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PERIODIC SOLUTIONS OF A CLASS OF SINGULAR RADIALLY SYMMETRIC PERTURBATIONS SYSTEMS

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Abstract. In this paper, we study the existence of infinitely many periodic solutions to planar radially symmetric systems with repulsive singular forces. The proof of the main result relies on topological degree theory and the global continuation principle of Leray-Schauder, together with a truncation technique. Recent result in the literature are generalized and significantly improved.

Keywords: Periodic solution; Singular systems; Topological degree.

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1. Introduction

In this work, we are concerned with the existence of positive periodic solutions for the following systems

(1) $$\ddot{x} = (f(t,|x|) - k^2|x|)\frac{x}{|x|}, \quad x \in \mathbb{R}^2 \setminus \{0\},$$

where $0 < k < \pi/T$ is a constant and the function $f : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ is $T-$ periodic in the time variable t for some $T > 0$, and satisfied L^1- Carathéodory condition, also $f(t,r)$ may be

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singular at $r = 0$, we therefore look for non-collision solutions, i.e., solutions which never attain the singularity.

Roughly speaking, systems (1) is singular at 0 means that $f(t,r)$ becomes unbounded when $r \to 0^+$. We say that (1) is of repulsive type (attractive type) if $f(t,r) \to +\infty$ (respectively $f(t,r) \to -\infty$) when $r \to 0^+$.

Such a type of singular systems appears in many problems of applications. Such as, if we take $f(t,r) = k^2 r - c/r^2 (c > 0)$, it is the famous Newtonian equation

$$\ddot{x} = -\frac{cx}{|x|^3}, \quad x \in \mathbb{R}^2 \setminus \{0\},$$

to describe the motion of a particle subjected to the gravitational attraction of a sun which lies at the origin, and if take $f(t,r) = k^2 r + c/r^2$, (1) may be used to model Rutherford's scattering of α particles by heavy atomic nuclei.

The question of existence of non-collision periodic orbits for scalar equations and dynamical systems with singularities has attracted much attention of many researchers over many years, such as [1, 3, 5, 8, 12, 18, 19, 25, 30]. There are two main lines of research in this area. The first one is the variational approach [2, 20, 21]. Usually, the proof requires some strong force condition, which was first introduced with this name by Gordon in [13], although the idea goes back at least to Poincaré[16]. Gordon's result, later improved by Capozzi, Greco and Salvatore [7], we stated it as follows.

Theorem 1.1. *Let $x(t) \in \mathbb{R}^2$ and the following assumptions hold .*

(A_1) *the function V to be $T-$ periodic in t, differentiable in $x \neq 0$ with continuous, and such that*

$$\lim_{x \to 0} V(t,x) = -\infty,$$

(A_2) *there exist $v \in [0,2)$ and positive constant c_1, c_2 such that*

$$V(t,x) \leq c_1 |x|^v + c_2$$

for every t and $x \neq 0$.

(A$_3$) *there are a C^1-function $U:\mathbb{R}^2\backslash\{0\}$, a neighborhood $\mathscr{N}$ of 0 and a positive constant c_3 such that*

$$\lim_{x\to 0} U(x) = -\infty \quad and \quad -V(t,x) \geq |\nabla U(x)|^2 - c_3,$$

for every $x \in \mathscr{N}\setminus\{0\}$, then, for every integer $k \geq 1$, system

$$\ddot{x} + \nabla V(t,x) = 0,$$

has a periodic solution with minimal period kT.

The strong force conditions $(A_2),(A_3)$ guarantees that the minimization procedure does not lead to a collision orbit. This similar condition has been widely used for a voiding collisions in the singularity case. For example, if we consider the system

$$\ddot{x} = \frac{1}{|x|^{\alpha}} + f(t),$$

the strong force condition corresponds to the case $\alpha \geq 2$.

Besides the variational approach, topological methods have been widely applied, starting with the pioneering paper of Lazer and Solimini [14]. In particular, some classical tools have been used to study singular differential equations and dynamical systems in the literature, including the degree theory [9, 26, 28, 29], the method of upper and lower solutions [4, 17], Schauder's fixed point theorem [11, 24], some fixed point theorems in cones for completely continuous operators [22, 23] and a nonlinear Leray-Schauder alternative principle [15, 31]. Contrasting with the variational setting, the strong force condition plays here a different role linked to repulsive singularities. A counterexamper in the paper of Lazer and Solimini [14] shows that a strong force assumption (unboundedness of the potential near the singularity) is necessary in some sense for the existence of positive periodic solutions in the scalar case.

However, compared with the case of strong singularities, the study of the existence of periodic solutions under the presence of weak singularities by topological methods is more recent and the number of references is much smaller, several existence results can be found in [18, 24].

Our main motivation is to obtain by the recent papers [10], we recall it as follows.

