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LACUNARY STATISTICAL CONVERGENCE OF DOUBLE GENERALIZED DIFFERENCE SEQUENCES ON PROBABILISTIC NORMED SPACE

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Abstract. In this paper we study define the concept of lacunary double Δ^m- statistical convergent sequences in probabilistic normed space and give some results. The main purpose of this paper is to generalize the results for double sequences on statistical convergence in probabilistic normed space given by Esi and Özdemir [4] earlier.

Keywords: double statistical convergence; double lacunary sequence; t-norm; probabilictic normed space.

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1. Introduction

A probabilistic normed space (PN space) is a natural generalization of an ordinary normed linear space. In PN space, the norms of vectors are represented by probability distribution functions rather than a positive number. Such spaces were first introduced by Serstnev [6] in 1963. In [13], Alsina et al. gave a new definition of PN-spaces which

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includes Serstnev's a special case and leads naturally to the identification of the principle class of PN-spaces, the Menger spaces. This definition becomes the standard one and has been adopted by many authors (for instance, [12], [7], [8], [9]) who have investigated the properties of PN spaces. The detailed history and the development of the subject up to 2006 can be found in [21].

On the other hand, statistical convergence was first introduced by Fast [8] as a generalization of ordinary convergence for real number sequences. Since then it has been studied by many authors (for instance, [26], [19], [20], [2]). Statistical convergence has also been discussed in more general abstract spaces such as the fuzzy number space [15], locally convex spaces [21] and Banach spaces [14]. Karakus [24] introduced and studied statistical convergence on PN spaces and followed by Karakus and Demirci [25] studied statistical convergence of double sequences on PN spaces. Recently Esi and Özdemir [4] introduced generalized Δ^m-statistical convergence in probabilistic normed space for single generalized difference sequences and Esi [3] has introduced lacunary statistical convergence of double sequences in probabilistic normed space.

It seems therefore reasonable to think if the concept of statistical convergence can be extended to probabilistic normed spaces and in that case enquire how the basic properties are affected. But basic properties do not hold on probabilistic normed spaces. The problem is that the triangle function in such spaces.

In this paper we extend the concept of lacunary statistical convergence of double generalized difference sequences to probabilistic normed spaces and observe that some basic properties are also preserved on probabilistic normed spaces. Since the study of convergence in PN-spaces is fundamental to probabilistic functional analysis, we feel that the concepts of Δ^m-statistical convergence and Δ^m-statistical Cauchy for double sequences in a PN-space would provide a more general framework for the subject.

2. Preliminaries

Now we recall some notations and definitions used in paper.

Definition 2.1.([13]) A function $f:\mathbb{R}\to\mathbb{R}_o^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t)=0$ and $\sup_{t\in\mathbb{R}} f(t)=1$. We will denote the set of all distribution functions by D.

Definition 2.2.([13]) A triangular norm, briefly t-norm, is a binary operation on $[0,1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, that is, it is the continuous mapping $\circledast:[0,1]\times[0,1]\to[0,1]$ such that for all $a,b,c\in[0,1]$:

(1) $a\circledast 1=a$,

(2) $a\circledast b=b\circledast a$,

(3) $c\circledast d\geq a\circledast b$ if $c\geq a$ and $d\geq b$,

(4) $(a\circledast b)\circledast c=a\circledast(b\circledast c)$.

Example 2.1. The $\circledast$ operations $a\circledast b=\max\{a+b-1,0\}$, $a\circledast b=a.b$ and $a\circledast b=\min\{a,b\}$ on $[0,1]$ are t-norms.

Definition 2.3.([10, 11]) A triple $(X,N,\circledast)$ is called a probabilistic normed space or shortly PN-space if X is a real vector space, N is a mapping from X into D (for $x\in X$, the distribution function $N(x)$ is denoted by N_x and $N_x(t)$ is the value of N_x at $t\in\mathbb{R}$) and $\circledast$ is a t-norm satisfying the following conditions:

(PN-1) $N_x(0)=0$,

(PN-2) $N_x(t)=1$ for all $t>0$ if and only if $x=0$,

(PN-3) $N_{\alpha x}(t)=N_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha\in R\backslash\{0\}$,

(PN-4) $N_{x+y}(s+t)\geq N_x(s)\circledast N_y(t)$ for all $x,y\in X$ and $s,t\in\mathbb{R}_o^+$.

