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THE EQUIVALENCE PROBLEM FOR VECTORS IN THE TWO-DIMENSIONAL MINKOWSKI SPACETIME AND ITS APPLICATION TO BÉZIER CURVES

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Abstract. Let $M(1,1)$ be the group of all transformations of the 2-dimensional Minkowski spacetime M generated by all pseudo-orthogonal transformations and parallel translations of M. Let $SM(1,1)$ is the proper subgroup of $M(1,1)$ and $SL(1,1)$ is the ortochoronous proper subgroup of $M(1,1)$. In this paper, conditions for the equivalence of two systems of vectors $\{x_1,x_2,\dots x_m\}$ and $\{y_1,y_2,\dots y_m\}$ are obtained for groups $G=M(1,1),SM(1,1),SL(1,1)$. Finally, we present a necessary and sufficient conditions for judging whether Bézier curves in M of degree m are G-equivalent.

Keywords: Invariant; Minkowski spacetime; Equivalence.

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1. Introduction

One of important problems in theory of invariants is finding necessary and sufficient conditions equivalence of systems of vectors $\{x_1,x_2,\dots x_m\}$ and $\{y_1,y_2,\dots y_m\}$ under the action of pseudo-orthogonal group(general Lorentz group) $O(1,1)$, special pseudo-orthogonal group(proper Lorentz group) $SO(1,1)$ and ortochoronous special pseudo-orthogonal group (Lorentz group) $L(1,1)$.

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Recently, all m-points invariants for different geometries is determined by a characterization of orbits of m-tuples of vectors in paper [21]. All scalar concomitants of vectors and all biscalars of a system of $s \leq n$ linearly independent contravariant vectors in n-dimensional Lorentz space is determined in papers [1, 5]. A solution of the problem of equivalence of a system of linearly independent vectors for pseudo-orthogonal group $O(n,1)$ in terms of Gram matrices of vectors $x_1, x_2, \dots x_m$ in the n- dimensional pseudo-Euclidean space of index 1 is given in [5]. But for a system of linearly dependent vectors for groups $G = O(1,1), SO(1,1), L(1,1)$, therefore mentioned papers do not contain a solution. For example, consider the following two systems: $V_x = \{x_1 = (1,1), x_2 = (2,2)\}$, $V_y = \{y_1 = (1,1), y_2 = (3,3)\}$. Clearly, vectors in V_x, V_y are linearly dependent and mentioned invariants are equal. But the systems are not $O(1,1)$-equivalent.

The paper presents a solution of the problem of G-equivalence of a system of vectors for groups $G = O(1,1), SO(1,1), L(1,1)$ in terms of invariants of vectors $x_1, x_2, \dots x_m$ in the two dimensional Minkowski spacetime geometry. Applications of the invariant theory and invariants in computer vision and pattern recognition are discussed in [3, 6, 14, 15, 16]. Transformations and invariants of curves, surfaces and graphical objects appear in computer aided geometric design and graphical applications in [7, 17]. The invariance of curves and surfaces relative to the Euclidean group, the affine group and other groups is investigated in [4, 12, 13, 14, 18]. Conditions for the coincidence of two Bézier curves of degree 3 and 4 in the Euclidean geometry are discussed in papers [11, 22, 23]. Differential invariants(the curvature, the torsion) of spacelike Bézier curves in the three dimensional Minkowski spacetime is given in paper [8]. In [19], the conditions of the global G- equivalence of curves are given in terms of the pseudo-Euclidean type and the system of polynomial differential G- invariant functions. In [20], the conditions of the global G- equivalence of null curves are given in terms of the pseudo-Euclidean type and the system of polynomial differential G- invariant functions. The solution of the equivalence problem, without using the methods in the aforementioned articles, is devoted to an application of control invariants of Bézier curves in M of degree m.

The paper is organized as follows. In Section 2, the definition of the system V_x type and the ratios of linearly dependence of vectors x_1, x_2 is given. The type and the ratios are $O(1,1)$-invariant($M(1,1)$-invariant, respectively). The conditions of $G = O(1,1), SO(1,1), L(1,1)$- e-quivalence of vectors are given in terms of the type and polynomial invariants of vectors x_1, x_2, $\ldots$ x_m functions. In Section 3, the definitions of a G-equivalence of Bézier curves, a control G-invariant of a Bézier curve are introduced. The conditions of $G = M(1,1), SM(1,1), SL(1,1)$-equivalence of Bézier curves in M of degree m is given.

2. The conditions of *G*-equivalence of vectors

Let R be the field of real numbers. The 2-dimensional pseudo-Euclidean space of index 1 will be denoted by M. M is 2-dimensional the Minkowski spacetime. $< u, v >$ is a referred to as a Lorentz inner product on M such that there exists an orthonormal basis $\{e_1, e_2\}$ for M with the property that if $u = u_1e_1 + u_2e_2$ and $v = v_1e_1 + v_2e_2$, then $< u, v >= u_1v_1 - u_2v_2$ for all $u, v \in M$ and denoted by $< u, v >$.

We define the matrix $A = \left(a_{ij}\right)_{i,j=1,2}$ associated with the pseudo-orthogonal transformations and the pseudo-orthogonal basis $\{e_i\}$ by $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ for all $a_{ij} \in R$. That is,

$$O(1,1) = \left\{ A \in G(2,\Re) : A^T \eta A = \eta, \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Then the group $M(1,1)$ of all pseudo-Euclidean motions of an 2-dimensional pseudo-Euclidean space has the form

$M(1,1) = \{F : M \to M : Fx = gx + b, g \in O(1,1), b \in M\}$, where gx is the multiplication of a matrix g and a column vector $x \in M$.

The following proposition is known in [18].

Proposition 2.1. *Let $O(1,1)$ be the pseudo-orthogonal group of index 1. Then, all elements of $O(1,1)$ as follows:*

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \text{ or } B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} \text{ for all } a, b \in R$$

The group of all proper pseudo-orthogonal transformations of M is denoted by $SO(1,1)$. It is a subgroup of $O(1,1)$.

That is, $SO(1,1)=\left\{A=\begin{pmatrix} a & b \\ b & a \end{pmatrix}\in O(1,1): detA=1\right\}$.

Put $SM(1,1)=\{F\in M(1,1): Fx=gx+b, g\in SO(1,1), b\in M\}$.

$SM(1,1)$ is a subgroup of $M(1,1)$.

The group of all ortochoronous proper pseudo-orthogonal transformations of M is denoted by $L(1,1)$.

We shall refer to $L(1,1)$ simply as the Lorentz group(see [9, p. 15-16]). That is, we denote $L(1,1)=\left\{A=\begin{pmatrix} a & b \\ b & a \end{pmatrix}\in O(1,1): detA=1, a\geq 1\right\}$.

