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k-GAMMA, k-BETA MATRIX FUNCTIONS AND THEIR PROPERTIES

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Abstract. The main aim of this paper is to define k-gamma and k-beta matrix functions, and derive the conditions for matrices M,N so that the k-beta matrix function $B_k(M,N)$ satisfies the relations $B_k(M,N) = B_k(N,M)$ and $B_k(M,N) = \Gamma_k(M)\Gamma_k(N)\Gamma_k^{-1}(M+N)$ in the form of k-symbol, where $k > 0$. A limit expression for the k-gamma function of a matrix is also established.

Keywords: k-gamma matrix function; k-beta matrix function; Factorial function, Matrix.

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1. Introduction

Many of the ordinary special functions of mathematical physics and most of their useful properties can be obtained from the theory of group representations. James [4] discussed that the special functions of a matrix argument appear in the study of spherical functions on certain symmetric spaces and multivariate analysis in statistics. Special functions of two diagonal matrix argument have been used in [5]. In [6], some properties of gamma and beta matrix functions are proved and analogue of the expression of the scalar gamma function as a limit is

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given for the gamma function of a matrix and it is also shown that the conditions for matrices M,N in $C^{r\times r}$ so that $B(M,N)$ is well defined and satisfy $B(M,N) = B(N,M)$, and $B(M,N) = \Gamma(M)\Gamma(N)\Gamma^{-1}(M+N)$ are established.

2. Preliminaries

Definition 2.1. The factorial function is denoted and defined by, $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$; for $n \geq 1, a \neq 0$ and $(a)_0 = 1$. The function $(a)_n$ is called the factorial function. It is also known as Pochhmmer's symbol.

Note that $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. It is an immediate generalization of the elementary factorial i.e., $n! = (1)_n$. In manipulations with $(a)_n$, it is important to keep in mind that $(a)_n$ is a product of n factors, starting with a and with each factor large by unity than the preceding factor.

Definition 2.2. Let $z \in \mathbb{C}$ ($\mathbb{C}$ is a set of complex numbers), the gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad Re(z) > 0.$$

In another way, it is defined as

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^{z-1}}{(z)_n}.$$

The relation between Pochhammer's symbol and gamma function is given below

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)},$$

see [3].

Definition 2.3. Let $k > 0$, then the Pochhammer k-symbol is defined by $(a)_{n,k} = a(a+k)(a+2k)\cdots(a+(n-1)k)$ for $n \geq 1, a \neq 0$ and$(a)_{0,k} = 0$.

Definition 2.4. For $k > 0$ and $z \in \mathbb{C}$, the k-gamma function Γ_k is defined by

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}}.$$

Its integral representation is also given by,

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{\frac{-t^k}{k}} dt.$$

The relation between Pochhammer *k*-symbol and *k*-gamma function is given as

$$(z)_{n,k} = \frac{\Gamma_k(z+nk)}{\Gamma_k(z)},$$

see [1].

Definition 2.5. If P is a matrix in $C^{r\times r}$ then by application of the matrix functional calculus, we define the pochhammar symbol for any matrix P in $C^{r\times r}$ as;

$$(P)_n = P(P+I)(P+2I)\cdots(P+(n-1)I), \qquad n>0, \qquad (P)_0 = I. \tag{1}$$

Definition 2.6. If P is a matrix in $C^{r\times r}$ and $k>0$ then by application of the matrix functional calculus, we define the pochhammar *k*-symbol for any matrix P in $C^{r\times r}$ as;

$$(P)_n = P(P+kI)(P+2kI)\cdots(P+(n-1)kI), \qquad n>0, \qquad (P)_{0,k} = I. \tag{2}$$

If P lies in $C^{r\times r}$, using decomposition and denoting $\alpha(P) = max_{z\in\sigma(P)}\mathbb{R}(z)$ (where $\sigma(P)$ is the set of all eigenvalue of P) for $t\in\mathbb{R}$, it follows that [7, pp. 336-556]:

$$\|e^{tP}\| \le e^{t\alpha(P)}[\frac{(\sum_{j=0}^{r-1}\|P\|\sqrt{r}t)^j}{j!}]. \tag{3}$$

Definition 2.7. Let M be a matrix and let $n\geq 1$, then $\Gamma(M)$ is defined by

$$\Gamma(M) = \lim_{n\to\infty}(n-1)!(M)_n^{-1}n^M, \tag{4}$$

where $(M)_n = M(M+I)\cdots(M+(n-1)I)$.

