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ANTI FUZZY IDEALS IN TERNARY SEMIGROUPS

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Abstract. We consider the ternary semigroup $\underline{S}$ of the fuzzy points of a ternary semigroup S, and discuss the relation between some anti fuzzy ideals of a ternary semigroup S and subsets $\underline{A}$ of the semigroup $\underline{S}$.

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1. Introduction

The concept of fuzzy set was initiated by L. Zadeh[14]. The study of fuzzy algebraic structures started with the introduction of the concepts of fuzzy groups in the pioneering paper of Rosenfeld [13]. Kuroki [6, 7, 8, 9] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them. M. Santiago and S. Bala developed the theory of ternary semigroups[12]. Kar and Sarkar defined fuzzy left (right, lateral) ideals of ternary semigroups and characterize regular and intra-regular ternary semigroups by using the concept of fuzzy ideals of ternary semigroups[3,4]. The concept of anti fuzzy interior ideals of ternary semigroups introduced in[2]. Kim considered the semigroup $\underline{S}$ of the fuzzy points of a semigroup S, and discussed the relation between some fuzzy ideals of a semigroup S and the subsets of $\underline{S}$ [5]. Hamouda considered the ternary semigroups of fuzzy points and investigated some relations between of ideals of fuzzy points and fuzzy ideals of ternary semigroups [1]. Based on the concept of anti fuzzy ideals of a ternary semigroup [13], in the present paper we consider the

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ternary semigroup $\underline{S}$ of the fuzzy points in a ternary semigroup S, and discuss the relation between some anti fuzzy ideals of a ternary semigroup S and the subsets of $\underline{S}$.

2. Preliminaries

Definition 2.1[12] *A ternary semigroup is a nonempty set S together with a ternary operation $(a, b, c) \to abc$ satisfying $(abc)de = a(bcd)e = ab(cde)$ for all $a; b; c; d; e \in S$.*

Example 2.2 *Let $\mathbb{Z}^-$ be the set of all negative integers. Then with the usual ternary multiplication, $\mathbb{Z}^-$ forms a ternary semigroup.*

Definition 2.3 [3] *A non-empty subset A of a ternary semigroup is called*

1) *A ternary subsemigroup if $A^3 = AAA \subseteq A$.*
2) *A left ideal of S if $SSA \subseteq A$.*
3) *A lateral ideal of S if $SAS \subseteq A$.*
4) *A right ideal of S if $ASS \subseteq A$.*
5) *An ideal of S if A is a left ideal, a lateral ideal and a right ideal of S.*

Definition 2.4 [4] *A ternary subsemigroup B of a ternary semigroup S is said to be a bi-ideal of S if $BSBSB \subseteq B$.*

Definition 2.5 [10] *A ternary subsemigroup B of a ternary semigroup S is called an interior ideal of S if $SSBSS \subseteq B$.*

Example 2.6 Let $S = \{(0,0), (0,1), (1,0), (1,1)\}$. Then S is a ternary semigroup with respect to ternary multiplication defined by

$$(i, j)(k, l)(m, n) = (i, n).$$

Let $A = \{(0,0), (0,1)\}$ be a subset of S.Then A is a right ideal of S, but not a lateral ideal nor a left ideal because

in SAS,

$$(1,0)(0,1)(1,1) = (1,1) \notin A,$$

in SSA,

$$(1,0)(1,1)(0,0) = (1,0) \notin A.$$

Let $B = \{(0,1),(1,1)\}$ be a subset of S. Then B is a left ideal of S, but not a lateral ideal nor a right ideal because

in SBS,

$$(1,0)(1,1)(1,0) = (1,0) \notin B,$$

in BSS,

$$(0,0)(1,1)(0,0) = (0,0) \notin B.$$

A function f from S to the closed interval [0, 1] is called a *fuzzy set* in S. The ternary semigroup S itself is a fuzzy set in S such that $S(x) = 1$ for all $x \in S$.

