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CHARACTERIZATION OF BEST SIMULTANEOUS APPROXIMATION OF A UNIFORMLY BOUNDED SET OF REAL VALUED FUNCTIONS

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Abstract. In this paper, we give generalized results of [1], who showed that the best simultaneous approximation to a set F of uniformly bounded real valued function on $[a,b]$ is equivalent to the best simultaneous approximation of the two functions $\sup_{f\in F} f$ and $\inf_{f\in F} f$. A simplified proof of this result is also provided.

Keywords: Real valued functions; Bounded functions; Best simultaneous approximation.

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1. Introduction

The problem of approximating a set of data in a given metric space by a single element of an approximating family arises naturally in many practical problems. A common procedure is to choose the best approximant by a least squares principle, which has the advantages of existence, uniqueness, stability and easy computability. However, in many cases the least deviation principle makes more sense. Geometrically this amounts to covering the given data set by a ball of minimal radius among those centred at points of the approximating family.

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The theory of best simultaneous approximants in this sense, called also Chebyshev centres was initiated by A.L Garkavi more than 50 years ago. It has drawn more attention in the last three decades, but still in the developing stage.

Let X be a normal linear space and K a subset of X. Given any bounded subset $F \subset X$, define $d(F,K) = \inf_{k\in K} \sup_{f\in F} \|f-k\|$. An element $k^* \in K$ is said to be a best simultaneous approximant to the set F, if $d(F,K) = \sup_{f\in F} \|f-k^*\|$. This definition was given by [4]. In [3] they considered the problem of simultaneous approximation of the case: $X = [a,b], \quad K$ a non-empty subset of X and $F = \{f_1, f_2\}$.

In [3], they studied the problem of X a normed linear space, K any subset and $F = \{f_1, f_2\}$. Using the same procedure as above, [4] extended the study to include $F = \{f_1, f_2, \dots, f_n\}$. Later [1] gave the same definition in [3] but in the set of real valued functions.

Definition 1.1. Let F be the set of uniformly bounded real valued functions in $[a,b]$ and S a non-empty family of real valued functions on $[a,b]$. If there exists an $s^* \in S$ such that

$$\inf_{s\in S} \sup_{f\in F} \|f-s\| = \sup_{f\in F} \|f-s^*\|,$$

then s^* is called a best simultaneous approximation to F.

In [3], they proved that the best simultaneous approximation of the two functions F^+ and F^- where

$$F^+ = \inf_{\delta>0} \sup_{0\leq|x-y|<\delta} \sup_{f\in F} f(y)$$

and

$$F^- = \sup_{\delta>0} \inf_{0\leq|x-y|<\delta} \inf_{f\in F} f(y)$$

is equivalent to the best simultaneous of F. The results we are generalizing in [1] showed that the best simultaneous approximation to F is equivalent to the best simultaneous approximation of the two functions $\sup_{f\in F} f$ and $\inf_{f\in F} f$.

2. Main results

The following lemma is as in [1].

Lemma 2.1. *Let A be a bounded set of real numbers and r any real number. Then*

$$\sup_{a\in A}|a-r| = \left|\frac{\alpha+\beta}{2}\right| + \frac{(\alpha-\beta)}{2}$$

where $\alpha = \sup\limits_{a\in A} a$ *and* $\beta = \inf\limits_{a\in A} a$.

Theorem 2.1. *Let F be a set of uniformly bounded, real valued functions in $[a,b]$ and S a non-empty family of real valued functions in $[a,b]$. If $s^* \in S$ is a best approximant to F, then*

$$\sup_{f\in F}\|f-s^*\| = \left\| \left| \frac{\sup\limits_{f\in F} f + \inf\limits_{f\in F} f}{2} - s^* \right| + \frac{\sup\limits_{f\in F} f - \inf\limits_{f\in F} f}{2} \right\|.$$

Proof. Let $s\in S$ and $x\in[a,b]$. Then by the lemma,

$$\sup_{f\in F}\|f(x)-s(x)\| = \left\| \left| \frac{\sup\limits_{f\in F} f(x) + \inf\limits_{f\in F} f(x)}{2} - s(x) \right| + \{\sup_{f\in F} f(x) - \inf_{f\in F} f(x)\} \right\|.$$

Now taking supremum of both sides over $[a,b]$, we get

$$\sup_{f\in F}\|f-s\| = \left\| \left| \frac{\sup\limits_{f\in F} f + \inf\limits_{f\in F} f}{2} - s \right| + \left(\frac{\sup\limits_{f\in F} f - \inf\limits_{f\in F} f}{2} \right) \right\|.$$