Example 2.2. Suppose that $(X,\|.\|)$ is a normed space $\mu\in D$ with $\mu(0)=0$ and $\mu\neq h$, where

$$h(t)=\begin{cases}0 & , \quad t\leq 0\\ 1 & , \quad t>0\end{cases}.$$

Define

$$N_x(t)=\begin{cases}h(t) & , \quad x=0\\ \mu\left(\frac{t}{\|x\|}\right) & , \quad x\neq 0\end{cases},$$

where $x \in X, t \in \mathbb{R}$. Then $(X, N, \circledast)$ is a PN-space. For example if we define the functions μ and ν on $\mathbb{R}$ by

$$\mu(x) = \begin{cases} 0 & , \quad x \leq 0 \\ \frac{x}{1+x} & , \quad x > 0 \end{cases}, \quad \nu(x) = \begin{cases} 0 & , \quad x \leq 0 \\ e^{-\frac{1}{x}} & , \quad x > 0 \end{cases}$$

then we obtain the following well-known $\circledast$ norms:

$$N_x(t) = \begin{cases} h(t) & , \quad x = 0 \\ \frac{t}{t+\|x\|} & , \quad x \neq 0 \end{cases}, \quad M_x(t) = \begin{cases} h(t) & , \quad x = 0 \\ e^{\left(-\frac{\|x\|}{t}\right)} & , \quad x \neq 0 \end{cases}.$$

We recall the concepts of convergence and Cauchy sequences for single sequences in a probabilistic normed space.

Definition 2.4.([1]) Let $(X, N, \circledast)$ is a PN-space. Then a sequence $x = (x_k)$ is said to be convergent to $l \in X$ with respect to the probabilistic norm N if, for every $\varepsilon > 0$ and $\theta \in (0,1)$, there exists a positive integer k_o such that $N_{x_k - l}(\varepsilon) > 1 - \theta$ whenever $k \geq k_o$. It is denoted by $N - \lim x = L$ or $x_k \overset{N}{\to} L$ as $k \to \infty$.

Definition 2.5.([1]) Let $(X, N, \circledast)$ is a PN-space. Then a sequence $x = (x_k)$ is called a Cauchy sequence with respect to the probabilistic norm N if, for every $\varepsilon > 0$ and $\theta \in (0,1)$, there exists a positive integer k_o such that $N_{x_k - x_l}(\varepsilon) > 1 - \theta$ for all $k, l \geq k_o$.

Definition 2.6.([1]) Let $(X, N, \circledast)$ is a PN-space. Then a sequence $x = (x_k)$ is said to be bounded in X, if there is a $r \in \mathbb{R}$ such that $N_{x_k}(r) > 1 - \theta$, where $\theta \in (0,1)$. We denote by l_∞^N the space of all bounded sequences in PN space.

The idea of statistical convergence for single sequences was introduced by Fast [18] and then studied by various authors, e.g., Salat [26], Fridy [19], Connor [20], Esi [2] and many others and in normed space by Kolk [14]. Recently Karakus [24] and Alotaibi [1] have studied the concept of statistical convergence in probabilistic normed spaces.

Firstly, we recall some definitions.

In 1900 Pringsheim presented the following definition for the convergence of double sequences.

Definition 2.7.([5]) A double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such an $x = (x_{k,l})$ more briefly as "P-convergent".

We shall denote the space of all P-convergent sequences by c^{u}. By a bounded double sequence we shall mean there exists a positive number K such that $|x_{k,l}| < K$ for all (k,l) and denote such bounded by $\|x\|_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty$. We shall also denote the set of all bounded double sequences by l_{∞}^{u}. We also note in contrast to the case for single sequence, a P-convergent double sequence need not be bounded.

