\ell^{p^*}, X)$. Since B is an isometry, we have $F(T) = \sum\limits_{n=1}^{\infty} (A(\delta_n) \otimes a_n B(x_n)) = \sum\limits_{n=1}^{\infty} (\delta_{\varphi(n)} \otimes a_n y_n) = \hat{T}$, where $\|y_n\| = \|x_n\|$.

Now,

$$\|F(T)\| = \left\|\hat{T}\right\| = \left(\sum_{n=1}^{\infty} |\lambda_n|^p\right)^{\frac{1}{p}} = \|T\|.$$

Hence F is an isometry by Theorem 3.1. To show that F is onto, let $S = \sum\limits_{n=1}^{\infty} \delta_n \otimes y_n \in C_p(\ell^{p^*}, X)$, where $y_n = a_n x_n$ such that $|a_n| = 1$. Let $\hat{S} = \sum\limits_{n=1}^{\infty} \delta_{\varphi^{-1}(n)} \otimes B^{-1} x_n$. Clearly $F(\hat{S}) = \sum\limits_{n=1}^{\infty} \delta_n \otimes a_n x_n = \sum\limits_{n=1}^{\infty} \delta_n \otimes y_n = S$. Then F is onto. This ends the proof.

Theorem 3.4. *Let $T = \delta_k \otimes x \in C_1(\ell^\infty, X)$ with $\|T\| = 1$.Then T is an extreme points of $C_1(\ell^\infty, X)$ if and only if x is extreme in $B[X]$.*

Proof. Let $x \in ext(B_1[X])$. We claim that $T = \delta_k \otimes x$ is an extreme points of $C_1(\ell^\infty, X)$. Without loss of generality, assume $T = \delta_1 \otimes x$ and assume that T is not an extreme point. Hence, there exist $T_1 = \sum\limits_{n=1}^{\infty} \delta_n \otimes x_n$ and $T_2 = \sum\limits_{n=1}^{\infty} \delta_n \otimes y_n \in C_1(\ell^\infty, X)$ such that $\delta_1 \otimes x = \frac{1}{2}(T_1 + T_2)$ and $\|T_1\| = \|T_2\| = 1$. Thus $\delta_1 \otimes x = \frac{1}{2}(T_1 + T_2)$. So, $(x, 0, 0, 0, \ldots) = \frac{1}{2} \sum\limits_{n=1}^{\infty} \delta_n \otimes (x_n + y_n) = (\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}, \ldots)$.Then $x = \frac{x_1+y_1}{2}$ which is a contradiction, since $x \in ext(B_1[X])$. Hence, $\delta_k \otimes x$ is an extreme points of $C_1(\ell^\infty, X)$.

The Converse is clear. This ends the proof.

For $X = \ell^p$, $1 \le p < \infty$, we have the following.

Theorem 3.5. *Let* $F: C_1(\ell^\infty, \ell^p) \to C_1(\ell^\infty, \ell^p)$ *be an isometric onto operator. Then F preserves basic atoms.*

Proof. Let F be an isometric onto operator of $C_1(\ell^\infty, \ell^p)$. Then as is known, F preserves the extreme points of the unit ball of $C_1(\ell^\infty, \ell^p)$. Now, let $\delta_k \otimes h \in C_1(\ell^\infty, \ell^p)$ be basic atom. Then by Theorem 3.3.8 $\delta_k \otimes \frac{h}{\|h\|} \in ext\ B_1(C_1(\ell^\infty, \ell^p))$. Hence $F(\delta_k \otimes \frac{h}{\|h\|}) = \delta_j \otimes g$ for some $g \in ext\ B_1(\ell^p)$. Since $\|h\|\, \delta_k \otimes \frac{h}{\|h\|} = \delta_k \otimes h$, then $F(\delta_k \otimes h) = \|h\|\, \delta_j \otimes g = \delta_j \otimes \hat{g}$, where $\left\|\hat{g}\right\| = \|h\|$.

Theorem 3.6. *Let* $F: C_1(\ell^\infty, \ell^p) \to C_1(\ell^\infty, \ell^p)$ *be a linear operator that preserves rank. Then* F *is an isometric onto operator, if and only if* $F(\sum_{n=1}^{\infty} \delta_n \otimes x_n) = \sum_{n=1}^{\infty} A(\delta_n) \otimes a_n B(x_n)$, *where* $A: \ell^1 \to \ell^1$ *is an isometric onto operator, and* $B: \ell^p \to \ell^p$ *is an isometric onto operator, and* (a_n) *is a sequence of reals such that* $|a_n| = 1$.

Proof. By using Theorem 3.5, We see that F preserves basic atoms, and by using Theorem 3.3m we can obtain the result immediately.

Conflict of Interests

The authors declare that there is no conflict of interests.

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