Theorem 1.2. *Assume the following hypotheses:*

$$(\mathrm{A}_1') \qquad \bar{e} := \frac{1}{T}\int_0^T e(t)dt \leq 0;$$

$$(\mathrm{A}_2') \qquad \lim_{r\to\infty} r^3 h(t,r) = +\infty;$$

$$(\mathrm{A}_3') \qquad \lim_{r\to\infty} \frac{h(t,r)}{r} = 0;$$

uniformly for almost every t. Then there exists $k_1 \geq 1$ such that, for every integer $k \geq k_1$, the equation

$$\ddot{x} = (-h(t,|x|) + e(t))\frac{x}{|x|},$$

has a periodic solution $x_k(t)$ with minimal period kT, which makes exactly one revolution around the origin in the period time kT. Moreover,

$$\lim_{k\to\infty}(\min|x_k|) = +\infty.$$

The aim of this paper is to show that the topological degree theorem can be applied to the periodic problem. We prove the existence of large-amplitude periodic solutions whose minimal period is an integer multiple of T.

The rest of this paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, by the use of topological degree theory, we will state and prove the main results. To illustrate the new results, some applications are also given.

Let us fix some notation to be used, we write C_T the $T-$ periodic continuous space, denote by p^* and p_* the essential supremum and infirmum of a given function $p \in L^1[0,T]$, if they exist, by $\|\cdot\|$ the supremum norm of $C[0,T]$.

2. Preliminaries

In this section, we present some results which will be need in sections 3. We may write the solutions of (1) in polar coordinates:

$$(1) \qquad x(t) = \rho(t)(\cos\varphi(t), \sin\varphi(t)).$$

Equation (1) is then equivalent to the system

$$\begin{cases} \ddot{\rho} + k^2\rho = f(t,\rho) + \frac{\mu^2}{\rho^3}, \\ \rho^2\dot{\varphi} = \mu, \end{cases} \tag{2}$$

where μ is the (scalar) angular momentum of $x(t)$. Recall that μ is constant in time along any solution. In the following, when considering a solution of (2) we will always implicitly assume that $\mu \geq 0$ and $\rho > 0$.

If x is a $T-$radially periodic, then ρ must be $T-$ periodic. In combination with the first equation of (2). We thus consider the boundary value problem

$$\begin{cases} \ddot{\rho} + k^2\rho = f(t,\rho) + \frac{\mu^2}{\rho^3}, \\ \rho(0) = \rho(T), \quad \dot{\rho}(0) = \dot{\rho}(T). \end{cases} \tag{3}$$

Lemma 2.1 [12] *Assume that $h : [0,T] \to \mathbb{R}$ is continuous, $k \in (0,\pi/T)$ is a constant. Then the equation*

$$u'' + k^2u = h(t),$$

with periodic boundary conditions

$$u(0) = u(T), \quad u'(0) = u'(T),$$

has a unique solution $u \in C^2[0,T]$ with the representation

$$u(t) = \int_0^T G(t,s)h(s)ds,$$

where

$$G(t,s) = \begin{cases} \frac{\sin k(t-s)+\sin k(T-t+s)}{2k(1-\cos kT)}, & 0 \leq s \leq t \leq T, \\ \frac{\sin k(s-t)+\sin k(T-s+t)}{2k(1-\cos kT)}, & 0 \leq t \leq s \leq T, \end{cases} \tag{4}$$

is the Green function.

It is obvious that $G(t,s) > 0$ for $0 \leq s,t \leq T$, and we denote

$$m = \min_{0\leq s,t\leq T} G(t,s), \quad M = \max_{0\leq s,t\leq T} G(t,s), \quad \sigma = m/M.$$

A direct calculation shows that

$$m = \frac{1}{2k}\cot\frac{kT}{2}, M = \frac{1}{2k}\csc\frac{kT}{2}, \sigma = \cos\frac{kt}{2} < 1, \int_0^T G(t,s)ds = \frac{1}{k^2}.$$

Let $X = C[0,T]$, we suppose that $f : [0,T] \times \mathbb{R} \to [0,\infty)$ is a continuous function. Define an operator:

$$(\mathscr{A}\rho)(t) = \int_0^T G(t,s) f(s,\rho(s)) ds$$

for $\rho \in X$ and $t \in [0,T]$. It is easy to prove that $\mathscr{A}$ is continuous and completely continuous.

3. Main results

In this section we dedicated the existence of one positive solution to the equation (1); here $k \in (0,\pi/T)$ is a constant and term $f(t,r)$ may be singular at $r = 0$, and satisfied L^1- Carathéodory condition, recall that L^1- Carathéodory means

(i) $f(\cdot,r)$ is measurable and $T-$ periodic, for every $r > 0$;

(ii) $f(t,\cdot)$ is continuous, for almost every $t \in \mathbb{R}$.

The proof is used the following global continuation principle of Leray-Schauder, which can be found in [27].

Lemma 3.1. *Let the operator $H : [\mu_1,\mu_2] \times \bar{G} \to X$ be a compact, where G is a bounded open set in the Banach space X. Then equation*

$$x - H(\mu,x) = 0, \quad \mu \in \mathbb{R}, \quad x \in X, \tag{1}$$

has a continuum $\mathscr{C}$ of solutions in $\mathbb{R} \times X$ which connects the set $\{\mu_1\} \times G$ with the set $\{\mu_2\} \times G$, if the following condition be satisfied.

(I) *$deg(H,G) \neq 0$,*

(II) *equation (1) has no solution on $[\mu_1,\mu_2] \times \partial G$.*

Now we present our main existence result of one positive solution to problem (1).