Definition 2.8.([15]) Let $K \subset \mathbb{N} \times \mathbb{N}$ be two-dimensional set of positive integers and let $K(n,m)$ be the numbers of (i,j) in K such that $i \leq n$ and $j \leq m$.Then the two-dimensional analogue of natural density can be defined as follows:

The lower asymptotic density of a set $K \subset \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = P - \liminf_{n,m} \frac{K(n,m)}{nm}.$$

In this case $\left(\frac{K(n,m)}{nm}\right)$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as

$$\delta_2(K) = P - \lim_{n,m} \frac{K(n,m)}{nm}.$$

For example, let $K = \{(i^2, j^2) : \ i,j \in \mathbb{N}\}$. Then

$$\delta_2(K) = P - \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0,$$

i.e., the set K has double natural density zero, while the set $L = \{(i,2j) : \ i,j \in \mathbb{N}\}$ has double natural density $\frac{1}{2}$.

Definition 2.9.([16]) The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that

$$k_o = 0, \ h_r = k_r - k_{r-1} \to \infty \ \text{ as } r \to \infty$$

and

$$l_o = 0, \ \overline{h_s} = l_s - l_{s-1} \to \infty \ \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h_s}$ and $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l) : \ k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \ \overline{q_s} = \frac{l_s}{l_{s-1}} \text{ and } q_{r,s} = q_r \overline{q_s}.$$

Definition 2.10. A real double sequence $x = (x_{k,l})$ is to be Δ^m-statistically convergent to L, provided that for each $\varepsilon > 0$

$$\{(k,l): \ k \leq m \text{ and } l \leq n, |\Delta^m x_{k,l} - L| \geq \varepsilon\}$$

has double natural density zero or equivalently

$$P - \lim_{m,n} \frac{1}{mn} |\{(k,l): \ k \leq m \text{ and } l \leq n, |\Delta^m x_{k,l} - L| \geq \varepsilon\}| = 0.$$

In this case we write $S^2_{\Delta^m} - \lim x = L$ or $x_{k,l} \to L\,(S^2_{\Delta^m})$.

Definition 2.11. A real double sequence $x = (x_{k,l})$ is said to be Δ^m-statistically Cauchy provided that, for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that for all $k,p \geq N$, $l,q \geq M$, the set

$$\{(k,l) \in I_{r,s}: \ |\Delta^m x_{k,l} - \Delta^m x_{p,q}| \geq \varepsilon\}$$

has double natural density zero or equivalently

$$P - \lim_{k,l} \frac{1}{kl} |\{(k,l): k,p \leq N \text{ and } l,q \leq M, \ |\Delta^m x_{k,l} - \Delta^m x_{p,q}| \geq \varepsilon\}| = 0.$$

Definition 2.12. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence, the double number sequence $(x_{k,l})$ is $S^\theta_{\Delta^m}-$convergent to L provided that for every $\varepsilon > 0$

$$\delta^\theta_{\Delta^m}\left(\{(k,l) \in I_{r,s}: \ |\Delta^m x_{k,l} - L| \geq \varepsilon\}\right) = 0$$

or equivalently

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s}: \ |\Delta^m x_{k,l} - L| \geq \varepsilon\}| = 0.$$

In this case we write $S^\theta_{\Delta^m} - \lim x = L$ or $x_{k,l} \to L\left(S^\theta_{\Delta^m}\right)$.

Definition 2.13. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence, the double number sequence $(x_{k,l})$ is said to be an $S^\theta_{\Delta^m}-$Cauchy sequence if there exists a double subsequence $\left\{x_{\bar{k}_r,\bar{l}_s}\right\}$ of $(x_{k,l})$ such that $\left(\bar{k}_r, \bar{l}_s\right) \in I_{r,s}$ for each (r,s), $P - \lim_{r,s} x_{\bar{k}_r,\bar{l}_s} = L$ and for every $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s}: \ \left|\Delta^m x_{k,l} - \Delta^m x_{\bar{k}_r,\bar{l}_s}\right| \geq \varepsilon\right\}\right| = 0.$$