Put $SL(1,1)=\{F\in SM(1,1): Fx=gx+b, g\in L(1,1), b\in M\}$.

$SL(1,1)$ is a subgroup of $M(1,1)$.

In [9, p.14-16], the groups $O(1,1)$, $SO(1,1)$ and $L(1,1)$ are named general Lorentz group, proper Lorentz group and orthocronous proper Lorentz group, respectively.

The following definition is known (see [9, p.10,12]).

Definition 2.1.

(i) A vector x in M will be called timelike vector if $\langle x,x\rangle<0$.

(ii) A vector x in M will be called spacelike vector if $\langle x,x\rangle>0$.

(iii) A non-zero vector x in M will be called null (or lightlike) vector if $\langle x,x\rangle=0$.

Let $V_x=\{x_1,x_2,\ldots x_m\}$ and $V_y=\{y_1,y_2,\ldots y_m\}$ be two systems of vectors in M. Let G be a subgroup of M(1,1).

Definition 2.2. V_x and V_y are called G-equivalent if there exists $F\in G$ such that $y_i=Fx_i$, $1\leq i\leq m$. This being the case, we write $x_i\overset{G}{\sim}y_i$. (shortly,$V_x\overset{G}{\sim}V_y$).

Definition 2.3. A function $f(x_0,x_1,\ldots,x_m)$ of vectors $x_0,x_1,\ldots,x_m$ in M will be called G-invariant if $f(Fx_0,Fx_1,\ldots,Fx_m)=f(x_0,x_1,\ldots,x_m)$ for all $F\in G$.

Example 2.1. *Since $< g(u), g(v) >=< u, v >$ for all $g \in O(1,1)$, we obtain that the scalar product $< u, v >$ of vectors $u, v \in M$ is $O(1,1)$-invariant. Similarly, the function $f(u,v,w) =< u-w, v-w >$ is $M(1,1)$-invariant.*

Example 2.2. *Let u_1, u_2 be vectors in M. Denote by $[u_1 u_2]$ determinant of the matrix $\|u_1 u_2\|$ of column-vectors u_1, u_2. Then $[u_1 u_2]$ is $SO(1,1)$-invariant. In fact, $[gu_1 gu_2] = \det g [u_1 u_2] = [u_1 u_2]$ for all $g \in SO(1,1)$.*

Proposition 2.2. *Let $V_x \overset{O(1,1)}{\sim} V_y$. Then $rank\,(V_x) = rank\,(V_y)$.*

Proof. It is obvious from Definition 2.2.

Corollary 2.1. *According to $O(1,1)-$equivalence, $rank\,(V_x)$ is an invariant.*

Example 2.3. The rank of a system V_x is $O(1,1)$-invariant, but it is not $M(1,1)$-invariant.

The number $T\,(V_x)$ will be called the type of the system V_x such that the type is determined the rank of the system V_x and the type of linearly independent vector(s) in V_x from Definition 2.2.

Definition 2.3.

- (i) The system V_x will be called first type if $rank(V_x) = 2$ and the linearly independent vectors in V_x are spacelike, timelike or null. This being case, denoted by $T\,(V_x) = 1$.
- (ii) The system V_x will be called second type if $rank(V_x) = 1$ and all vectors in V_x are timelike. This being case, denoted by $T\,(V_x) = 2$.
- (iii) The system V_x will be called third type if $rank(V_x) = 1$ and all vectors in V_x are spacelike. This being case, denoted by $T\,(V_x) = 3$.
- (iv) The system V_x will be called fourth type if $rank(V_x) = 1$ and all vectors in V_x are null. This being case, denoted by $T\,(V_x) = 4$.

Proposition 2.3. *Let $V_x \overset{O(1,1)}{\sim} V_y$. Then $T\,(V_x) = T\,(V_y)$.*

Proof. It is obvious from Definition 2.2.

Corollary 2.2. *According to $O(1,1)-$equivalence, the type is an invariant.*

Let V_x be a system of vectors in M. We consider the case $T\,(V_x) = 1$. Since $T\,(V_x) = 1$, for simplicity, we assume that there exist two linearly independent vectors x_1, x_2 in V_x such that

$x_i = \lambda_{i1}x_1 + \lambda_{i2}x_2$ for all $i \geq 3$ and $\lambda_{i1}, \lambda_{i2} \in R$. Here, the ordered pair $(\lambda_{i1}, \lambda_{i2})$ will be called the ratios of linearly dependence of vectors $x_i, 2 < i \leq m$ and denoted by L_x^1. Similarly, we consider the case $T(V_x) = r$ for all $r = 2,3,4$. Since $T(V_x) = r$ for all $r = 2,3,4$, for simplicity, we assume that there exists linearly independent vector x_1 in V_x such that $x_i = \lambda_{i1}x_1$ for all $i \geq 2$ and $\lambda_{i1} \in R$. Here, the number λ_{i1} will be called the ratio of linearly dependence of $x_i, 1 < i \leq m$ and denoted by L_x^2.

Proposition 2.4. *Let V_x and V_y be two systems of vectors in M and $V_x \overset{O(1,1)}{\sim} V_y$. Then $L_x^k = L_y^k$ for $k = 1,2$.*

Proof. The proof follows easy from Definition 2.2 and Proposition 2.3.

Corollary 2.3. *According to $O(1,1)-$equivalence, L_x^k is an invariant.*

Let $x_1, x_2, \ldots x_m \in M$. Denote the matrix $\left\| < x_i, x_j > \right\|_{i,j=1,2,\ldots,m}$ by $Gr(x_1, x_2, \ldots x_m)$ and its determinant by $detGr(x_1, x_2, \ldots x_m)$.

Proposition 2.5. *Vectors $x_1, x_2, \ldots x_m \in M$ are linearly depended if and only if $detGr(x_1, x_2, \ldots x_m) = 0$.*

Proof. A proof is given [10, p.75].

Proposition 2.6. *Let V_x be a system of vectors in M and $T(V_x) = 1$. Then element $(\lambda_{i1}, \lambda_{i2})$ of L_x^1 as follows:*

$$\lambda_{i1} = \frac{\begin{bmatrix} < x_1, x_i > & < x_1, x_2 > \\ < x_2, x_i > & < x_2, x_2 > \end{bmatrix}}{detGr(x_1, x_2)}, \lambda_{i2} = \frac{\begin{bmatrix} < x_1, x_1 > & < x_1, x_i > \\ < x_2, x_1 > & < x_2, x_i > \end{bmatrix}}{detGr(x_1, x_2)}$$

for all $3 \leq i \leq m$.