Theorem 3.2. *Let the following assumptions hold .*

$$\frac{a(t)}{x^\alpha} \leq f(t,x) \leq \frac{b(t)}{x^\alpha}. \tag{H}$$

where $\alpha > 0, a(t), b(t)$ are continuous positive $T-$ periodic function. Then there exist a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, equation (1) has a periodic solution with minimal period kT. Moreover, one of the following two alternative holds

(i) *if μ_k denotes the angular momentum associated to $x_k(t)$ then*

$$\lim_{k\to\infty} \min |x_k(t)| = +\infty \quad and \quad \lim_{k\to\infty} \mu_k = +\infty;$$

(ii) *exist constant $C > 0$ (independent of μ and k) such that*

$$\frac{1}{C} < |x_k(t)| < C, \quad for\ every\ t \in \mathbb{R} \quad and\ every \quad k \geq k_1,$$

and

$$\lim_{k\to\infty} \mu_k = 0.$$

Now we begin by showing that Theorem 3.2(i), for μ large enough, the conclusion hold. By Lemma 2.1 , we know ρ is a positive $T-$periodic solution of (3) if and only if ρ is a positive fixed point in C_T of the following operator

$$(T\rho)(t) = \int_0^T G(t,s)[f(t,\rho) + \frac{\mu^2}{\rho^3}]ds,$$

where $G(t,s)$ is the Green function given by (4).

Let us define the truncation modified $f_n : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ as follows:

$$f_n(t,\rho) = \begin{cases} f(t,\rho), & \text{if } \rho \geq 1/n, \\ f(t,1/n), & \text{if } \rho \leq 1/n. \end{cases} \tag{2}$$

We consider the equation

$$\ddot{\rho} + k^2\rho = f_n(t,\rho) + \frac{\mu^2}{\rho^3} + \frac{k^2}{n}, \tag{3}$$

and define an operator T_n by

$$(T_n\rho)(t) = \int_0^T G(t,s)[f_n(t,\rho) + \frac{\mu^2}{\rho^3} + \frac{k^2}{n}]ds.$$

We will prove that

$$\lim_{n\to\infty} T_n = T. \tag{4}$$

We use topological degree theory, and will eventually find appropriate open set Ω such that

$$deg(T_n, \Omega, 0) = deg(T, \Omega, 0).$$

So we only show that

$$deg(T_n, \Omega, 0) \neq 0,$$

i.e, Eq.(3) has solution. To this aim, the solution existence of the problem (3) is proved by deforming Eq.(3) to a simpler $T-$periodic problem

$$\ddot{\rho}+k^2\rho=\frac{\mu^2}{\rho^3}+\frac{c}{\rho^\alpha}+\frac{k^2}{n}$$

where $c=a_*$. We establish the homotopy equation is

$$\ddot{\rho}+k^2\rho=f_n(t,\rho;\lambda)+\frac{\mu^2}{\rho^3}+\frac{k^2}{n}, \tag{5}$$

where

$$f_n(t,\rho;\lambda)=\lambda f_n(t,\rho)+(1-\lambda)\frac{c}{\rho^\alpha}.$$

Thus we need to find a priori estimates for the possible $T-$periodic solutions of the corresponding homotopy equation (5). Note that $f_n(t,\rho;\lambda)$ also satisfies the condition (H) uniformly with respect to $\lambda\in[0,1]$, and the L^1- Carathéodory condition.

Lemma 3.3. *Assume ρ is a $T-$perodic solution of (5), and $\lambda\in[0,1]$, then*

$$\min_{0\le t\le T}\rho(t)\ge\sigma\|\rho\|.$$

Proof. Let T' be the operator associated with equation (5). Following from Lemma 2.1, we know that

$$\rho(t)=(T'\rho)(t)=\int_0^T G(t,s)[f_n(t,\rho;\lambda)+\frac{\mu^2}{\rho^3}+\frac{k^2}{n}].$$

Then

$$\begin{aligned}
\min_{0\le t\le T}\rho(t) &= \min_{0\le t\le T}(T'\rho)(t)\\
&= \min_{0\le t\le T}\int_0^T G(t,s)[f_n(t,\rho;\lambda)+\frac{\mu^2}{\rho^3}+\frac{k^2}{n}]ds\\
&\ge m\int_0^T[f_n(t,\rho;\lambda)+\frac{\mu^2}{\rho^3}+\frac{k^2}{n}]ds\\
&\ge \sigma\max_{0\le t\le T}\int_0^T G(t,s)[f_n(t,\rho;\lambda)+\frac{\mu^2}{\rho^3}+\frac{k^2}{n}]ds\\
&= \sigma\|T'\rho\|=\sigma\|\rho\|.
\end{aligned}$$

This completes the proof. □

Lemma 3.4. *For every $\Gamma > 0$, let $\mu(\Gamma) \geq K > 0$ (K is a constant), if $\mu \geq \mu(\Gamma), \lambda \in [0,1]$, and ρ is a $T-$perodic solution of (5), then $\|\rho\| > \Gamma$.*