Using the concepts, we extend the double lacunary generalized difference statistical convergence and $S^{\theta}_{\Delta^m}-$Cauchy double sequence to the setting of sequences in a PN space endowed with the strong topology as follows:

Definition 2.14. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. Then a double sequence $(x_{k,l})$ in X is double lacunary Δ^m-statistically convergent to θ in X if for every $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \ \Delta^m x_{k,l} \notin \Lambda_x(\varepsilon)\}| = 0,$$

where $\Lambda_x(\varepsilon) = \left\{(x_{k,l}) \in X : \ N_{\Delta^m x_{k,l} - L}(\varepsilon) > 1 - \varepsilon\right\}$ is the neighborhood of θ. In this case we write $S^{\theta}_{N_{\Delta^m}} - \lim x_{k,l} = \theta$ or $x_{k,l} \to \theta \left(S^{\theta}_{N_{\Delta^m}}\right)$ and we will call θ, as the double lacunary statistically Δ^m-limit of the sequence $(x_{k,l})$. We shall use $S^{\theta}_{N_{\Delta^m}}$ to denote the set of all double lacunary Δ^m-convergent sequences from X. Of course, there is nothing about θ as a limit, if one wishes to consider the double lacunary statistically convergent of the sequence $(x_{k,l})$ to the vector L, then it suffices to consider the sequence $(\Delta^m x_{k,l} - L)$ and it is double lacunary statistically Δ^m-convergent to θ.

Definition 2.11. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. Then a double sequence $(x_{k,l})$ in X is said to be double lacunary Δ^m-statistically Cauchy sequence if there exists a double subsequence $\left\{x_{\bar{k}_r, \bar{l}_s}\right\}$ of $(x_{k,l})$ such that $\left(\bar{k}_r, \bar{l}_s\right) \in I_{r,s}$ for each (r,s), $P - \lim_{r,s} x_{\bar{k}_r, \bar{l}_s} = x$ and for every $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} : \ \Delta^m x_{k,l} - \Delta^m x_{\bar{k}_r, \bar{l}_s} \notin \Lambda_x(\varepsilon)\right\}\right| = 0.$$

3. Main results

Now we give the analogues of these definitions with respect to probabilistic norm N.

Definition 3.1. Let $(X, N, \circledast)$ is a PN-space. A double sequence $x = (x_{k,l})$ is said to be Δ^m-convergent to $L \in X$ in X with respect to probabilistic norm N, that is, $x_{k,l} \overset{\Delta^m}{\to} L$ if for every $\varepsilon > 0$ and $\vartheta \in (0,1)$, there is a positive integer k_o such that $N_{\Delta^m x_{k,l} - L}(\varepsilon) > 1-$

ϑ whenever $k \geq k_o$ and $l \geq k_o$. In this case we write $N_{\Delta^m} - \lim x_{k,l} = L$, where

$$\Delta^m x = (\Delta^m x_{k,l}) = \left(\Delta^{m-1} x_{k,l} - \Delta^{m-1} x_{k,l+1} - \Delta^{m-1} x_{k+1,l} + \Delta^{m-1} x_{k+1,l+1}\right)$$

$$\left(\Delta^1 x_{k,l}\right) = (\Delta x_{k,l}) = (x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1})$$

and $\Delta^0 x = (x_{k,l})$ and also this generalized difference double notion has the following binomial representation:

$$\Delta^m x_{k,l} = \sum_{i=0}^{m} \sum_{j=0}^{m} (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{k+i,l+j}.$$

Definition 3.2. Let $(X, N, \circledast)$ is a PN-space. Then a double sequence $x = (x_{k,l})$ is lacunary statistically Δ^m-convergent to $L \in X$ with respect to probabilistic norm N provided that, for every $\varepsilon > 0$ and $\vartheta \in (0,1)$

$$\delta^{\theta}_{N_{\Delta^m}} \left(\left\{(k,l) \in I_{r,s} : \ N_{\Delta^m x_{k,l} - L} (\varepsilon) \leq 1 - \vartheta\right\}\right) = 0$$

or equivalently

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} : \ N_{\Delta^m x_{k,l} - L} (\varepsilon) \leq 1 - \vartheta\right\}\right| = 0.$$

In this case we write $\delta^{\theta}_{N_{\Delta^m}} - \lim x_{k,l} = L$ or $x_{k,l} \to L \left(\delta^{\theta}_{N_{\Delta^m}}\right)$.