Proof. Since $T(V_x) = 1$, we have $rank(V_x) = 2$. Then there exist linearly independent vectors x_1, x_2 in V_x such that $x_i = \lambda_{i1}x_1 + \lambda_{i2}x_2$ for all $3 \leq i \leq m$ and $\lambda_{i1}, \lambda_{i2} \in R$.

Hence, we have

(1) $$< x_i, x_1 >= \lambda_{i1} < x_1, x_1 > + \lambda_{i2} < x_2, x_1 >$$

(2) $$< x_i, x_2 >= \lambda_{i1} < x_1, x_2 > + \lambda_{i2} < x_2, x_2 >$$

for all $3 \leq i \leq m$.

For linearly independent vectors x_1, x_2 in V_x, we have $detGr(x_1, x_2) \neq 0$. Then there exists an unique solution of equalities (1) and (2). This solution is given in proposition.

Proposition 2.7. *Let V_x be a system of vectors in M and $T(V_x) = r$ for all $r = 2, 3$. Then element λ_{i1} of L_x^1 as follows:*

$\lambda_{i1} = \frac{<x_1, x_i>}{<x_1, x_1>}$ *for all $2 \leq i \leq m$.*

Proof. It follows from Proposition 2.6.

Corollary 2.4. *Let V_x be a system of vectors in M and $T(V_x) = r$ for all $r = 1, 2, 3$. According to Propositions 2.6. and 2.7., components of elements of L_x^1, L_x^2 are given in terms of scalar products of vectors $x_1, x_2, \ldots x_m$.*

Let $x_i = (x_{i1}, x_{i2}) \in M$ for all $1 \leq i \leq m$.

Proposition 2.8. *Let V_x be a system of vectors in M and $T(V_x) = 4$. Then element λ_{i1} of L_x^2 as follows: $\lambda_{i1} = \frac{x_{i2}}{x_{12}}$ for $2 \leq i \leq m$.*

Proof. It follows from Propositions 2.6. and 2.7.

Corollary 2.5. *Let V_x be a system of vectors in M and $T(V_x) = 4$. According to Proposition 2.8., components of elements of L_x^2 are not given in terms of scalar products of vectors $x_1, x_2, \ldots, x_m$.*

Theorem 2.1.labelthe2.2 *Let V_x and V_y be two system of vectors in M. Assume that $T(V_x) = T(V_y) = 1$. Then following two conditions are equivalent:*

(i)

$$V_x \overset{O(1,1)}{\sim} V_y$$

(ii)

$$< x_i, x_j >=< y_i, y_j >$$

for all $i = 1,2;\ j = 1,2,\dots,m$ *and* $i \leq j$

Proof.

$(i) \rightarrow (ii)$: Let V_x be a system of vectors in M and $T(V_x) = 1$. Since the function $f(x_j, x_k) =< x_j, x_k >$ is $O(1,1)$-invariant, condition (i) implies (ii).

$(ii) \rightarrow (i)$: Assume that condition (ii) is valid.

We have the case $T(V_x) = T(V_y) = 1$. Then there exist vectors $x_1, x_2 \in V_x$ which are linearly independent. We prove that vectors $y_1, y_2 \in V_y$ are linearly independent. Let $X = \|x_1 x_2\|$ and $Y = \|y_1 y_2\|$ be two matrix of column-vectors x_1, x_2 and y_1, y_2, respectively. Linearly independence of x_1, x_2 implies $det X \neq 0$. Let $X^\top$ be the transpose matrix of X and $Gr(x_1, x_2)$ is the Gram matrix of vectors x_1, x_2. Then it is easy to see that

$$X^\top \eta X = Gr(x_1, x_2). \tag{3}$$

Since $< x_i, x_j >=< y_i, y_j >$ for all $i = 1,2;\ j = 1,2$ and $i \leq j$, we have

$$Gr(x_1, x_2) = Gr(y_1, y_2). \tag{4}$$

Equalities (3) and (4) imply

$$X^\top \eta X = Y^\top \eta Y, \tag{5}$$

whence

$$(det X)^2 = (det Y)^2. \tag{6}$$

Since $det X \neq 0$, equality (6) implies $det Y \neq 0$. That is, vectors y_1, y_2 are linearly independent. Then there exists the 2×2-matrix g such that $det g \neq 0$ and

$$Y = gX. \tag{7}$$

Equalities (4) and (7) imply

$$X^\top \eta X = Y^\top g^\top \eta g Y. \tag{8}$$

Since $detX \neq 0$, equality (8) implies $g^{\top}\eta g = \eta$. This means that $g \in O(1,1)$. Equalities (7) and (8) imply $y_j = gx_j$ for all $j = 1,2$.

Let $j > 2$. Condition (ii) of our theorem and equalities

$$X^{\top}\eta x_j = \begin{pmatrix} < x_1, x_j > \\ < x_2, x_j > \end{pmatrix}, Y^{\top}\eta y_j = \begin{pmatrix} < y_1, y_j > \\ < y_2, y_j > \end{pmatrix}$$

imply

$$X^{\top}\eta x_j = Y^{\top}\eta y_j \tag{9}$$

Using equalities (7) and (9), we obtain

$$X^{\top}\eta x_j = X^{\top}g^{\top}\eta y_j \tag{10}$$

Since $g \in O(1,1)$, we have $g\eta g^{\top} = \eta$. Hence equality (10) implies $y_j = gx_j$ for all $j > 2$. Our theorem is proved in the case $T(V_x) = 1$.

Theorem 2.2. *Let V_x and V_y be two system of vectors in M. Assume that $T(V_x) = T(V_y) = r$ for all $r = 2,3$. Then following two conditions are equivalent:*

(i)

$$V_x \overset{O(1,1)}{\sim} V_y$$

(ii)

$$< x_1, x_j > = < y_1, y_j >$$

for all $j = 1,2,\ldots,m$.

Proof.

$(i) \to (ii)$: Let V_x be a system of vectors in M and $T(V_x) = r$ for all $r = 2,3$. Since the function $f(x_j, x_k) = < x_j, x_k >$ is $O(1,1)$-invariant, condition (i) implies (ii).

$(ii) \to (i)$: Assume that condition (ii) is valid.

(a) We consider the case $T(V_x) = T(V_y) = 2$. Since $T(V_x) = T(V_y) = 2$, we have $rank(V_x) = rank(V_y) = 1$. Then there exists vector $x_1 \in V_x$ which is $x_1 \neq 0$ and $< x_1, x_1 > \neq 0$. Since $< x_1, x_1 > = < y_1, y_1 > \neq 0$ and $T(V_y) = 2$, there exists vector $y_1 \in V_y$ which is $y_1 \neq 0$.