Definition 2.8. Let M and N be two matrices in $C^{r\times r}$ such that $Re(z)>0$, $Re(w)>0$, for all $z\in\sigma(M)$ and $w\in\sigma(N)$, then

$$B(M,N) = \int_0^\infty t^{M-I}(1-t)^{N-I}dt$$

and

$$B(M,N) = \Gamma_k(M)\Gamma_k(N)\Gamma_k^{-1}(M+N),$$

see [6].

Lemma 2.1. *If* $0 \leq \alpha < 1$ *and* $k > 0$*, then*

$$1+\alpha \leq e^{\alpha} \leq (1-\alpha)^{-1}.$$

Lemma 2.2. *If* $0 \leq \alpha < 1$*, n is a positive integer, then*

$$(1-\alpha)^n \geq 1-n\alpha.$$

Lemma 2.3. *If* $0 \leq t < n$*, n is a positive integer, then*

$$0 \leq e^{-t} - (1-\frac{t}{n}) \leq \frac{te^{-t}}{n},$$

see [3].

3. Derivation of k-gamma and k-beta matrix functions

To derive the k-gamma matrix function, first we have to prove the above Lemma 2.3 in terms of k, which is given by

Lemma 3.1. *If* $0 \leq t < n$*, n is a positive integer and* $k > 0$*, then*

$$0 \leq e^{-\frac{t^k}{k}} - (1-\frac{t^k}{nk}) \leq \frac{t^{2k}e^{-t^k}}{nk^2}.$$

Proof. Using $\alpha = \frac{t^k}{nk}$ in Lemma 2.1, we get

$$1+\frac{t^k}{nk} \leq e^{\frac{t^k}{nk}} \leq (1-\frac{t^k}{nk})^{-1},$$

from which it follows that

$$(1+\frac{t^k}{nk})^n \leq e^{\frac{t^k}{k}} \leq (1-\frac{t^k}{nk})^{-n}$$
$$\Rightarrow (1+\frac{t^k}{nk})^{-n} \geq e^{-\frac{t^k}{k}} \geq (1-\frac{t^k}{nk})^n.$$

Hence, we have $e^{-\frac{t^k}{k}} - (1-\frac{t^k}{nk})^n \geq 0$ and

$$e^{-\frac{t^k}{k}} - (1-\frac{t^k}{n})^n = e^{-\frac{t^k}{k}}[1-e^{\frac{t^k}{k}}(1-\frac{t^k}{nk})^n].$$

Since $e^{\frac{t^k}{k}} \geq (1+\frac{t^k}{nk})^n$, we have

$$e^{-\frac{t^k}{k}} - (1-\frac{t^k}{nk})^n \leq e^{-\frac{t^k}{k}}[1-(1-\frac{t^{2k}}{n^2k^2})^n]. \tag{5}$$

Now using $\alpha = \frac{t^{2k}}{n^2k^2}$ in Lemma 2.2, we obtain

$$(1-\frac{t^{2k}}{n^2k^2})^n \geq 1-\frac{t^{2k}}{nk^2}.$$

Using this result in equation (5), we get

$$e^{-\frac{t^k}{k}} - (1-\frac{t^k}{nk})^n \leq e^{\frac{-t^k}{k}}[1-1+\frac{t^{2k}}{nk^2})^n] = \frac{t^{2k}}{nk^2}e^{-\frac{t^k}{k}},$$

which is the required result.

Lemma 3.2. *If M is a matrix in $C^{r\times r}$, $k>0$ and $\mathbb{R}e(z)>0$ for all $z \in \sigma(M)$, then by application of matrix calculus, we have*

$$\Gamma_k(M) = \lim_{n\to\infty} \int_0^{(nk)^{\frac{1}{k}}} (1-\frac{t^k}{nk})^n t^{M-I} dt = \lim_{n\to\infty} n!k^n (nk)^{\frac{M}{k}-1}(M)^{-1}_{n,k}. \tag{6}$$