Proof. Using a contradiction argument, assume that the result were wrong. Then there are some $\Gamma > 0$ and sequence $(\lambda_n)_n, (\mu_n)_n$ and $(\rho_n)_n$ such that ρ_n is a $T-$periodic solution of (5), and $(\lambda_n)_n \in [0,1], \lim_{n\to\infty} \mu_n = +\infty$ for $\lambda = \lambda_n$ and $\mu = \mu_n$, with $\|\rho_n\| < \Gamma$. Multiplying (5) by $\rho_n^3(t)$ and integrating from 0 to T, we obtain

$$-3\int_0^T \dot{\rho}_n^2(t)\rho_n^2(t)dt + \int_0^T k^2\rho_n^4(t)dt = \int_0^T f_n(t,\rho_n(t);\lambda)\rho_n^3(t)dt + \int_0^T \frac{k^2\rho_n^3(t)}{n}dt + \mu_n^2 T.$$

Then, we have from condition (H), for all t,

$$\begin{aligned} \mu_n^2 T &= \int_0^T k^2\rho_n^4 dt - 3\int_0^T \dot{\rho}_n^2\rho_n^2(t)dt - \int_0^T f_n(t,\rho_n(t);\lambda)\rho_n^3(t)dt - \int_0^T \frac{k^2\rho_n^3}{n}dt \\ &\leq \int_0^T k^2\rho_n^4 dt = \frac{k^2\Gamma^5}{5}. \end{aligned}$$

This is a contradiction to the fact of $\mu_n \to +\infty$. $\square$

Let us now fix $\bar{\mu} = \mu(\Gamma) = K$.

Lemma 3.5. *Give A,B, with $\bar{\mu} \leq A \leq B$, if $\mu \in [A,B], \lambda \in [0,1]$. Then exist two constants $C', \delta > 0$ such that any positive $T-$perodic solution $\rho(\cdot)$ to (5) satisfies*

$$\delta < \rho(t) < C', \quad |\dot{\rho}(t)| < C'.$$

for every $t \in [0,T]$.

Proof. $T-$perodic of (5) is equivalent to the following fixed point problem in $C[0,T]$,

$$\rho = T'\rho. \tag{6}$$

Let

$$R = \max\{\sqrt[3]{\frac{B^2+b^*}{k^2\sigma^3}}, \sqrt[\alpha]{\frac{B^2+b^*}{k^2\sigma^\alpha}}\} + 2.$$

We claim that any fixed point ρ of (6) must satisfy $\rho(t) < R$, i.e., $\|\rho\| < R$. Otherwise, assume that ρ is a solution of (6) such that $\|\rho\| \geq R$. Note that $\int_0^T G(t,s)ds = \frac{1}{k^2}$ and $f_n(t,\rho) \geq 0$. By Lemma 3.3, for all t. We have immediately that

$$\rho(t) \geq \frac{1}{n} \quad \text{and} \quad \rho(t) \geq \sigma\|\rho\| \geq \sigma R. \tag{7}$$

Using (7) and the choice of $\bar{\mu}$. We have from condition (H),

$$\begin{aligned}
\rho(t) &= (T'\rho)(t) = \int_0^T G(t,s)[f_n(t,\rho(s);\lambda) + \frac{\mu^2}{\rho^3(s)}]ds + \frac{1}{n} \\
&= \int_0^T G(t,s)[\lambda f_n(t,\rho(s)) + \frac{\mu^2}{\rho^3(s)} + (1-\lambda)\frac{c}{\rho(s)^\alpha}]ds + \frac{1}{n} \\
&= \int_0^T G(t,s)[\lambda f(t,\rho(s)) + \frac{\mu^2}{\rho^3(s)} + (1-\lambda)\frac{c}{\rho(s)^\alpha}]ds + \frac{1}{n} \\
&\le \int_0^T G(t,s)[\frac{\lambda b(s)}{\rho(s)^\alpha} + \frac{\mu^2}{\rho(s)^3} + (1-\lambda)\frac{c}{\rho(s)^\alpha}]ds + \frac{1}{n} \\
&\le \int_0^T G(t,s)[\frac{\lambda b(s)}{(\sigma R)^\alpha} + \frac{\mu^2}{(\sigma R)^3} + (1-\lambda)\frac{c}{(\sigma R)^\alpha}]ds + \frac{1}{n} \\
&\le \frac{\lambda b^*}{k^2(\sigma R)^\alpha} + \frac{B^2}{k^2(\sigma R)^3} + \frac{(1-\lambda)c}{k^2(\sigma R)^\alpha} + \frac{1}{n} \\
&\le \frac{B^2}{k^2(\sigma R)^3} + \frac{b^*}{k^2(\sigma R)^\alpha} + 1 < R.
\end{aligned}$$

Therefore,

$$R \ge \|\rho\| < R.$$

This is a contradiction and the claim is proved, i.e., any solution of (6) satisfy $\rho(t) < R$.