Since $T(V_x) = T(V_y) = 2$, we have $< x_1, x_1 > = < y_1, y_1 > = k$ and $k < 0$.

We define $e_1 = \frac{x_1}{\sqrt{|k|}}$ such that $< e_1, e_1 > = -1$. By [2, Lemma2, p.234], e_1 can be extended to a pseudo-orthonormal basis $\{e_1, e_2\}$ of index 1 such that $< e_2, e_2 > = 1$. Similarly, for $x_1 \neq y_1$, we define $f_1 = \frac{y_1}{\sqrt{|k|}}$ such that $< f_1, f_1 > = -1$. By [2, Lemma2, p.234], f_1 can be extended to a pseudo-orthonormal basis $\{f_1, f_2\}$ of index 1 such that $< f_1, f_1 > = 1$.

Otherwise, there exist $F \in O(1,1)$ such that $F(e_i) = f_i$ for $i = 1, 2$. Hence, we have $F(x_1) = F\left(e_1(\sqrt{|k|})\right) = (\sqrt{|k|})F(e_1) = y_1$. Since x_1, y_1 are non-zero vectors, the vectors can be written $x_i = \lambda_i x_1$ and $y_i = \beta_i y_1$ for $i > 1$. From Proposition 2.7., we have $\lambda_i = \beta_i$ for $i > 1$. Hence, for $F \in O(1,1)$, we have $F(x_i) = F(\lambda_i x_1) = \lambda_i F(x_1) = \lambda_i y_1 = y_i$ for $i > 1$. This means that systems V_x, V_y are $O(1,1)-equivalent$.

(b) We consider the case $T(V_x) = T(V_y) = 3$. Then the proof is similar to the case (a).

Theorem 2.3. *Let V_x and V_y be two system of vectors in M. Assume that $T(V_x) = T(V_y) = 4$. Then following two conditions are equivalent:*

(i)

$$V_x \overset{O(1,1)}{\sim} V_y$$

(ii)

$$< x_1, x_1 > = < y_1, y_1 >$$
$$L_x^2 = L_y^2$$

Proof. $(i) \rightarrow (ii)$: Using Proposition 2.4. and Theorem 2.1., condition (i) implies (ii).

$(ii) \rightarrow (i)$: Assume that condition (ii) is valid.

Since $T(V_x) = T(V_y) = 4$, we have $rank(V_x) = rank(V_y) = 1$. Then there exists vector $x_1 \in V_x$ which is $x_1 \neq 0$ and $< x_1, x_1 > = 0$. Since $< x_1, x_1 > = < y_1, y_1 > = 0$ and $T(V_y) = 4$, y_1 is a null vector in V_y.

Let $x_1 = (x_{11}, x_{12})$, $y_1 = (y_{11}, y_{12}) \in M$. Since x_1 is a null vector, we have $x_1 = (x_{11}, x_{11})$ or $x_1 = (x_{11}, -x_{11})$. Assume that $\bar{x_1} = (1,1)$, $y_1 = (y_{11}, y_{12}) \in M$ and $\bar{x_1} \neq y_1$.

Then there exist $g_1 \in O(1,1)$ such that $g_1\bar{x_1} = y_1$. Similarly, from Proposition 2.1., there exist $g_2 \in O(1,1)$ such that $g_2x_1 = \bar{x_1}$ for all $x_1 = (x_{11}, x_{12}) \in M$. That is there exist $g = g_1g_2 \in O(1,1)$ such that $gx_1 = y_1$. We prove that there exist $F \in O(1,1)$ such that $Fx_1 = y_1$ for $x_1 = (x_{11}, x_{11})$ and $y_1 = (y_{11}, y_{11})$.

Now we show that there exist $g \in O(1,1)$ such that $gx_1 = y_1$ for $x_1 = (x_{11}, x_{11})$ and $y_1 = (y_{11}, -y_{11})$. Let $x_1 = (1,1)$. From Proposition 2.1., there is no $A \in O(1,1)$ such that $Ax_1 = y_1$. But there exist $B \in O(1,1)$ such that $Bx_1 = \tilde{x_1}$ for $x_1 = (x_{11}, x_{11})$ and $\tilde{x_1} = (x_{11}, -x_{11})$. So there exist $F \in O(1,1)$ such that $Fx_1 = y_1$ for $x_1 = (x_{11}, x_{11})$ and $y_1 = (y_{11}, y_{11})$.

Since x_1 and y_1 are non-zero vectors, we have $x_i = \lambda_i x_1$ and $y_i = \beta_i y_1$ for all $i > 1$. According to condition (ii) of our theorem, since $L_x^2 = L_y^2$, we have $\lambda_i = \beta_i$ for all $i = 2,3,\ldots,m$. Hence, for $F \in O(1,1)$, we have $Fx_i = \lambda_i F x_1 = \lambda_i y_1 = y_i$ for all $i > 1$. This means that systems V_x and V_y are $O(1,1)$-equivalent.

Theorem 2.4. *Let V_x and V_y be two systems of vectors in M. Assume that $T(V_x) = T(V_y) = 1$. Then following two conditions are equivalent:*

(i)

$$V_x \overset{SO(1,1)}{\sim} V_y$$

(ii)

$$< x_i, x_j > = < y_i, y_j >$$

$$[x_1x_2] = [y_1y_2]$$

for all $i = 1,2; j = 1,2,\ldots,m,\ i \le j$.

Proof. $(i) \to (ii)$: Let V_x be a system of vectors in M and $T(V_x) = 1$. Since the function $f(x_i, x_j) = < x_i, x_j >$ and $g(x_k, x_l) = [x_kx_l]$ for all $1 \le i \le j \le m$ and $1 \le k < l \le m$ is $SO(1,1)$-invariant, condition (i) implies (ii).

$(ii) \to (i)$: Assume that condition (ii) is valid.

Let $T(V_x) = T(V_y) = 1$. Then there exist vectors $x_1, x_2 \in V_x$ which are linearly independent. This equivalent to $[x_1x_2] \neq 0$. Condition (ii) imply $[x_1x_2] = [y_1y_2] \neq 0$. That is vectors $y_1, y_2 \in V_y$ are linearly independent. By Theorem 2.1., equalities $< x_j, x_k > = < y_j, y_k >$ for all $j = 1,2$ and $k = 1,2,\ldots,m$ imply the existence $g \in O(1,1)$ such that $y_i = gx_i$ for all $1 \le i \le m$. Using the equalities $[x_1x_2] = [y_1y_2]$ and $y_i = gx_i$ for all $1 \le i \le 2$, we have $[y_1y_2] = [gx_1gx_2] = det g[x_1x_2] =$

$[x_1 x_2]$. Hence we obtain that $detg = 1$. That is $g \in SO(1,1)$. This means that systems V_x and V_y are $SO(1,1)$-equivalent.