Definition 2.7[13] *Let f be a fuzzy set in a nonempty set S. For any $t \in [0,1]$; the subset $f_t = \{x \in S: f(x) \leq t\}$ of S is called anti level subset of f.*

Let A and Bbe two fuzzy sets in$S$. Then theinclusion relation $A \subseteq B$is defined by $A(x) \leq B(x)$ for all $x \in S$. $A \cap B$ and $A \cup B$ are fuzzy sets in S defined by $(A \cap B)(x) = min\{A(x), B(x)\} = A(x) \wedge B(x)$,$(A \cup B)(x) = max\{A(x), B(x)\} = A(x) \vee B(x)$, for all $x \in S$.

Definition 2.8 *Let S be a non-empty set and $x \in S, t \in [0,1)$. An anti fuzzy point x_t of S is a fuzzy set in S, defined by,*

$$x_t(y) = \begin{cases} t & if\ x = y, \\ 1 & otherwise, \end{cases}$$

for all $y \in S$.

Definition 2.9.[13] *A non-empty fuzzy set A in a ternary semigroup S is called an anti fuzzy ternary subsemigroup of S if $A(xyz) \leq A(x) \vee A(y) \vee A(z)$ for all $x, y, z \in S$.*

Definition 2.10.[13] *A non-empty fuzzy set* A *in a ternary semigroup S is called an anti fuzzy left (resp. lateral, right) ideal of S if* $A(xyz) \leq A(z)$ *(resp.* $A(xyz) \leq A(y), A(xyz) \leq A(x)$*) for all* $x, y, z \in S$.

If A is an anti fuzzy left ideal, a fuzzy lateral ideal and a fuzzy right ideal of S*, then* A *is called an anti fuzzy ideal of* S.

Definition 2.11.[13] *An anti fuzzy ternary subsemigroup* B *in a ternary semigroup* S *is called an anti fuzzy interior ideal of* S *if* $B(xsary) \leq B(a)$*for all* $x, a, r, s, y \in S$.

Example 2.12. In example 2.6, $S = \{(0,0), (0,1), (1,0), (1,1)\}$ is a ternary semigroup and $A = \{(0,0), (0,1)\}$ is a right ideal of S. Define a fuzzy set f in S as follows:

$$f(x) = \begin{cases} 0.6 & if\ x \in A; \\ 1 & otherwise. \end{cases}$$

It is clear that f is an anti fuzzy right ideal, but not an anti fuzzy lateral ideal nor an anti fuzzy left ideal because

$$f\big((1,1)(0,1)(1,1)\big) = f\big((1,1)\big) = 1 \nleq f\big((0,1)\big) = 0.6,$$

and

$$f\big((1,1)(0,1)(0,1)\big) = f\big((1,1)\big) = 1 \nleq f\big((0,1)\big) = 0.6,$$

Similarly, for the left ideal $B = \{(0,1), (1,1)\}$we can define an anti fuzzy left ideal f which is neither an anti fuzzy lateral ideal nor an anti fuzzy right ideal.

3. Main Results

Let $\boldsymbol{\mathcal{F}}(\boldsymbol{S})$ be the set of all fuzzy sets in a ternary semigroup S. For each $A, B, C \in \boldsymbol{\mathcal{F}}(\boldsymbol{S})$, the anti product of A , B, C is a fuzzy set $A * B * C$ defined as follows:

$$(A * B * C)(x) = \begin{cases} \bigwedge_{x=abc} \{A(a) \vee B(b) \vee C(c)\} & if\ abc = x \\ 1 & otherwise. \end{cases}$$

Proposition 3.1. $(\mathcal{F}(S),\ *)$ *is a ternary semigroup*.