On the other hand, using again (7) and the condition (H),

$$\begin{aligned}
\rho(t) &= (T'\rho)(t) = \int_0^T G(t,s)[f_n(t,\rho(s);\lambda) + \frac{\mu^2}{\rho^3(s)}]ds + \frac{1}{n} \\
&= \int_0^T G(t,s)[\lambda f(t,\rho(s)) + \frac{\mu^2}{\rho^3(s)} + (1-\lambda)\frac{c}{\rho(s)^\alpha}]ds + \frac{1}{n} \\
&\ge \int_0^T G(t,s)[\frac{\lambda a(s)}{\rho(s)^\alpha} + \frac{\mu^2}{\rho(s)^3} + (1-\lambda)\frac{c}{\rho(s)^\alpha}]ds \\
&> \int_0^T G(t,s)[\frac{\lambda a(s)}{R^\alpha} + \frac{\mu^2}{R^3} + (1-\lambda)\frac{c}{R^\alpha}]ds \\
&\ge \frac{\lambda a_*}{k^2 R^\alpha} + \frac{A^2}{k^2 R^3} + \frac{(1-\lambda)c}{k^2 R^\alpha} \\
&\ge \frac{A^2}{k^2 R^3} + \frac{a_*}{k^2 R^\alpha} := \delta.
\end{aligned}$$

Next, we prove the fact

$$|\dot{\rho}(t)| \le L,$$

for some constant $L > 0$. To this end, by the boundary condition $\rho(0) = \rho(t)$, $\dot{\rho}(t_0) = 0$ for some $t_0 \in [0,T]$. Integrate (5) from 0 to T, we obtain

$$\int_0^T k^2\rho(t)dt = \int_0^T [f_n(t,\rho(t);\lambda) + \frac{\mu^2}{\rho^3} + \frac{k^2}{n}]dt.$$

Then

$$\begin{aligned}
\|\dot{\rho}\| &= \max_{0\le t\le T} |\dot{\rho}(t)| = \max_{0\le t\le T} |\int_{t_0}^t \ddot{\rho}(s)ds| \\
&= \max_{0\le t\le T} |\int_{t_0}^t [f_n(s,\rho(s);\lambda) + \frac{\mu^2}{\rho^3(s)} + \frac{k^2}{n} - k^2\rho(s)]ds \\
&\le \int_0^T [f_n(s,\rho(s);\lambda) + \frac{\mu^2}{\rho^3(s)} + \frac{k^2}{n} + k^2\rho(s)]ds \\
&\le 2\int_0^T k^2\rho(s)ds \\
&< 2k^2RT := L.
\end{aligned}$$

Defining $C' = \max\{R,L\}$,the proof is completed. $\square$

If ρ is a fixed point of $\rho = T_n\rho$, we denoted by ρ_n, i.e., Eq.(5) (with $\lambda = 1$) have periodic solutions with $\delta < \rho_n(t) < C'$ and $|\dot{\rho_n}| < C'$, this shows that $\{\rho_n\}$ is a bounded and equi-continuous family on $[0,T]$. The Arzela-Ascoli Theorem guarantees that $\{\rho_n\}$ has a subsequence $\{\rho_{n_i}\}_{i\in\mathbb{N}}$, converging uniformly on $[0,T]$ to a function $\rho \in C[0,T]$. Moreover ρ_{n_i} satisfies the integral equation

$$\rho_{n_i}(t) = \int_0^T G(t,s)[f(s,\rho_{n_i}(s)) + \frac{\mu^2}{\rho_{n_i}^3(s)}]ds + \frac{1}{n_i}.$$

Let $i \to \infty$, we arrive at

$$\rho(t) = \int_0^T G(t,s)[f(s,\rho(s)) + \frac{\mu^2}{\rho^3(s)}]ds, \tag{8}$$

where the uniform continuity of $f(t,\rho) + \frac{\mu^2}{\rho}$ on $[0,T] \times [\delta, C']$ is used. Therefore, (4) can be easily obtained.

The following lemma gives us an important information concerning Eq.(3). Let us use C_T^1 denote the set of $T-$perodic C^1- functions with the usual norm of $C^1[0,T]$.

Lemma 3.6. *Given A,B, with $\bar{\mu} \le A \le B$, there is a continuum $\mathscr{C}_{A,B}$ in $[A,B] \times C_T^1$, connecting $\{A\} \times C_T^1$ with $\{B\} \times C_T^1$, whose elements (μ,ρ) is solution of Eq.(3).*

Proof. Let us define the following operators:

$$L : D(L) \subset C^1([0,T]) \to L^1(0,T),$$

$$D(L) = \{\rho \in W^{2,1}(0,T) : \rho(0) = \rho(T), \dot{\rho(0)} = \dot{\rho(T)}\},$$

$$L\rho = \ddot{\rho} + k^2\rho,$$

and

$$N : [A,B] \times C^1([0,T] \to L^1(0,T)),$$

$$N(\mu,\rho)(t) = f_n(t,\rho) + \frac{\mu^2}{\rho^3} + \frac{k^2}{n}.$$

Thus $T-$periodics for Eq.(3.3) is equivalent to

(9) $$\rho - L^{-1}N(\mu,\rho) = 0.$$

since L is invertible.

Define Ω to be the following open subset of $C^1([0,T])$:

$$\Omega = \{\rho \in C^1([0,T]) : \delta < \rho(t) < C' \quad \text{and} \quad |\dot{\rho}(t)| < C' \quad \text{for every} \quad t \in [0,T]\}.$$

where C' be the constant given by Lemma 3.5. By Lemma 3.5, Eq.(9) has no solutions (μ,ρ) on $[A,B] \times \partial\Omega$.