Definition 3.3. Let $(X, N, \circledast)$ is a PN-space. Then a double sequence $x = (x_{k,l})$ is said to be double lacunary Δ^m-statistically Cauchy with respect to the probabilistic norm N provided that, for every $\varepsilon > 0$ and $\vartheta \in (0,1)$, there exists a double subsequence $\left\{x_{\bar{k}_r, \bar{l}_s}\right\}$ of $x = (x_{k,l})$ such that $\left(\bar{k}_r, \bar{l}_s\right) \in I_{r,s}$ for each (r,s), $P - \lim_{r,s} x_{\bar{k}_r, \bar{l}_s} = L$ and

$$\delta^{\theta}_{N_{\Delta^m}} \left(\left\{(k,l) \in I_{r,s} : \ N_{\Delta^m x_{k,l} - \Delta^m x_{\bar{k}_r, \bar{l}_s}} (\varepsilon) \leq 1 - \vartheta\right\}\right) = 0$$

or equivalently

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} : \ N_{\Delta^m x_{k,l} - \Delta^m x_{\bar{k}_r, \bar{l}_s}} (\varepsilon) \leq 1 - \vartheta\right\}\right| = 0.$$

By using (3.1) and well-known density properties, we easily get the following lemma.

Lemma 3.1. Let $(X, N, \circledast)$ is a PN-space. Then for every $\varepsilon > 0$ and $\vartheta \in (0,1)$, the following statements are equivalent:

(i) $P-\lim_{r,s}\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}:\ N_{\Delta^m x_{k,l}-L}(\varepsilon)\le 1-\vartheta\right\}\right|=0,$

(ii) $\delta^{\theta}_{N_{\Delta^m}}\left(\left\{(k,l)\in I_{r,s}:\ N_{\Delta^m x_{k,l}-L}(\varepsilon)\le 1-\vartheta\right\}\right)=0,$

(iii) $\delta^{\theta}_{N_{\Delta^m}}\left(\left\{(k,l)\in I_{r,s}:\ N_{\Delta^m x_{k,l}-L}(\varepsilon)> 1-\vartheta\right\}\right)=1,$

(iv) $P-\lim_{r,s}\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}:\ N_{\Delta^m x_{k,l}-L}(\varepsilon)> 1-\vartheta\right\}\right|=1.$

Proof. The first three parts are equivalent from Definition 2.5. It follows from Definition 2.4 that

$$\left\{(k,l)\in I_{r,s}:\ N_{\Delta^m x_{k,l}-L}(\varepsilon)\ge \vartheta\right\}$$

$$\left\{(k,l)\in I_{r,s}:\ N_{\Delta^m x_{k,l}-L}(\varepsilon)\ge 1+\vartheta\right\}\cup\left\{(k,l)\in I_{r,s}:\ N_{\Delta^m x_{k,l}-L}(\varepsilon)\ge 1-\vartheta\right\}.$$

Also, Definition 2.3 implies that (ii) and (iv) are equivalent.

Theorem 3.2. Let $(X,N,\circledast)$ is a PN-space. If a double sequence $x=(x_{k,l})$ is Δ^m-statistically convergent with respect to the probabilistic norm N, then $S^{\theta}_{N_{\Delta^m}}-\lim x_{k,l}$ is unique.