Proof. It is obvious that $\mathcal{F}(S)$ is closed under the ternary operation $*$. Let A, B, C, E, F be fuzzy sets in $\mathcal{F}(S)$. Let x be any element of S such that it is not expreeible as product of three elements in S, then

$$\big((A*B*C)*E*F\big)(x) = 1 = (A*(B*C*E)*F)(x) = \big(A*B*C*(E*F)\big)(x).$$

If $x = abc$ for some a, b, c in S, then

$$\big((A*B*C)*E*F\big)(x) = \bigwedge_{x=abc}\{((A*B*C)(a) \vee E(b) \vee F(c)\} =$$

$$= \bigwedge_{x=abc}\left\{\bigwedge_{a=pqr}\{A(p) \vee B(q) \vee C(r)\} \vee E(b) \vee F(c)\right\}$$

$$= \bigwedge_{x=(pqr)bc}\{A(p) \vee B(q) \vee C(r)\} \vee E(b) \vee F(c)\}$$

$$= \bigwedge_{x=pq(rbc)}\{A(p) \vee B(q) \vee (C(r) \vee E(b) \vee F(c))\}$$

$$= \bigwedge_{x=abc}\left\{\bigwedge_{w=rbc} A(p) \vee B(q) \vee (C*E*F)(w)\right\}$$

$$= \bigwedge_{x=pqw} A(p) \vee B(q) \vee (C*E*F)(w)$$

$$= (A*B*(C*E*F))(x)$$

In similar argument, we show that $\big(A*B*(C*E*F)\big)(x) = (A*(B*C*E)*F)(x)$. Hence $(\mathcal{F}(S),\ *)$ is a ternary semigroup. □

Let $\underline{S}$ be the set of all anti fuzzy points in a ternary semigroup S.

Proposition 3.2. *If* x_α, y_β *and* $z_\gamma \in \underline{S}$, *then* $x_\alpha * y_\beta * z_\gamma = (xyz)_{\alpha\vee\beta\vee\gamma}$.

Proof. Let $w \in S$. If $w \neq abc$ for any $a, b, c \in S$, then

$$(x_\alpha * y_\beta * z_\gamma)(w) = 1 = (xyz)_{\alpha\vee\beta\vee\gamma}(w).$$

If $w = abc$ for some $a, b, c \in S$, then we have

$$(x_\alpha * y_\beta * z_\gamma)(w) = \bigwedge_{w=abc} \{x_\alpha(a) \vee y_\beta(b) \vee z_\gamma(c)\}$$

1) If $x = a, y = b$ and $z = c$, then $w = xyz$ and $x_\alpha(x) = \alpha, y_\beta(y) = \beta$ and $z_\gamma(z) = \gamma$. Therefore, $(x_\alpha * y_\beta * z_\gamma)(w) = x_\alpha(x) \vee y_\beta(y) \vee z_\gamma(z) = \alpha \vee \beta \vee \gamma = (xyz)_{\alpha\vee\beta\vee\gamma}$.
2) If either $x \neq a$ *or* $y \neq b$ or $z \neq c$, then either $x_\alpha(a) = 1$ *or* $y_\beta(b) = 1$ or $z_\gamma(c) = 1$ and hence $(x_\alpha * y_\beta * z_\gamma)(w) = 1 = (xyz)_{\alpha\vee\beta\vee\gamma}(w)$.

Therefore, we conclude that $x_\alpha * y_\beta * z_\gamma = (xyz)_{\alpha\vee\beta\vee\gamma}$. □

It is clear that $(x_\alpha * y_\beta * z_\gamma) * w_\sigma * u_\tau = x_\alpha * \left(y_\beta * z_\gamma * w_\sigma\right) * u_\tau = x_\alpha * y_\beta * (z_\gamma * w_\sigma * u_\tau)$ for $x_\alpha, y_\beta, z_\gamma, w_\sigma, u_\tau \in \underline{S}$. Thus $\underline{S}$ is a ternary subsemigroup of $\boldsymbol{\mathcal{F}}(\boldsymbol{S})$. For any $A \in \boldsymbol{\mathcal{F}}(\boldsymbol{S})$, we denote $\underline{A} = \{x_\alpha \in \underline{S}: A(x) \leq \alpha\}$. For any $A, B, C \subseteq \underline{S}$, we define the product of $A, B$and C as $A * B * C = \left\{x_\alpha * y_\beta * z_\gamma: x_\alpha \in A, y_\beta \in B, z_\gamma \in C\right\}$.