Then taking infimum over S, we obtain

$$\inf_{s\in S}\sup_{f\in F}\|f-s\| = \inf_{s\in S}\left\| \left| \frac{\sup\limits_{f\in F} f + \inf\limits_{f\in F} f}{2} - s \right| + \left(\frac{\sup\limits_{f\in F} f - \inf\limits_{f\in F} f}{2} \right) \right\|.$$

By definition of best simultaneous approximant, we have

$$\sup_{f\in F}\|f-s^*\| = \left\| \left| \frac{\sup\limits_{f\in F} f + \inf\limits_{f\in F} f}{2} - s^* \right| + \left(\frac{\sup\limits_{f\in F} f - \inf\limits_{f\in F} f}{2} \right) \right\|.$$

Remark 2.3. If f_1 and f_2 are any two real valued functions in $[a,b]$ then for every $x\in[a,b]$

$$\sup_{i=1,2} f_i(x) + \inf_{i=1,2} f_i(x) = f_1(x)+f_2(x)$$

and

$$\sup_{i=1,2} f_i(x) - \inf_{i=1,2} f_i(x) = \left|\frac{f_1(x)+f_2(x)}{2}\right|.$$

Therefore, we obtain the following result, which was proved in [2] as a special case of the above theorem :

$$\inf_{s\in S}\max\{\|f_1-s\|,\|f_2-s\|\}=\inf_{s\in S}\left\|\left|\frac{f_1+f_2}{2}-s\right|+\frac{f_1-f_2}{2}\right\|.$$

With F as in the above theorem, we have the following.

Theorem 2.4. *s^* is a best simultaneous approximant to F if and only if it is a best simultaneous approximant to $\sup\limits_{f\in F} f$ and $\inf\limits_{f\in F} f$.*

Proof. In [2],

$$\inf_{s\in S}\max\{\|f_1-s\|,\|f_2-s\|\}=\inf_{s\in S}\left\|\left|\frac{f_1+f_2}{2}-s\right|+\frac{f_1-f_2}{2}\right\|.$$

Substituting $f_1=\sup_{f\in F} f$ and $f_2=\inf_{f\in F} f$ in the above expression, we get

$$\max\{\|\sup_{f\in F} f-s^*\|,\inf_{f\in F} f-s^*\}=\left\|\left|\frac{\sup_{f\in F}+\inf_{f\in F}}{2}-s^*\right|+\sup_{f\in F} f-\inf_{f\in F} f\right\|.$$

By previous theorem, we obtain

$$\sup_{f\in F}\|f-s^*\|=\max\{\|\sup_{f\in F} f-s^*\|,\|\inf_{f\in F} f-s^*\|\}.$$

Conversely, suppose

$$\sup_{f\in F}\|f-s^*\|=\max\{\|\sup_{f\in F} f-s^*\|,\|\inf_{f\in F} f-s^*\|\},$$

$$\sup_{f\in F}\|f-s^*\|=\max\{\|\sup_{f\in F} f-s^*\|,\|\inf_{f\in F} f-s^*\|\}.$$

Without loss of generality, suppose

$$\max\{\|\sup_{f\in F} f-s^*\|,\|\inf_{f\in F} f-s^*\|\}=\|\sup_{f\in F} f-s^*\|.$$

Thus

$$\sup_{f\in F}\|f-s^*\|=\|\sup_{f\in F} f-s^*\|\le\inf_{s\in S}\sup_{f\in F}\|f-s\|.$$

Hence

$$\sup_{f\in F}\|f-s^*\|=\inf_{s\in S}\sup_{f\in F}\|f-s\|.$$

Now we consider simultaneous approximation problem in L_1 norm. Let f be a set of uniformly bounded integrable function in $[a,b]$ and S be a non-empty set of integrable functions in $[a,b]$. Then s^* is said to be a best simultaneous approximant to F in L_1 norm if

$$\inf_{s\in S}\int_a^b \sup_{f\in F}|f(x)-s(x)|dx = \int_a^b \sup_{f\in F}|f(x)-s^*(x)|dx.$$

Theorem 2.5. *s^* is a best simultaneous approximant to F in L_1 norm if and only if*

$$\int_a^b \sup_{f\in F}|f(x)-s^*(x)|dx = \int_a^b \max\{|\sup_{f\in F} f(x)-s^*(x)|,|\inf_{f\in F} f(x)-s^*(x)|\}dx.$$

That is, simultaneous approximation to F in L_1 norm is equivalent to simultaneous approximation of the two functions $\sup\limits_{f\in F} f$ and $\inf\limits_{f\in F} f$ in L_1 norm.