Proof. Suppose that the double sequence $x=(x_{k,l})$ is Δ^m-statistically convergent to two distinct points L_1 and L_2 (say) with respect to the probabilistic norm N. Let $\varepsilon>0$ and $\beta>0$. Choose $\vartheta\in(0,1)$ such that $(1-\vartheta)\circledast(1-\vartheta)\ge 1-\beta$. Then, we define the following sets:

$$K_1(\vartheta,\varepsilon)=\left\{(k,l)\in I_{r,s}:\ N_{\Delta^m x_{k,l}-L_1}(\varepsilon)\le 1-\vartheta\right\}$$

and

$$K_2(\vartheta,\varepsilon)=\left\{(k,l)\in I_{r,s}:\ N_{\Delta^m x_{k,l}-L_2}(\varepsilon)\le 1-\vartheta\right\}.$$

Then, clearly

$$P-\lim_{r,s}\frac{|K_1(\vartheta,\varepsilon)\cap K_2(\vartheta,\varepsilon)|}{h_{r,s}}=1$$

so $K_1(\Delta^m,\gamma,\varepsilon)\cap K_2(\Delta^m,\gamma,\varepsilon)$ is a non-empty set. Since $\delta^{\theta}_{N_{\Delta^m}}-\lim x_{k,l}=L_1$ and $\delta^{\theta}_{N_{\Delta^m}}-\lim x_{k,l}=L_2$ we have $\delta^{\theta}_{N_{\Delta^m}}(K_1(\vartheta,\varepsilon))=0$ and $\delta^{\theta}_{N_{\Delta^m}}(K_2(\vartheta,\varepsilon))=0$ for all $\varepsilon>0$, respectively. Let

$$K(\vartheta,\varepsilon)=K_1(\vartheta,\varepsilon)\cap K_2(\vartheta,\varepsilon).$$

Then we observe that $\delta^{\theta}_{N_{\Delta^m}}(K(\vartheta,\varepsilon))=0$ which implies $\delta^{\theta}_{N_{\Delta^m}}(\mathbb{N}\times\mathbb{N}-K(\vartheta,\varepsilon))=1$. If $(k,l)\in\mathbb{N}\times\mathbb{N}-K(\vartheta,\varepsilon)$, then we have

$$N_{L_1-L_2}(\varepsilon)\geq N_{\Delta^m x_{k,l}-L_1}\left(\frac{\varepsilon}{2}\right)\circledast N_{\Delta^m x_{k,l}-L_2}\left(\frac{\varepsilon}{2}\right)$$

$$>(1-\vartheta)\circledast(1-\vartheta)\geq 1-\beta.$$

Since $\beta>0$ was arbitrary, we get $N_{L_1-L_2}(\varepsilon)=1$ for all $\varepsilon>0$, which yields $L_1-L_2=0$. Therefore $L_1=L_2$ and the proof is completed.

Theorem 3.3. Let $(X,N,\circledast)$ is a PN-space. If $N_{\Delta^m}-\lim x_{k,l}=L$ then $\delta^{\theta}_{N_{\Delta^m}}-\lim x_{k,l}$ $=L$.

Proof. Let $N_{\Delta^m}-\lim x_{k,l}=L$.Then for every $\vartheta\in(0,1)$ and $\varepsilon>0$, there is a number $k_o\in\mathbb{N}$ such that $N_{\Delta^m_{k,l}-L}(\varepsilon)>1-\vartheta$ for all $k\geq k_o$ and $l\geq k_o$. Hence the set

$$\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\ N_{\Delta^m x_{k,l}-L}(\varepsilon)\leq 1-\vartheta\right\}$$

has at most finitely many terms. Since every finite subset of the natural numbers has double density zero, we immediately see that

$$\delta^{\theta}_{N_{\Delta^m}}\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\ N_{\Delta^m x_{k,l}-L}(\varepsilon)\leq 1-\vartheta\right\}\right)=0,$$

whence the result.

Theorem 3.4. Let $(X,N,\circledast)$ is a PN-space and $x=(x_{k,l})$ be a double sequence. Then $\delta^{\theta}_{N_{\Delta^m}}-\lim x_{k,l}=L$ if and only if there exists a subset

$$K=\{(k,l):\ k,l=1,2,3,...\}\subset\mathbb{N}\times\mathbb{N}$$

such that $\delta^{\theta}_{N}(K)=1$ and $N_{\Delta^m}-\lim_{\substack{k,l\to\infty\\(k,l)\in K}}x_{k,l}=L$.