Lemma 3.3. *Let A, B and C be fuzzy sets in a ternary semigroup S. Then*

a) *If* $A \subseteq B$, then $\underline{B} \subseteq \underline{A}$.

b) $\underline{A \cup B} = \underline{A} \cup \underline{B}$.

c) $\underline{A \cap B} = \underline{A} \cap \underline{B}$.

d) $\underline{A * B * C} \supseteq \underline{A} * \underline{B} * \underline{C}$.

Proof. (a) Straightforward.

b) Let $z_\alpha \in \underline{A \cup B}$, then

$$(A \cup B)(z) = A(z) \vee B(z) \leq \alpha.$$

Hence, $A(z) \leq \alpha$ *or* $B(z) \leq \alpha$, that is , $z_\alpha \in \underline{A} \cup \underline{B}$. This implies that $\underline{A \cup B} \subseteq \underline{A} \cup \underline{B}$. Let $z_\alpha \in \underline{A} \cup \underline{B}$, then $(z) \leq \alpha$ *or* $B(z) \leq \alpha$ and hence$(A \cup B)(z) \leq \alpha$. This implies that $z_\alpha \in \underline{A \cup B}$ and consequently, $\underline{A} \cup \underline{B} \subseteq \underline{A \cup B}$. Therefore, $\underline{A \cup B} = \underline{A} \cup \underline{B}$.

(c) the proof is similar to (a), by considering the suitable modifications.

(d) Let $z \in S$and$z_\omega \in \underline{A} * \underline{B} * \underline{C}$, then $z_\omega = a_\alpha * b_\beta * c_\gamma$ such that $a_\alpha \in \underline{A}$, $b_\beta \in \underline{B}$ and $c_\gamma \in \underline{C}$. If $z = pqr$ for some$p, q, r \in S$, then $A(p) \leq a_\alpha(p)$, $B(q) \leq b_\beta(q)$ and $C(r) \leq c_\gamma(r)$. Hence, we have $A(p) \leq \wedge_{a_\alpha \in \underline{A}}\ a_\alpha(p)$, $B(q) \leq \wedge_{b_\beta \in \underline{B}}\ b_\beta(q)$ and $C(r) \leq \wedge_{c_\gamma \in \underline{C}}\ c_\gamma(r)$. Thus

$$(A * B * C)(z) = \bigwedge_{z=pqr} A(p) \vee B(q) \vee C(r)$$

$$\leq \bigwedge_{z=pqr} \bigwedge_{a_\alpha \in \underline{A},\, b_\beta \in \underline{B},\, c_\gamma \in \underline{C}} a_\alpha(p) \vee b_\beta(q) \vee c_\gamma(r)$$

$$= \bigwedge_{a_\alpha \in \underline{A},\, b_\beta \in \underline{B},\, c_\gamma \in \underline{C}} \bigwedge_{z=pqr} a_\alpha(p) \vee b_\beta(q) \vee c_\gamma(r)$$

$$= \bigwedge_{a_\alpha \in \underline{A},\, b_\beta \in \underline{B},\, c_\gamma \in \underline{C}} \left(a_\alpha * b_\beta * c_\gamma\right)(z) = \bigwedge_{a_\alpha \in \underline{A},\, b_\beta \in \underline{B},\, c_\gamma \in \underline{C}} z_\omega(z) = \omega.$$

This implies that $z_\omega \in \underline{A * B * C}$ and hence $\underline{A * B * C} \supseteq \underline{A} * \underline{B} * \underline{C}$. □

Theorem 3.4. *Let A be a fuzzy set in a ternary semigroup S. then the following conditions are equivalent:*

a) A is an anti fuzzy left (lateral, right) ideal of S.

b) $\underline{A}$ *is a left (lateral, right)ideal of* $\underline{S}$.