Theorem 2.5. *Let V_x and V_y be two systems of vectors in M. Assume that $T(V_x) = T(V_y) = r$ for $r = 2,3$. Then following two conditions are equivalent:*

(i)

$$V_x \overset{SO(1,1)}{\sim} V_y$$

(ii)

$$< x_1, x_j > = < y_1, y_j >$$

for all $j = 1,2,\ldots,m$.

Proof. $(i) \to (ii)$: Let V_x be a system of vectors in M and $T(V_x) = r$ for all $r = 2,3$. Since the function $f(x_j, x_k) = < x_j, x_k >$ is $SO(1,1)$-invariant, condition (i) implies (ii).

$(ii) \to (i)$: Assume that condition (ii) is valid.

(a) We consider the case $T(V_x) = T(V_y) = 2$. Since $T(V_x) = T(V_y) = 2$, we have $rank(V_x) = rank(V_y) = 1$. Then there exists vector $x_1 \in V_x$ such that x_1 is a timelike vector. Since $< x_1, x_1 > = < y_1, y_1 >$ and $T(V_y) = 2$, there exists vector $y_1 \in V_y$ such that y_1 is a timelike vector.

From Theorem 2.2. and equality $< x_1, x_1 > = < y_1, y_1 >$, there exist $g \in O(1,1)$ such that $gx_1 = y_1$. We prove that $g \in SO(1,1)$. Assume that $g \in O(1,1)$ and $detg = -1$. Then we can be written $g = g_1 \eta$ such that $g_1 \in SO(1,1)$ and $\eta \in O(1,1)$. Put $x = (x_1, x_2), \bar{x} = (x_1, -x_2) \in M$. Since $gx_1 = y_1$ and $g = g_1 \eta$, we have $gx_1 = (g_1 \eta)x_1 = g_1(\eta x_1) = g_1 \bar{x_1} = y_1$. So there exist $g_1 \in SO(1,1)$ such that $g_1 \bar{x_1} = y_1$. Now we prove that the existence $h \in SO(1,1)$ such that $hx_1 = \bar{x_1}$. Assume that $h = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. From equality $hx_1 = \bar{x_1}$, there exist $a, b \in R$ such that $a^2 - b^2 = 1$. That is $h \in SO(1,1)$. Since $hx_1 = \bar{x_1}$ and $g_1 \bar{x_1} = y_1$, we have $(g_1 h) \in SO(1,1)$ such that $(g_1 h)x_1 = y_1$. Let $m > 1$. From Proposition 2.7. and Theorem 2.2., we have $(g_1 h)x_i = y_i$ for all $i > 1$. This means that systems V_x and V_y are $SO(1,1)$-equivalent.

(b) We consider the case $T(V_x) = T(V_y) = 3$. Then the proof is similar to the case (a).

Theorem 2.6. *Let V_x and V_y be two systems of vectors in M. Assume that $T(V_x) = T(V_y) = 4$. Then*

$$V_x \overset{SO(1,1)}{\sim} V_y \Leftrightarrow \begin{array}{c} \langle x_1, x_1 \rangle = \langle y_1, y_1 \rangle \\ sgn(x_{11}x_{12}) = sgn(y_{11}y_{12}) \\ L_x^2 = L_y^2 \end{array}$$

for $x_1 = (x_{11}, x_{12}), y_1 = (y_{11}, y_{12}) \in M$

Proof. $(i) \to (ii)$: Using Proposition 2.8. and Theorem 2.3., condition (i) imply $< x_1, x_1 >=< y_1, y_1 >$ and $L_x^2 = L_y^2$. We prove that $sgn(x_{11}x_{12}) = sgn(y_{11}y_{12})$. Since $V_x \overset{SO(1,1)}{\sim} V_y$, there exist $g \in SO(1,1)$ such that $gx_i = y_i$ for all $1 \le i \le m$. Let $x_1 = (x_{11}, x_{12}), y_1 = (y_{11}, y_{12}) \in M$. Since x_1 is a null vector, we have $x_1 = (x_{11}, x_{11})$ or $x_1 = (x_{11}, -x_{11})$. Since $g \in SO(1,1)$, we have $g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and $a^2 - b^2 = 1$. Assume that $x_1 = (x_{11}, x_{11})$. Using equality $gx_1 = y_1$, we have $y_1 = ((a+b)x_{11}, (a+b)x_{11})$. Hence, we have $sgn(x_{11}x_{12}) = sgn\left(x_{11}^2\right) > 0$ and $sgn(y_{11}y_{12}) = sgn\left((a+b)^2 x_{11}^2\right) > 0$. That is $sgn(x_{11}x_{12}) = sgn(y_{11}y_{12})$. Similarly, assume that $x_1 = (x_{11}, -x_{11})$. Using equality $gx_1 = y_1$, we have $y_1 = ((a+b)x_{11}, -(a+b)x_{11})$. Hence, we have $sgn(x_{11}x_{12}) = sgn\left(-x_{11}^2\right) < 0$ and $sgn(y_{11}y_{12}) = sgn\left(-(a+b)^2 x_{11}^2\right) < 0$. That is $sgn(x_{11}x_{12}) = sgn(y_{11}y_{12})$.

$(ii) \to (i)$: Assume that condition (ii) is valid.

Since $T(V_x) = T(V_y) = 4$, we have $rank(V_x) = rank(V_y) = 1$. Then there exists vector $x_1 \in V_x$ which is $x_1 \neq 0$ and $< x_1, x_1 >= 0$. Since $< x_1, x_1 >=< y_1, y_1 >= 0$ and $T(V_y) = 4$, y_1 is a null vector in V_y.

Let $x_1 = (x_{11}, x_{12})$, $y_1 = (y_{11}, y_{12}) \in M$. Since x_1 is a null vector, we have $x_1 = (x_{11}, x_{11})$ or $x_1 = (x_{11}, -x_{11})$. Similarly, since y_1 is a null vector, we have $y_1 = (y_{11}, y_{11})$ or $y_1 = (y_{11}, -y_{11})$. From equality $sgn(x_{11}x_{12}) = sgn(y_{11}y_{12})$, we have $x_1 = (x_{11}, x_{11})$ and $y_1 = (y_{11}, y_{11})$ or $x_1 = (x_{11}, -x_{11})$ and $y_1 = (y_{11}, -y_{11})$. Then there exist $g \in SO(1,1)$ such that $gx_1 = y_1$.