Proof. Suppose that $\delta^{\theta}_{N_{\Delta^m}}-\lim x_{k,l}=L$. Now for any $\varepsilon>0$ and $r\in\mathbb{N}$, let

$$K(r,\varepsilon)=\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:N_{\Delta^m x_{k,l}-L}(\varepsilon)\leq 1-\frac{1}{r}\right\},$$

(3.1)

$$M(r,\varepsilon)=\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:N_{\Delta^m x_{k,l}-L}(\varepsilon)>1-\frac{1}{r}\right\}$$

Then $\delta^{\theta}_{N_{\Delta^m}}\{K(r,\varepsilon)\}=0$ and

$$M(1,\varepsilon)\supset M(2,\varepsilon)\supset M(3,\varepsilon)\supset\cdots\supset M(i,\varepsilon)\supset M(i+1,\varepsilon)\supset\cdots \tag{1}$$

$$\delta^{\theta}_{N_{\Delta^m}}\{M(r,\varepsilon)\}=1,\ r=1,2,3,\ldots. \tag{2}$$

Now we have to show that for $(k,l)\in M(r,\varepsilon)$, $x=(x_{k,l})$ is $N_{\Delta^m}-$convergent to L. Suppose that $x=(x_{k,l})$ is not $N_{\Delta^m}-$convergent to L. Therefore there is $\theta>0$ such that the set

$$\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:N_{\Delta^m x_{k,l}-L}(\varepsilon)\leq 1-\vartheta\ \right\}$$

has infinitely many terms. Let

$$M(\vartheta,\varepsilon)=\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:N_{\Delta^m x_{k,l}-L}(\varepsilon)>1-\vartheta\right\},\ \vartheta>\frac{1}{r}\ (r=1,2,3,\ldots).$$

Then $\delta^{\theta}_{N}\{M(\vartheta,\varepsilon)\}=0$ and by (1), we have $M(r,\varepsilon)\subset M(\vartheta,\varepsilon)$. Hence $\delta^{\theta}_{N}\{M(r,\varepsilon)\}=0$ which contradicts (2). Therefore $x=(x_{k,l})$ is $N_{\Delta^m}-$convergent to L.

Conversely, suppose that there exists a subset $K=\{(k,l):\ k,l=1,2,3,...\}\subset\mathbb{N}\times\mathbb{N}$ such that $\delta^{\theta}_{N}(K)=1$ and $N_{\Delta^m}-\lim_{\substack{k,l\to\infty\\(k,l)\in K}}x_{k,l}=L$. Then there exists $k_o\in\mathbb{N}$ such that for every $\vartheta\in(0,1)$ and $\varepsilon>0$

$$N_{\Delta^m x_{k,l}-L}(\varepsilon)>1-\vartheta \text{ for all } k,l\geq k_o.$$

Now

$$M(\vartheta,\varepsilon)=\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:N_{\Delta^m x_{k,l}-L}(\varepsilon)\leq 1-\vartheta\right\}$$

$$\subset\mathbb{N}\times\mathbb{N}-\{(k_{k_o+1},l_{k_o+1}),(k_{k_o+2},l_{k_o+2}),(k_{k_o+3},l_{k_o+3}),\ldots\}$$

Therefore $\delta^{\theta}_{N_{\Delta^m}}\{M(\vartheta,\varepsilon)\}\leq 1-1=0$. Hence $\delta^{\theta}_{N_{\Delta^m}}-\lim x_{k,l}=L$. This completes the proof of the theorem.

Theorem 3.5. Let $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence with $\limsup_r q_r<\infty$ and $\limsup_s \overline{q}_s<\infty$ then $S^{\theta}_{N_{\Delta^m}}\subset S^2_{\Delta^m}$.