Proof. Let A is an anti fuzzy left ideal in S, and let $x_p \in \underline{A}$ and $y_q, z_r \in \underline{S}$. Then $y_q * z_r * x_p = (yzx)_{q\vee r\vee p} \in \underline{S} * \underline{S} * \underline{A}$. Since A is an anti fuzzy left ideal, we have $A(yzx) \leq A(x) \leq p \leq q \vee r \vee p$. Hence $y_q * z_r * x_p = (yzx)_{q\vee r\vee p} \in \underline{A}$. This implies that $\underline{S} * \underline{S} * \underline{A} \subseteq \underline{A}$, thus $\underline{A}$ is a left ideal of $\underline{S}$. Conversely, assume that $\underline{A}$ is a left ideal of $\underline{S}$. Let $x, y, z \in S$, if $A(z) = 1$, then $A(xyz) \leq 1 = A(z)$. If $A(z) \neq 1$ then $z_{A(z)} \in \underline{A}$ and $x_{A(z)}, y_{A(z)} \in \underline{S}$. Since $\underline{A}$ is a left ideal of $\underline{S}$, we have $x_{A(z)} * y_{A(z)} * z_{A(z)} = (xyz)_{A(z)} \in \underline{S} * \underline{S} * \underline{A} \subseteq \underline{A}$. This implies that $A(xyz) \leq A(z)$, and hence A is an anti fuzzy left ideal of S. By a similar argument, one can prove the other cases.
□

Lemma 3.5. *Let A and B be any anti fuzzy interior ideals of a ternary semigroup S.Then*

a) $A \cup B$ *is also an anti fuzzy interior ideal of S (provided* $A \cup B \neq \emptyset$*) .*

b) $\underline{A} \cup \underline{B}$ *is also an interior ideal of* $\underline{S}$.

Proof. a) Since A and B are anti fuzzy ternary subsemigroups of S, $A \cup B$ is an anti fuzzy ternary subsemigroup of S [13]. Let $x, a, r, s, y \in S$ be arbitrary elements of S. Since A and B are anti fuzzy interior ideals of S, then

$$(A \cup B)(xsary) = A(xsary) \vee B(xsary)$$
$$\geq A(a) \vee B(a) = (A \cup B)(a).$$

Hence $A \cup B$ is an anti fuzzy interior ideal of S .

b) At first, it is an easy exercise to show that: A is an anti fuzzy ternary subsemigroup of S if and only if $\underline{A}$ is a ternary subsemigroup of $\underline{S}$. From lemma 3.3, we have $\underline{A} \cup \underline{B} = \underline{A \cup B}$ and so it is a ternary subsemigroup of $\underline{S}$. Let $a_\alpha \in \underline{A \cup B}$ and $x_p, \acute{x}_r, \acute{y}_s, y_q \in \underline{S}$, then

$$(x\acute{x}a\acute{y}y)_{p \vee r \vee \alpha \vee s \vee q} = x_p * \acute{x}_r * a_\alpha * \acute{y}_s * y_q \in \underline{S} * \underline{S} * \underline{A \cup B} * \underline{S} * \underline{S}.$$

Since A ∪ B is a fuzzy interior ideal of S, then

$$(A \cup B)(x\acute{x}a\acute{y}y) \leq (A \cup B)(a) = A(a) \wedge B(a) \leq \alpha \wedge \alpha = \alpha$$
$$\leq p \vee r \vee \alpha \vee s \vee q.$$

This implies that

$$x_p * \acute{x}_r * a_\alpha * \acute{y}_s * y_q = (x\acute{x}a\acute{y}y)_{p \vee r \vee \alpha \vee s \vee q} \in \underline{A \cup B}.$$