By the homotopic invariant property of the topological degree, let us take $\lambda = 0$, so that (5) becomes

$$\ddot{\rho} + k^2\rho = \frac{\mu^2}{\rho^3} + \frac{c}{\rho^\alpha} + \frac{k^2}{n}.$$

Define the systems

$$\begin{cases} \dot{\rho} = u, \\ \dot{u} = -k^2\rho + \frac{\mu^2}{\rho^3} + \frac{c}{\rho^\alpha} + \frac{k^2}{n}. \end{cases}$$

Let

$$Y = \begin{pmatrix} \rho \\ u \end{pmatrix}, \quad \dot{Y} = F(Y) = \begin{pmatrix} u \\ -k^2\rho + \frac{\mu^2}{\rho^3} + \frac{c}{\rho^\alpha} + \frac{k^2}{n} \end{pmatrix}$$

It is easy to know F has a unique zero (ρ_0,u_0), and computer determinant of Jacobian matrix

$$
\begin{aligned}
|J_F(\rho_0,u_0)| &= \begin{vmatrix} 0 & 1 \\ -\frac{3\mu^2}{\rho^4}-\frac{\alpha c}{\rho^{\alpha+1}}-k^2 & 0 \end{vmatrix} \\
&= k^2+\frac{\alpha c}{\rho^{\alpha+1}}+\frac{3\mu^2}{\rho^4}>0.
\end{aligned}
$$

By a result of Capietto,Mawhin and Zanolin [6], the Leray-Schauder degree of $I-L^{-1}N(\mu)$ equal to the Brouwer degree of F, i.e.,

$$d_{LS}(I-L^{-1}N(\mu,\cdot,)\Omega,0)=d_B(F,(\delta,C')\times(-C',C'),0)=1$$

Since $L^{-1}N(\mu,\cdot)$ is a compact operator, and by the global continuation principle of Leray-Schauder, the proof is completed. $\square$

We can deduce from Lemma 3.6 that there is a connected set $\mathscr{C}$, contained in $[\bar{\mu},+\infty]\times C^1_T$, which connects $\{\bar{\mu}\times C^1_T\}$ with $\{\mu^*\times C^1_T\}$, for every $\mu^*>\bar{\mu}$, whose element (μ,ρ) is the solution of Eq.(3.3).

Lemma 3.7. *For every $\varepsilon>0$, if $(\mu,\rho)\in\mathscr{C}$ with $\mu\geq\bar{\mu}$, then*

$$\int_0^T\frac{\mu}{\rho^2(t)}dt\leq\varepsilon.$$

Proof. Following from (H), we know that

$$\lim_{r\to\infty}\frac{k^2r-f_n(t,r)}{r^2}=0,$$

there exists ε' such that

$$r\geq\varepsilon'\Rightarrow k^2r-f_n(t,r)\leq\varepsilon' r^2,\quad \text{for}\quad t\in[0,T].$$

Let (μ,ρ) be an element of $\mathscr{C}$, with $\mu\geq\bar{\mu}$, $\|\rho\|\leq C'$. Hence

$$\frac{1}{T}\int_0^T k^2\rho(t)dt-\frac{1}{T}\int_0^T f_n(t,\rho(t))dt\leq\frac{1}{T}\int_0^T\varepsilon'\rho^2(t)dt\leq\varepsilon'\|\rho\|^2.$$

Note that

$$\frac{1}{T}\int_0^T\frac{\mu^2}{\rho^3(t)}dt\geq\frac{\mu^2}{\|\rho\|^3},$$

and integrating in (3), we obtain

$$\frac{1}{T}\int_0^T \frac{\mu^2}{\rho^3(t)}dt = \frac{1}{T}\int_0^T k^2\rho(t)dt - \frac{1}{T}\int_0^T f_n(t,\rho(t))dt - \frac{1}{T}\int_0^T \frac{k^2}{n}dt \le \varepsilon'\|\rho\|^2,$$

so we have

$$\frac{\mu^2}{\|\rho\|^5} \le \varepsilon'.$$

Then, we using Lemma 3.3,

$$\int_0^T \frac{\mu}{\rho^2(t)}dt \le \frac{T\mu}{(\min\rho)^2} \le \frac{T\mu}{(\sigma\|\rho\|)^2} = \frac{T\sqrt{\varepsilon' C'}}{\sigma^2}.$$

Set

$$\varepsilon' = \frac{\sigma^4\varepsilon^2}{C'T^2},$$

thus proving the lemma. □

Defined the function

$$\Phi(\mu,\rho) \mapsto \int_0^T \frac{\mu}{\rho^2(t)}dt.$$

By the Lebesgue controlled convergence theorem, it is continuous from $\mathscr{C}$ to $\mathbb{R}$, and $\mathscr{C}$ is connected, its image is an interval. By Lemma 3.6 and Lemma 3.7, this interval is of the type $(0,\bar{\theta})$ for some $\bar{\theta} > 0$.

Lemma 3.8. *For every $\theta \in (0,\bar{\theta}]$, there are (μ,ρ,φ), verifying system (2), for which $(\mu,\rho) \in \mathscr{C}$ and*

$$\rho(t+T) = \rho(t), \quad \varphi(t+T) = \varphi(t) + \theta,$$

for every $t \in \mathbb{R}$.