Since x_1 and y_1 are non-zero vectors, we have $x_i = \lambda_i x_1$ and $y_i = \beta_i y_1$ for all $i > 1$. According to condition (ii) of our theorem, since $L_x^2 = L_y^2$, we have $\lambda_i = \beta_i$ for all $i = 2, 3, \ldots, m$. Hence, for $g \in SO(1,1)$, we have $gx_i = \lambda_i g x_1 = \lambda_i y_1 = y_i$ for all $i > 1$. This means that systems V_x and V_y are $SO(1,1)$-equivalent.

Theorem 2.7. *Suppose that* $v = (v_1, v_2) \in M$ *is spacelike and* $w = (w_1, w_2) \in M$ *is either spacelike or null. Then,*

(i) $v_1 w_1 > 0$*, in which case* $\langle v, w \rangle > 0$

(ii) $v_1 w_1 < 0$*, in which case* $\langle v, w \rangle < 0$

Proof. The proof is similar to the proof of theorem in [9, Theorem 1.3.1].

Theorem 2.8. *Let A be an element of* $O(1,1)$. *Then following two conditions are equivalent:*

(i) $A \in L(1,1)$

(ii) *A preserves the space orientation of all null vectors and spacelike vectors.*

Proof. The proof is similar to the proof of theorem in [9, Theorem 1.3.3].

Theorem 2.9. *Let* V_x *and* V_y *be two systems of vectors in* M *and* $T(V_x) = T(V_y) = 1$. *Then*

(i) *if* x_1 *is one of linearly independent vectors in* V_x *which is a timelike(or null) vector, then*

$$V_x \overset{L(1,1)}{\sim} V_y \Leftrightarrow \begin{array}{c} \langle x_i, x_j \rangle = \langle y_i, y_j \rangle \\ [x_1 x_2] = [y_1 y_2] \\ sgn(x_{12}) = sgn(y_{12}) \end{array}$$

for all $i = 1,2; j = 1,2,\ldots,m,\ i \leq j$.

(ii) *if* x_1 *is one of linearly independent vectors in* V_x *which is a spacelike vector , then*

$$V_x \overset{L(1,1)}{\sim} V_y \Leftrightarrow \begin{array}{c} < x_i, x_j >=< y_i, y_j > \\ [x_1 x_2] = [y_1 y_2] \\ sgn(x_{11}) = sgn(y_{11}) \end{array}$$

for all $i = 1,2; j = 1,2,\ldots,m,\ i \leq j$.

Proof. It follow from [9, Theorem 1.3.1], [9, Theorem 1.3.3], Theorems 2.4., 2.7., 2.8.

Theorem 2.10. *Let* V_x *and* V_y *be two systems of vectors in* M. *Assume that* $T(V_x) = T(V_y) = 2$. *Then*

$$V_x \overset{L(1,1)}{\sim} V_y \Leftrightarrow \begin{array}{c} < x_1, x_1 >=< y_1, y_1 > \\ sgn(x_{12}) = sgn(y_{12}) \end{array}$$

Proof. It follow from Theorems 2.5., 2.8., [9, Theorem 1.3.1] and [9, Theorem 1.3.3].

Theorem 2.11. *Let V_x and V_y be two systems of vectors in M. Assume that $T(V_x) = T(V_y) = 3$. Then*

$$V_x \overset{L(1,1)}{\sim} V_y \Leftrightarrow \begin{array}{c} <x_1,x_1>=<y_1,y_1> \\ sgn(x_{11}) = sgn(y_{11}) \end{array}$$

Proof. It follow from Theorems 2.5., 2.7. and 2.8.

Theorem 2.12. *Let V_x and V_y be two systems of vectors in M. Assume that $T(V_x) = T(V_y) = 4$. Then*

$$V_x \overset{L(1,1)}{\sim} V_y \Leftrightarrow \begin{array}{c} <x_1,x_1>=<y_1,y_1> \\ sgn(x_{11}x_{12}) = sgn(y_{11}y_{12}) \\ sgn(x_{12}) = sgn(y_{12}) \\ L_x^2 = L_y^2 \end{array}$$

Proof. It follow from Theorems 2.6., 2.7. and 2.8.

3. The equivalence of Bézier curves

Definition 3.1. Bézier curves $\alpha(t)$ and $\beta(t)$ in M will be called G -equivalent and written $\alpha \overset{G}{\sim} \beta$ if there exists $F \in G$ such that $\beta(t) = F\alpha(t)$ for all $t \in [0,1]$.

Remark 3.1. In this definition, Bézier curves are considered as paths (see [13, p. 796]; [19, Definition 3].

Definition 3.2. A G-invariant function $f(x_0,x_1,\ldots,x_m)$ of control points $x_0,x_1,\ldots,x_m$ of a Bézier curve $\alpha(t) = \sum_{j=0}^{m} x_j B_{j,m}(t)$ will be called a control G-invariant of $\alpha(t)$, where $B_{j,m}(t)$ are Bernstein basis polynomials.

Example 3.1. Let $\alpha(t)$ and $\beta(t)$ be Bézier curves of degrees of m and k, respectively. Assume that $\alpha \overset{O(1,1)}{\sim} \beta$. Then $m = k$ that is the degree of a Bézier curve $\alpha(t)$ is $O(1,1)$-invariant.

Theorem 3.1. *Let $\alpha(t) = \sum_{j=0}^{m} x_j B_{j,m}(t)$ and $\beta(t) = \sum_{j=0}^{m} y_j B_{j,m}(t)$ be Bézier curves in M of degree m. Then following four conditions are equivalent:*

(i) $\alpha \overset{M(1,1)}{\sim} \beta$

(ii) $\{x_0,x_1,\ldots,x_m\} \overset{M(1,1)}{\sim} \{y_0,y_1,\ldots,y_m\}$

(iii) $\{x_1 - x_0, x_2 - x_0, \ldots, x_m - x_0\} \overset{O(1,1)}{\sim} \{y_1 - y_0, y_2 - y_0, \ldots, y_m - y_0\}$

Proof. $(i) \leftrightarrow (ii)$: According to the property of the affine invariance ([4, p. 137]),

$$F\left(\sum_{j=0}^{m} x_j B_{j,m}(t)\right) = \sum_{j=0}^{m} F(x_j) B_{j,m}(t) \tag{11}$$

for every $F \in M(1,1)$. Assume that $\alpha \overset{M(1,1)}{\sim} \beta$. Then $\beta(t) = F\alpha(t)$ for some $F \in M(1,1)$. Using (11), we obtain $y_j = Fx_j$ for all $j = 0, 1, \ldots, m$ that is $\{x_0, x_1, \ldots, x_m\} \overset{M(1,1)}{\sim} \{y_0, y_1, \ldots, y_m\}$. Conversely, suppose that $\{x_0, x_1, \ldots, x_m\} \overset{M(1,1)}{\sim} \{y_0, y_1, \ldots, y_m\}$. Then there exists $F \in M(1,1)$ such that $y_j = Fx_j$ for all $j = 0, 1, \ldots, m$. Using (11), we obtain $\beta(t) = F\alpha(t)$ that is $\alpha \overset{M(1,1)}{\sim} \beta$.