Therefore, $\underline{A} \cup \underline{B}$ is also an interior deal of $\underline{S}$. □

Theorem 3.6. *Let A be a fuzzy set in a ternary semigroup S. Then* $\underline{A}$ *is an interior ideal of* $\underline{S}$ *if and only if A is an anti fuzzy interior ideal of S.*

Proof. Let A is an anti fuzzy interior ideal of S, then $\underline{A}$ is a ternary subsemigroup of $\underline{S}$. Suppose that $x_p, \acute{x}_r, \acute{y}_s, y_q \in \underline{S}$ and $z_\alpha \in \underline{A}$. Then $A(z) \leq \alpha$, and $A(x\acute{x}z\acute{y}y) \leq A(z) \leq \alpha \leq p \vee r \vee \alpha \vee s \vee q$.Hence $\underline{S} * \underline{S} * \underline{A} * \underline{S} * \underline{S} \ni (x_p * \acute{x}_r * a_\alpha * \acute{y}_s * y_q) = (x\acute{x}z\acute{y}y)_{p \vee r \vee \alpha \vee s \vee q} \in \underline{A}$. This implies that $\underline{S} * \underline{S} * \underline{A} * \underline{S} * \underline{S} \subseteq \underline{A}$, thus $\underline{A}$ is an interior ideal of $\underline{S}$. Conversely, suppose that $\underline{A}$ is an interior ideal of $\underline{S}$. For all $, y, z \in S$, the elements $x_{A(x)}, y_{A(y)}, z_{A(z)}$ belong to $\underline{A}$. Since $\underline{A}$ is an interior ideal of $\underline{S}$, we have

$$x_{A(x)} * y_{A(y)} * z_{A(z)} = (xyz)_{A(x) \vee A(y) \vee A(z)} \in \underline{A}.$$

Thus $A(xyz) \leq A(x) \vee A(y) \vee A(z)$. Therefore A is an anti fuzzy ternary subsemigroup in S. Let $x, \acute{x}, z, \acute{y}, y \in S$, if $A(z) \neq 1$, then $z_{A(z)} \in \underline{A}$ and $x_{A(z)}, \acute{x}_{A(z)}, y_{A(z)}, \acute{y}_{A(z)} \in \underline{S}$. Since $\underline{A}$ is an interior ideal of $\underline{S}$, we have $(x\acute{x}z\acute{y}y)_{A(z)} = (x\acute{x}z\acute{y}y)_{A(z) \vee A(z) \vee A(z) \vee A(z) \vee A(z)} = x_{A(z)} * x'_{A(z)} *$

$z_{A(z)} * y'_{A(z)} * y_{A(z)} \in \underline{A}$. This implies tha $A(x\acute{x}z\acute{y}y) \leq A(z)$, and hence A is an anti fuzzy interior ideal of S. □

Let S be a ternary semigroup. An element $x \in S$ is called *regular* if there exists an element $a \in S$ such that $x = xax$. A ternary semigroup iscalled *regular* if all its elements are regular [3].

Theorem 3.8. *Let A be a fuzzy set in a regular ternary semigroup S. Then the following conditions are equivalent:*

a) A is an anti fuzzy ideal of S.

b) $\underline{A}$ is an interior ideal of $\underline{S}$.