Proof. Given $\theta \in (0,\bar{\theta}]$, there are $(\mu,\rho) \in \mathscr{C}$, such that

$$\Phi(\mu,\rho) = \theta.$$

Obviously, ρ is $T-$perodic satisfies the first equation in (2), and defining

$$\varphi(t) = \int_0^t \frac{\mu}{\rho^2(s)}ds,$$

it is also satisfies the second equation in (2). Moreover

$$\varphi(t+T)-\varphi(t)=\int_t^{t+T}\frac{\mu}{\rho^2(s)}ds=\int_0^T\frac{\mu}{\rho^2(s)}ds=\theta. \quad \square$$

For every $\theta\in(0,\bar{\theta}]$, let $x(t)$ be a solution of system (2.2). Then it follows from Lemma 3.8 that

$$\begin{aligned} x(t+T) &= \rho(t+T)e^{i\varphi(t+T)} \\ &= \rho(t)e^{i\varphi(t)}\cdot e^{i\theta} \\ &= x(t)e^{i\theta}. \end{aligned}$$

In particular, if $\theta=2\pi/k$ for some integer $k\geq 1$, then $x(t)$ is periodic with minimal period kT. Hence, for every integer $k\geq 2\pi/\bar{\theta}$, we have such a $kT-$periodic solution, which we denote by $x_k(t)$. Let $(\rho_k(t),\varphi_k(t))$ be its polar coordinates, and μ_k be its angular momentum. By the above construction, (μ_k,ρ_k,φ_k) satisfy system (2.2), $(\mu_k,\rho_k)\in\mathscr{C}$, and

$$\int_0^T\frac{\mu_k}{\rho_k^2(t)}dt=\frac{2\pi}{k}. \tag{10}$$

Assume (μ_{k_j}) be abounded subsequence, with $(\mu_{k_j})\in[\bar{\mu},B]$ for some B, using Lemma 3.5 with $\lambda=1$, there exist a constant $C_1>0$ such that $\|\rho_{k_j}\|<C_1$, and hence

$$\int_0^T\frac{\mu_{k_j}}{\rho_{k_j}^2(t)}dt>\frac{\bar{\mu}T}{C_1^2},$$

for every j, in contradiction with (10), so

$$\lim_{k\to\infty}\min\mu_k(t)=+\infty \tag{11}$$

By (11) and Lemma 3.4, with $\lambda=1$, we have

$$\lim_{k\to\infty}\rho_k=+\infty,$$

thus concluding the proof of Theorem 3.2(i).

Next we will prove Theorem 3.2(ii), for every $\mu>0$ sufficiently small, equation (1) has a periodic solution.

Let $\mu=0$, (2) can be written the $T-$perodic problem

$$\ddot{\rho}+k^2\rho=f(t,\rho). \tag{12}$$

We consider the following $T-$periodic problem

$$\ddot{\rho}+k^2\rho = f_n(t,\rho)+\frac{k^2}{n}, \tag{13}$$

where the truncation function $f_n(t,\rho)$ is defined by (2).

Lemma 3.9. *There exist $C>0$ such that, if $\rho(t)$ is a solution of (13), then*

$$\frac{1}{C}<\rho(t)<C, \quad |\dot{\rho}(t)|<C. \tag{14}$$

Proof. By the same argument as in the proof of Lemma 3.5, let

$$r=\sqrt[\alpha]{\frac{b^*}{k^2\sigma^\alpha}}+2.$$

we obtain

$$\frac{a_*}{k^2r^\alpha}<\rho(t)\le r, \quad \text{and} \quad |\dot{\rho}(t)|\le 2k^2rT.$$

Let

$$C=\max\{k^2r^\alpha/a_*, r, 2k^2rT\},$$

we have the desired results. $\square$

Same as (8), we can obtain the solutions of (12) also satisfied (14). Define Ω' to be the following open and bounded subset of $C^1[0,T]$:

$$\Omega'=\{\rho\in C^1([0,T]):\frac{1}{C}<\rho(t)<C \quad \text{and} \quad |\dot{\rho}(t)|<C \quad \text{for every} \quad t\in[0,T]\}, \tag{15}$$

and define the operator

$$N_1 : C^1([0,T]\to L^1(0,T)),$$

$$(N_1\rho)(t)=f_n(t,\rho)+\frac{k^2}{n}.$$

The $T-$periodic problem for Eq.(13) is equivalent to

$$\rho-L^{-1}N_1\rho=0 \tag{16}$$

In [6] shows that the degree of (16) is equal to the Brower degree

$$deg(I-L^{-1}N_1,\Omega,0)=1.$$

Thus define the operator

$$N_2 : C^1([0,T] \to L^1(0,T)),$$

$$(N_2\rho)(t) = f(t,\rho),$$

and obtain

$$deg(I - L^{-1}N_2, \Omega, 0) = deg(I - L^{-1}N_1, \Omega, 0) = 1.$$

Let X be a Banach space of functions, such that

$$C^1([0,T]) \subseteq X \subseteq C[0,T],$$

with continuous immersionsand set

$$X^+ = \{\rho \in X : \min \rho > 0\}.$$

Lemma 3.10. *Let Ω' be an open bounded subset of X given for (15). Assume that there is a periodic solution of (12) in Ω', then there exists a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, equation (1) has a periodic solution $x_k(t)$ with minimal period kT.*