$(ii) \leftrightarrow (iii)$: Assume that $\{x_0, x_1, \ldots, x_m\} \overset{M(1,1)}{\sim} \{y_0, y_1, \ldots, y_m\}$. Then there exists $F \in M(1,1)$, where F has the form $Fz = gz + p$, $g \in O(1,1)$, $p \in M$ for all $z \in M$ such that $y_j = Fx_j = gx_j + p$ for all $j = 0, 1, \ldots, m$. These equalities imply $y_j - y_0 = g(x_j - x_0)$ for all $j = 1, 2, \ldots, m$. This means that $\{x_i - x_0, 1 \leq i \leq m\} \overset{O(1,1)}{\sim} \{y_i - y_0, 1 \leq i \leq m\}$. Conversely, assume that $\{x_i - x_0, 1 \leq i \leq m\} \overset{O(1,1)}{\sim} \{y_i - y_0, 1 \leq i \leq m\}$. Then there exists $g \in O(1,1)$ such that $y_j - y_0 = g(x_j - x_0)$ for all $j = 1, 2, \ldots, m$. Put $p = y_0 - gx_0$. Then $y_j = gx_j + p$ for all $j = 0, 1, \ldots, m$. This means that $\{x_0, x_1, \ldots, x_m\} \overset{M(1,1)}{\sim} \{y_0, y_1, \ldots, y_m\}$.

Corollary 3.1. *Let $\{x_0, x_1, \ldots, x_m\}$ be a system of vectors in M. Then the type $T\,(x_1 - x_0, \ldots, x_m - x_0)$ is $O(1,1)$-invariant.*

Definition 3.3. Let $\alpha(t) = \sum_{j=0}^{m} x_j B_{j,m}(t)$ be Bézier curves in M of degree m. The type $T(x_1 - x_0, x_2 - x_0 \ldots, x_m - x_0)$ of the system $\{x_1 - x_0, x_2 - x_0 \ldots, x_m - x_0\}$ will be called the control points type of the Bézier curve α and will be denoted by $T(\alpha)$.

Since the control points type of a Bézier curve is $O(1,1)$-invariant, in the case $T(\alpha) \neq T(\beta)$, Bézier curves α and β are not $0(1,1)$-equivalent. Therefore, for an investigation of $0(1,1)$-equivalence of Bézier curves α and β, we assume that $T(\alpha) = T(\beta)$.

Theorem 3.2. *Let $\alpha(t) = \sum_{j=0}^{m} x_j B_{j,m}(t)$ and $\beta(t) = \sum_{j=0}^{m} y_j B_{j,m}(t)$ be Bézier curves in M of degree m. Assume that $T(\alpha) = T(\beta) = 1$. Then*

$$\alpha \overset{M(1,1)}{\sim} \beta \Leftrightarrow < x_i - x_0, x_j - x_0 > = < y_i - y_0, y_j - y_0 >$$

for all $i = 1,2, j = 1,2,\dots,m; i \le j$

Proof. It follows from Theorem 2.1. and Theorem 3.1.

Theorem 3.3. *Let* $\alpha(t) = \sum_{j=0}^{m} x_j B_{j,m}(t)$ *and* $\beta(t) = \sum_{j=0}^{m} y_j B_{j,m}(t)$ *be Bézier curves in* M *of degree* m*. Assume that* $T(\alpha) = T(\beta) = r$ *for* $r = 2,3$*. Then*

$$\alpha \overset{M(1,1)}{\sim} \beta \Leftrightarrow < x_1 - x_0, x_j - x_0 > = < y_1 - y_0, y_j - y_0 >$$

for all $j = 1,2,\dots,m$*.*

Proof. It follows from Theorem 2.2. and Theorem 3.1.

Theorem 3.4. *Let* $\alpha(t) = \sum_{j=0}^{m} x_j B_{j,m}(t)$ *and* $\beta(t) = \sum_{j=0}^{m} y_j B_{j,m}(t)$ *be Bézier curves in* M *of degree* m*. Assume that* $T(\alpha) = T(\beta) = 4$*. Then*

$$\alpha \overset{M(1,1)}{\sim} \beta \Leftrightarrow \begin{array}{c} \langle x_1 - x_0, x_1 - x_0 \rangle = \langle y_1 - y_0, y_1 - y_0 \rangle \\ L^2_{x-x_0} = L^2_{y-y_0} \end{array}$$

Proof. It follows from Theorem 2.3. and Theorem 3.1.

Theorem 3.5. *Let* $\alpha(t) = \sum_{j=0}^{m} x_j B_{j,m}(t)$ *and* $\beta(t) = \sum_{j=0}^{m} y_j B_{j,m}(t)$ *be Bézier curves in* M *of degree* m*. Then following three conditions are equivalent:*

(i) $\alpha \overset{SM(1,1)}{\sim} \beta$

(ii) $\{x_0, x_1, \dots, x_m\} \overset{SM(1,1)}{\sim} \{y_0, y_1, \dots, y_m\}$

(iii) $\{x_1 - x_0, x_2 - x_0, \dots, x_m - x_0\} \overset{SO(1,1)}{\sim} \{y_1 - y_0, y_2 - y_0, \dots, y_m - y_0\}$

Proof. It is similar to the proof of Theorem 3.1.

Theorem 3.6. *Let* $\alpha(t) = \sum_{j=0}^{m} x_j B_{j,m}(t)$ *and* $\beta(t) = \sum_{j=0}^{m} y_j B_{j,m}(t)$ *be Bézier curves in* M *of degree* m*. Assume that* $T(\alpha) = T(\beta) = 1$*. Then*

$$\alpha \overset{SM(1,1)}{\sim} \beta \Leftrightarrow \begin{array}{c} \langle x_i - x_0, x_j - x_0 \rangle = \langle y_i - y_0, y_j - y_0 \rangle \\ [(x_1 - x_0)(x_2 - x_0)] = [(y_1 - y_0)(y_2 - y_0)] \end{array}$$

for all $i = 1,2, j = 1,2,\dots,m; i \le j$

Proof. It follows from Theorem 2.4. and Theorem 3.5.