Proof. Since $\limsup_r q_r<\infty$ and $\limsup_s \overline{q}_s<\infty$ there exists $H>0$ such that $q_r<H$ and $\overline{q}_s<H$ for all r and s. Suppose that $x_{k,l}\to\theta\,(S^2_{\Delta^m})$ and

$$N_{r,s}=\left|\{(k,l)\in I_{r,s}:\Delta^m x_{k,l}\notin\Lambda_{\theta}(\varepsilon)\}\right|$$

By the definition of $S^{\theta}_{\Delta^m} - \lim x_{k,l} = \theta$, for given $\varepsilon > 0$ there exists $r_o \in \mathbb{N}$ such that $\frac{N_{r,s}}{h_{r,s}} < \varepsilon$ for all r and $s > r_o$. Let

$$M = \max\left\{N_{r,s} :\ 1 \leq r \leq r_o \text{ and } 1 \leq s \leq r_o\right\}.$$

Let m and n be such that $k_{r-1} < m \leq k_r$ and $l_{s-1} < n \leq l_s$. Then we can write

$$\begin{aligned} &\frac{1}{mn}\left|\left\{(k,l) :\ k \leq m \text{ and } l \leq n, \Delta^m x_{k,l} \notin \Lambda_\theta(\varepsilon)\right\}\right| \\ &\leq \frac{1}{k_{r-1}l_{s-1}}\left|\left\{k \leq k_r, l \leq l_s : \Delta^m x_{k,l} \notin \Lambda_\theta(\varepsilon)\right\}\right| \\ &= \frac{1}{k_{r-1}l_{s-1}}\sum_{i,j=1,1}^{r,s} N_{i,j} \leq \frac{M r_o^2}{k_{r-1}l_{s-1}} + \varepsilon H^2 \end{aligned}$$

and the result follows immediately.

Theorem 3.6. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence with $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$ then $S^2_{\Delta^m} \subset S^{\theta}_{N_{\Delta^m}}$.

Proof. Suppose that $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$ then there exists $\delta > 0$ such that $q_r > 1 + \delta$ and $\overline{q}_s > 1 + \delta$. This implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\overline{h}_s}{l_s} \geq \frac{\delta}{1+\delta}$. Since $h_{r,s} = k_r l_s - k_{r-1}l_{s-1}$, we are granted the following

$$\frac{k_r l_s}{h_{r,s}} \leq \frac{1+\delta}{\delta}$$

and

$$\frac{k_{r-1}l_{s-1}}{h_{r,s}} \leq \frac{1}{\delta}.$$

Let $S^{\theta}_{\Delta^m} - \lim x_{k,l} = \theta$. We are going to prove $S^2_{\Delta^m} - \lim x_{k,l} = \theta$. We can write

$$\begin{aligned} &\frac{1}{k_r l_s}\left|\left\{(k,l) \in I_{r,s} : k \leq k_r \text{ and } l \leq l_s, \Delta^m x_{k,l} \notin \Lambda_\theta(\varepsilon)\right\}\right| \\ &\geq \frac{1}{k_r l_s}\left|\left\{(k,l) \in I_{r,s} : \Delta^m x_{k,l} \notin \Lambda_\theta(\varepsilon)\right\}\right| \\ &\geq \left(\frac{\delta}{1+\delta}\right)^2 \frac{1}{h_{r,s}}\left|\left\{(k,l) \in I_{r,s} : x_{k,l} \notin \Lambda_\theta(\varepsilon)\right\}\right| \end{aligned}$$

which proves the theorem.

Theorem 3.7. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence, then $S^{\theta}_{N_{\Delta^m}} = S^2_{\Delta^m}$ if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ and $1 < \liminf_s \overline{q}_s \leq \limsup_s \overline{q}_s < \infty$.

Proof. The proof is an immediate consequence of the Theorem 3.5 and Theorem 3.6.

4. Conclusion

In this paper, we obtained some results on lacunary statistical convergence for double generalized difference sequences on probabilistic normed spaces. The obtained results here are more general than the results of Esi [3].

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