Proof. Assume s^* is a best simultaneous approximant to F in L_1 norm. But Theorem 3 of [1] shows that if s^* is a best simultaneous to F in l_1 norm, then

$$\int_a^b \sup_{f\in F}|f(x)-s^*(x)|dx = \int_a^b \left|\frac{\sup\limits_{f\in F} f+\inf\limits_{f\in F} f}{2}-s^*\right|dx + \int_a^b \frac{\sup\limits_{f\in F} f-\inf\limits_{f\in F} f}{2}dx.$$

In [5] they proved that for two functions f_1 and f_2, if $s^*\in S$ is a best simultaneous approximant to f_1 and f_2 in l_1 norm, then

$$\int_a^b \max\{|f_1(x)-s^*(x)|,|f_2(x)-s^*(x)|\}dx = \int_a^b \left|\frac{f_1+f_2}{2}-s^*\right|dx + \int_a^b \left|\frac{f_1-f_2}{2}\right|dx.$$

Substituting $f_1=\sup\limits_{f\in F} f$ and $f_2=\inf\limits_{f\in F} f$, we obtain

$$\int_a^b \max\{|\sup_{f\in F} f(x)-s^*(x)|,|\inf_{f\in F} f(x)-s^*(x)|\}dx =$$
$$\int_a^b \left|\frac{\sup\limits_{f\in F} f+\inf\limits_{f\in F} f}{2}-s^*\right|dx + \int_a^b \left|\frac{\sup\limits_{f\in F} f-\inf\limits_{f\in F} f}{2}\right|dx.$$

Then together with

$$\int_a^b \sup_{f\in F}|f(x)-s^*(x)|dx = \int_a^b \left|\frac{\sup\limits_{f\in F} f+\inf\limits_{f\in F} f}{2}-s^*\right|dx + \int_a^b \frac{\sup\limits_{f\in F} f-\inf\limits_{f\in F} f}{2}dx,$$

we get

$$\int_a^b \sup_{f\in F}|f(x)-s^*|dx = \int_a^b \max\{|\sup_{f\in F} f(x)-s^*(x)|,|\inf_{f\in F} f(x)-s^*(x)|\}dx.$$

Conversely, suppose

$$\int_a^b \sup_{f\in F}|f(x)-s^*|dx = \int_a^b \max\{|\sup_{f\in F} f(x)-s^*(x)|,|\inf_{f\in F} f(x)-s^*(x)|\}dx.$$

If

$$\max\{|\sup_{f\in F} f(x)-s^*(x)|,|\inf_{f\in F} f(x)-s^*(x)|\} = \sup_{f\in F}|f(x)-s^*|,$$

then

$$\begin{aligned}\int_a^b \max\{|\sup_{f\in F} f(x)-s^*(x)|,|\inf_{f\in F} f(x)-s^*(x)|\}dx &= \int_a^b \sup_{f\in F}|f(x)-s^*(x)|dx \\ &\leq \inf_{s\in S}\int_a^b \sup_{f\in F}|f(x)-s(x)|dx.\end{aligned}$$

Thus

$$\int_a^b \sup_{f\in F}|f(x)-s^*(x)|dx = \inf_{s\in S}\int_a^b \sup_{f\in F}|f(x)-s(x)|dx.$$

Therefore $s^* \in S$ is a best simultaneous approximant to F in L_1 norm.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] A. Sermin, Simultaneous approximation of a uniformly bounded set of real valued functions, J. Approx. Theory 45 (1985), 129-132.

[2] J.B. Diaz, H.W. Mc Laughlin, Simultaneous approximation of a set of bounded real functions, Math. Comput. 23 (1969), 584-594.

[3] J.B. Diaz, H.W. Mc Laughlin, On simultaneous Chebyshev approximation and Chebyshev approximation with additive weight, J. Approx. Theory 6 (1972), 68-71.

[4] A.S.B. Hollan, J.H. Mc Cabe, G.M. Philips, B.N. Sahney, Best simultaneous L_1-approximation, J. Approx. Theory 24 (1978), 361-375.

[5] G.M. Philips, B.N. Sahney, On best simultaneous approximation in L_1 and L_2 norms. "Theory of approximations with applications", Law and Sahney, Eds, Academic Press, New York, 1976.