Theorem 3.7. *Let $\alpha(t)=\sum_{j=0}^{m}x_jB_{j,m}(t)$ and $\beta(t)=\sum_{j=0}^{m}y_jB_{j,m}(t)$ be Bézier curves in M of degree m. Assume that $T(\alpha)=T(\beta)=r$ for $r=2,3$. Then*

$$\alpha \overset{SM(1,1)}{\sim} \beta \Leftrightarrow \left\langle x_1-x_0,x_j-x_0\right\rangle=\left\langle y_1-y_0,y_j-y_0\right\rangle$$

for all $j=1,2,\ldots,m$.

Proof. It follows from Theorem 2.5. and Theorem 3.5.

Theorem 3.8. *Let $\alpha(t)=\sum_{j=0}^{m}x_jB_{j,m}(t)$ and $\beta(t)=\sum_{j=0}^{m}y_jB_{j,m}(t)$ be Bézier curves in M of degree m. Assume that $T(\alpha)=T(\beta)=4$. Then*

$$\alpha \overset{SM(1,1)}{\sim} \beta \Leftrightarrow \begin{array}{c} \langle x_1-x_0,x_1-x_0\rangle=\langle y_1-y_0,y_1-y_0\rangle \\ sgn\left((x_{11}-x_{01})(x_{12}-x_{02})\right)=sgn\left((y_{11}-y_{01})(y_{12}-y_{02})\right) \\ L^2_{x-x_0}=L^2_{y-y_0} \end{array}$$

Proof. It follows from Theorem 2.6. and Theorem 3.5.

Theorem 3.9. *Let $\alpha(t)=\sum_{j=0}^{m}x_jB_{j,m}(t)$ and $\beta(t)=\sum_{j=0}^{m}y_jB_{j,m}(t)$ be Bézier curves in M of degree m. Then following three conditions are equivalent:*

- (i) $\alpha \overset{SL(1,1)}{\sim} \beta$
- (ii) $\{x_0,x_1,\ldots,x_m\} \overset{SL(1,1)}{\sim} \{y_0,y_1,\ldots,y_m\}$
- (iii) $\{x_1-x_0,x_2-x_0,\ldots,x_m-x_0\} \overset{SL(1,1)}{\sim} \{y_1-y_0,y_2-y_0,\ldots,y_m-y_0\}$

Proof. It is similar to the proof of Theorem 3.5.

Theorem 3.10. *Let $\alpha(t)=\sum_{j=0}^{m}x_jB_{j,m}(t)$ and $\beta(t)=\sum_{j=0}^{m}y_jB_{j,m}(t)$ be Bézier curves in M of degree m. Assume that$T(\alpha)=T(\beta)=1$.*

(i) *if x_1 is one of control points in $\alpha(t)$ which is a timelike(or null) vector, then*

$$\alpha \overset{SL}{\sim} \beta \Leftrightarrow \begin{array}{c} \left\langle x_i-x_0,x_j-x_0\right\rangle=\left\langle y_i-y_0,y_j-y_0\right\rangle \\ [(x_1-x_0)(x_2-x_0)]=[(y_1-y_0)(y_2-y_0)] \\ sgn\left(x_{12}-x_{02}\right)=sgn\left(y_{12}-y_{02}\right) \end{array}$$

for all $i=1,2, j=1,2,\ldots,m; i\leq j$

(ii) *if x_1 is one of control points in $\alpha(t)$ which is a spacelike vector, then*

$$\alpha \overset{SL(1,1)}{\sim} \beta \Leftrightarrow \begin{array}{c} \langle x_i - x_0, x_j - x_0 \rangle = \langle y_i - y_0, y_j - y_0 \rangle \\ [(x_1 - x_0)(x_2 - x_0)] = [(y_1 - y_0)(y_2 - y_0)] \\ sgn\,(x_{11} - x_{01}) = sgn\,(y_{11} - y_{01}) \end{array}$$

for all $i = 1,2, j = 1,2,\ldots,m; i \leq j$

Proof. It follows from Theorem 2.9. and Theorem 3.9.

Theorem 3.11. *Let $\alpha(t) = \sum_{j=0}^{m} x_j B_{j,m}(t)$ and $\beta(t) = \sum_{j=0}^{m} y_j B_{j,m}(t)$ be Bézier curves in M of degree m. Assume that $T(\alpha) = T(\beta) = 2$. Then*

$$\alpha \overset{SL(1,1)}{\sim} \beta \Leftrightarrow \begin{array}{c} \langle x_1 - x_0, x_1 - x_0 \rangle = \langle y_1 - y_0, y_1 - y_0 \rangle \\ sgn\,(x_{12} - x_{02}) = sgn\,(y_{12} - y_{02}) \end{array}$$

Proof. It follows from Theorem 2.10 and Theorem 3.9.

Theorem 3.12. *Let $\alpha(t) = \sum_{j=0}^{m} x_j B_{j,m}(t)$ and $\beta(t) = \sum_{j=0}^{m} y_j B_{j,m}(t)$ be Bézier curves in M of degree m. Assume that $T(\alpha) = T(\beta) = 3$. Then*

$$\alpha \overset{SL(1,1)}{\sim} \beta \Leftrightarrow \begin{array}{c} \langle x_1 - x_0, x_j - x_0 \rangle = \langle y_1 - y_0, y_j - y_0 \rangle \\ sgn\,(x_{11} - x_{01}) = sgn\,(y_{11} - y_{01}) \end{array}$$

Proof. It follows from Theorem 2.11 and Theorem 3.9.

Theorem 3.13. *Let $\alpha(t) = \sum_{j=0}^{m} x_j B_{j,m}(t)$ and $\beta(t) = \sum_{j=0}^{m} y_j B_{j,m}(t)$ be Bézier curves in M of degree m. Assume that $T(\alpha) = T(\beta) = 4$. Then*

$$\alpha \overset{SL(1,1)}{\sim} \beta \Leftrightarrow \begin{array}{c} \langle x_1 - x_0, x_1 - x_0 \rangle = \langle y_1 - y_0, y_1 - y_0 \rangle \\ sgn\,((x_{11} - x_{01})(x_{12} - x_{02})) = sgn\,((y_{11} - y_{01})(y_{12} - y_{02})) \\ sgn\,(x_{12} - x_{02}) = sgn\,(y_{12} - y_{02}) \\ L^2_{x - x_0} = L^2_{y - y_0} \end{array}$$

Proof. It follows from Theorem 2.12 and Theorem 3.9.

Conflict of Interests

The author declares that there is no conflict of interests.

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