Proof. Define the operator

$$N' : X^+ \to L^1(0,T),$$

$$(N'\rho)(t) = f(t,\rho) + \frac{\mu^2}{\rho^3(t)},$$

problem (3) is then equivalent to

$$L\rho = N'\rho. \tag{17}$$

We claim that there exists K (K is a constant given by Lemma 3.4) such that for every $\mu \in [0,K]$ there is no solution of (17) on the boundary $\partial\Omega'$. Otherwise, suppose that there are two sequences $(\mu_n)_n$ and $(\rho_n)_n$ such that $\mu_n \to 0, \rho_n \in \partial\Omega'$ and

$$\rho_n = L^{-1}N'\rho_n.$$

Sine Ω' is uniformly positively bounded, the closure of Ω' is contained is X^+, then $(\rho_n)_n$ and $(1/\rho_n)_n$ are uniformly bounded, so that $L^{-1}N'\rho_n$ is bounded in $L^1(0,T)$. Being $L^{-1}: L^1(0,T) \to X$ is a compact operator, there exists a subsequence, still denoted by $(\rho_n)_n$, for which $L^{-1}N'\rho_n$ converges to some $\bar{\rho} \in X$. Hence $\rho_n \to \bar{\rho}$, as well, and being $\partial\Omega'$ closed, $\bar{\rho} \in \partial\Omega'$. Since $\rho_n \to \rho$ uniformly, and $\bar{\rho} \in X^+$, we deduce from the definition of N_μ that $\bar{\rho} = L^{-1}N'\bar{\rho}$, so that ρ is solve of (17), a contradiction with the assumptions.

By Lemma 3.1, there is a continuum $\mathscr{C}$ in $[0,K] \times \Omega$ connecting $\{0\} \times \{\Omega\}$ with $\{K\} \times \{\Omega\}$, whose element (μ, ρ) satisfy (3).

Similar to (10), we have

$$\int_0^T \frac{\mu_k}{\rho_k^2(t)} dt = \frac{2\pi}{k}.$$

Since $\rho_k \in \Omega'$, and Ω' is bounded in $C([0,T])$, there is a constant $C_2 > 0$ such that $\rho_k(t) < C_2$, for every $t \in [0,T]$, hence

$$\frac{2\pi}{k} = \int_0^T \frac{\mu_k}{\rho_k^2(t)} dt > \frac{\mu_k T}{C_2^2},$$

so that $\lim_{k \to \infty} \mu_k = 0$. $\square$

The proof of Theorem 3.2(ii) is completed.

We now present an example to show that our results.

Example Consider the equation

$$\ddot{x} + k^2 x = \frac{c(t)x}{|x|^\gamma} + d(t)|x|^\beta x, \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{18}$$

where $0 < k < \pi/T$, and $\gamma, \beta > 0, c(t), d(t)$ are continuous positive $T-$periodic functions. We write the solutions of (18) in polar coordinates (1), and consider the $T-$periodic problem

$$\ddot{\rho} + k^2 \rho = \frac{c(t)}{\rho^{\gamma-1}} + d(t)\rho^{\beta+1} + \frac{\mu^2}{\rho^3}.$$

By Lemma 2.1 and Lemma 3.3, and a direct calculation obtain that

(1) if $\gamma > 1, \beta > 0$, then

$$\min\{\sqrt[\beta]{\frac{k^2\sigma^\beta}{3d^*}}, \sqrt[\gamma]{\frac{c_*\sigma^{\gamma-1}}{k^2}}\} \le \rho(t) \le \max\{\sqrt[\gamma]{\frac{3c^*}{\sigma^{\gamma-1}k^2}}, \sqrt[4]{\frac{3\mu^2}{k^2\sigma^3}}\};$$

(2) if $1 > \gamma > 0, \beta > 0$, then

$$\min\{\sqrt[\beta]{\frac{k^2\sigma^\beta}{3d^*}}, \sqrt[\gamma]{\frac{c_*}{k^2}}\} \leq \rho(t) \leq \max\{\sqrt[\gamma]{\frac{3c^*}{k^2}}, \sqrt[4]{\frac{3\mu^2}{k^2\sigma^3}}\}.$$

we apply Theorem 3.2. If we take

$$a(t) = L_1 c(t), \qquad b(t) = L_2^{\alpha+\beta+1} d(t), \qquad \alpha = \gamma,$$

where

$$L_1 = \min\{\sqrt[\beta]{\frac{k^2\sigma^\beta}{3d^*}}, \sqrt[\gamma]{\frac{c_*\sigma^{\gamma-1}}{k^2}}, \sqrt[\gamma]{\frac{c_*}{k^2}}\}, \quad L_2 = \max\{\sqrt[\gamma]{\frac{3c^*}{\sigma^{\gamma-1}k^2}}, \sqrt[4]{\frac{3\mu^2}{k^2\sigma^3}}, \sqrt[\gamma]{\frac{3c^*}{k^2}}\}.$$

Then the condition (H) is satisfied. So there exist a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, equation (18) has a periodic solution with minimal period kT. Moreover, one of the following two alternative holds

(i) if μ_k denotes the angular momentum associated to $x_k(t)$ then

$$\lim_{k\to\infty} \min |x_k(t)| = +\infty \quad \text{and} \quad \lim_{k\to\infty} \mu_k = +\infty;$$

(ii) exist $C > 0$ (independent of μ and k) such that

$$\frac{1}{C} < |x_k(t)| < C, \quad \text{for every } t \in \mathbb{R} \quad \text{and every} k \geq k_1$$

and

$$\lim_{k\to\infty} \mu_k = 0.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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