Proof. Let A be an anti fuzzy ideal of S. Then A is an anti fuzzy ternary subsemigroup of S, and consequently $\underline{A}$ is a ternary subsemigroup of $\underline{S}$. Since any anti fuzzy ideal of S is an anti fuzzy interior ideal of S[13], then theorem 3.6 implies that $\underline{A}$ is an interior ideal of $\underline{S}$. Assume that (b) holds. Let $x \in S$, then there exists $a \in S$ such that $x = xax$ (since S is regular). If $A(x) = 1, A(xyz) \leq 1 = A(x)$. If $A(x) \neq 1$, then $x_{A(x)} \in \underline{A}$ and $y_{A(x)}, z_{A(x)} \in \underline{S}$. Since $\underline{A}$ is an interior ideal of $\underline{S}$, then

$$(xyz)_{A(x)} = (xaxyz)_{A(x)} = x_{A(x)} * a_{A(x)} * x_{A(x)} * y_{A(x)} * z_{A(x)} \in \underline{A}.$$

This implies that $A(xyz) \leq A(x)$, and hence A is an anti fuzzy right ideal of S. In a similar argument we prove that A is an anti fuzzy left ideal of S. It remains to show that A is an anti fuzzy lateral ideal of S. For this purpose, assume that $y, a \in S$ such that $y = yay$ (since S is regular). Since $A(y) = A(yay) \leq A(y) \vee A(a) \vee A(y)$, $A(a) \leq A(y)$. If $A(y) \neq 1$, then $y_{A(y),} a_{A(y)} \in \underline{A}$ and $x_{A(y)}, z_{A(y)} \in \underline{S}$. Since $\underline{A}$ is an interior ideal of $\underline{S}$, we have $(xyz)_{A(y)} = (xyayz)_{A(y)} = x_{A(y)} * y_{A(y)} * a_{A(y)} * y_{A(y)} * z_{A(y)} \in \underline{A}$. This implies that $A(xyz) \leq A(y)$, and hence A is an anti fuzzy lateral ideal of S.This completes that A is an anti fuzzy ideal of .□

A ternary semigroup S is called *intra-regular* if for each element $a \in S$, there exist elements $x, y \in S$ such that $a = xa^3y$ [3].For example, let $S = \{i, 0, -i\}$. Then S is a ternary semigroup under the multiplication over complex numbers. In S, we have$(-i)(i^3)(-i) = i$, $(i)(0^3)(-i) = 0$ and $(i)(-i)^3(i) = -i$. Therefore, $S = \{i, 0, -i\}$ is intra-regular.

Theorem 3.9. *A ternary semigroup S is intra-regular if and only if $\underline{S}$ is intra-regular.*

Proof.(⇒)Let a_α be an element in $\underline{S}$. Since S is intra-regular and $a \in S$, there exist $x, y \in S$ such that $a = xa^3y$.Thus x_α , $y_\alpha \in \underline{S}$ and $x_\alpha * a_\alpha * a_\alpha * a_\alpha * y_\alpha = (xa^3y)_\alpha = a_\alpha$. Hence $\underline{S}$ is intra-regular.

(⇐) Assume $\underline{S}$ is intra-regular and $a \in S$. Then for any $\alpha \in [0,1)$, there exist $x_\beta, y_\gamma \in \underline{S}$ such that $a_\alpha = x_\beta * a_\alpha * a_\alpha * a_\alpha * y_\gamma = (xa^3y)_{\beta\vee\alpha\vee\gamma}$. This implies that $a = xa^3y$ for $x, y \in S$, hence S is intra-regular.

An anti fuzzy ternary subsemigroup A of a ternary semigroup S is called *an anti fuzzy bi-ideal* of S if $A(xaybz) \le A(x) \vee A(y) \vee A(z)$ for all $x; a; y; b; y \in S$[10].

Theorem 3.10 *An anti fuzzy ternary subsemigroup B of a ternary semigroup S is an anti fuzzy bi-ideal of S if and only if* $(B * \Theta * B * \Theta * B) \supseteq B$.

Where Θ is the fuzzy subset of S mapping every element of S on 0.

Proof. Let B be an anti fuzzy bi-ideal of a ternary semigroup S and $x \in S$. If $x \neq abc$ for any $a, b, c \in S$, then $(B * \Theta * B * \Theta * B)(x) = 1 \ge B(x)$. If such exists, let $x \neq abc$ for some $a, b, c \in S$, then

$$(B * \Theta * B * \Theta * B)(x)$$

$$= \bigwedge_{x=abc} \{(B * \Theta * B)(a) \vee \Theta(b) \vee B(c)\}$$

$$= \bigwedge_{x=abc} \left\{ \{ \bigwedge_{a=pqr} B(p) \vee \Theta(q) \vee B(r)\} \Theta(b) \vee B(c) \right\}$$

$$= \bigwedge_{x=abc} \left\{ \{ \bigwedge_{a=pqr} B(p) \vee B(r)\} \vee B(c) \right\} \ge \bigwedge_{x=pqrbc} \{B(p) \vee B(r)\} \vee B(c)\}$$

$$\ge \bigwedge_{x=pqrbc} \{B(pqrbc)\} = \bigwedge_{x=pqrbc} \{B(x)\} = B(x).$$

This implies that $(B * \Theta * B * \Theta * B) \supseteq B$.

Conversely, let B be an anti fuzzy ternary subsemigroup of S such that $(B * \Theta * B * \Theta * B) \supseteq B$. Let $u, v, w, x, y \in S$. Then

$$B(uvwxy) \leq (B * \Theta * B * \Theta * B)(uvwxy)$$

$$= \bigwedge_{uvwxy=abc} \{(B * \Theta * B)(a) \vee \Theta(b) \vee B(c)\}$$

$$\leq (B * \Theta * B)(uvw) \vee \Theta(x) \vee B(y) = \bigwedge_{uvw=pqr} \{B(p) \vee \Theta(q) \vee B(r)\} \vee B(y)$$

$$\leq B(u) \vee \Theta(v) \vee B(w) \vee B(y) = B(u) \vee B(w) \vee B(y).$$

Hence B is an anti fuzzy bi-ideal of S. □

Theorem 3.11.(see [13, Theorem 3.8]). *An anti fuzzy ternary subsemigroup$f$ of a semigroup S is an anti fuzzy bi-ideal of S if and only if the anti level set of f, f_t is a bi-ideal of S for $t \in Im\, f$.*

Theorem 3.112. *Let A be a fuzzy set in a ternary semigroup S. Then A is an anti fuzzy bi- ideal of S if and only if $\underline{A}$ is a bi- ideal of $\underline{S}$.*

Proof. Let A be an anti fuzzy bi- ideal of S, then by theorem 3.10, $A * \Theta * A * \Theta * A \supseteq A$. From lemma 3.3, we have $\underline{A} * \underline{\Theta} * \underline{A} * \underline{\Theta} * \underline{A} \subseteq \underline{A * \Theta * A * \Theta * A} \subseteq \underline{A}$. Since $\underline{A}$ is a ternary subsemigroup of $\underline{S}$, we conclude that $\underline{A}$ is a bi-ideal of $\underline{S}$. Conversely, we assume that $\underline{A}$ is a bi-ideal of $\underline{S}$. In order to prove that A be an anti fuzzy bi- ideal of S , by theorem 3.11, it is sufficient to show that the anti level set of A, $A_t = \{x \in S : A(x) \leq t\}$, is bi-ideal. So, for $x, y, z \in A_t$, it is clear that $x_t, y_t, z_t \in \underline{A}$. Since $\underline{A}$ is a bi-ideal of $\underline{S}$, then $w_t = (xaybz)_t = x_t * a_t * y_t * b_t * z_t \in \underline{A} * \underline{\Theta} * \underline{A} * \underline{\Theta} * \underline{A} \subseteq \underline{A}$. Hence $A(xaybz) \leq t$ and consequently $(xaybz) \in A_t$ for $a, b \in S$. this means that $A_t S A_t S A_t \subseteq A_t$, that is, A_t is a bi-ideal of S. Now, and by the fact that A is an anti fuzzy ternary semigroup of S , it follows that A is an anti fuzzy bi- ideal of S. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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