Proof. In the integral on right hand side in (6) put $\frac{t^k}{nk} = \beta$, this implies that $t = (nk\beta)^{\frac{1}{k}}$ (where β is a matrix, so this means that $\beta^{\frac{1}{k}} = \beta^{\frac{I}{k}}$). Thus after simplification we obtain

$$\int_0^{(nk)^{\frac{1}{k}}} (1-\frac{t^k}{nk})^n t^{M-I} dt = \frac{(nk)^{\frac{M}{k}}}{k}\int_0^1 (1-\beta)^n \beta^{\frac{M}{k}-I} d\beta. \tag{7}$$

An integrating by parts gives us the reduction formula, we get

$$\begin{aligned}
\int_0^1 (1-\beta)^n \beta^{\frac{M}{k}-I} d\beta &= \frac{k^{n-1}n(n-1)(n-2)\cdots 1}{M(M+kI)(M+2kI)\cdots(M+(n-1)kI)}\int_0^1 \beta^{\frac{M}{k}+n-I} d\beta \\
&= \frac{k^{n+1}n(n-1)(n-2)\cdots 1}{M(M+kI)(M+2kI)\cdots(M+(n-1)kI)(M+nkI)}[\beta^{\frac{M}{k}+n}]_0^1 \\
&= \frac{k^{n+1}n!}{M(M+kI)(M+2kI)\cdots(M+(n-1)kI)(M+nkI)}.
\end{aligned}$$

Therefore, (7) becomes

$$\int_0^{(nk)^{\frac{1}{k}}} (1-\frac{t^{nk}}{n})^n t^{M-I} dt = \frac{(nk)^{\frac{M}{k}} k^n n!}{M(M+kI)(M+2kI)\cdots(M+(n-1)kI)(M+nkI)}$$

so that

$$\lim_{n\to\infty} \int_0^{(nk)^{\frac{1}{k}}} (1-\frac{t^k}{nk})^n t^{M-I} dt = \lim_{n\to\infty} n! k^n (nk)^{\frac{M}{k}-I} (M)_{n,k}^{-1}.$$

Furthermore, we write

$$\Gamma_k(M) = \lim_{n\to\infty} n! k^n (nk)^{\frac{M}{k}-I} (M)_{n,k}^{-1}.$$

Theorem 3.1. *If M is a matrix in $C^{r\times r}$ and $k>0$, then by matrix functional calculus gamma matrix as:*

$$\Gamma_k(M) = \int_0^\infty t^{M-I} e^{-\frac{t^k}{k}} dt. \tag{8}$$

Proof. The integral on right hand side in (8) converges. With the aid of above (6) and (8), we write

$$\begin{aligned}\int_0^\infty e^{-\frac{t^k}{k}} t^{M-I} dt - \Gamma_k(M) &= \lim_{n\to\infty}[\int_0^\infty e^{-\frac{t^k}{k}} t^{M-I} dt - \int_0^{(nk)^{\frac{1}{k}}} (1-\frac{t^k}{nk})^n t^{M-I} dt] \\ &= \lim_{n\to\infty}[\int_0^{n^{\frac{1}{k}}} [e^{-\frac{t^k}{k}} - (1-\frac{t^k}{nk})^n] t^{M-I} dt - \int_{n^{\frac{1}{k}}}^\infty e^{-\frac{t^k}{k}} t^{M-I} dt].\end{aligned}$$

Since $\int_0^\infty e^{-t^k} t^{z-1} dt$ is convergent, so this implies that

$$\lim_{n\to\infty} \int_{(nk)^{\frac{1}{k}}}^\infty e^{-\frac{t^k}{k}} t^{M-I} dt = 0.$$

Hence

$$\int_0^\infty e^{-\frac{t^k}{k}} t^{M-I} dt - \Gamma_k(z) = \lim_{n\to\infty} \int_0^{(nk)^{\frac{1}{k}}} [e^{-\frac{t^k}{k}} - (1-\frac{t^k}{nk})^n] t^{M-I} dt.$$

Next, we prove

$$\lim_{n\to\infty}\int_{0}^{(nk)^{\frac{1}{k}}}[e^{-\frac{t^k}{k}}-(1-\frac{t^k}{nk})^n]t^{M-I}dt=0. \tag{9}$$

By lemma 3.1, $0\leq e^{-\frac{t^k}{k}}-(1-\frac{t^k}{nk})^n\leq\frac{t^{2k}e^{-\frac{t^k}{k}}}{nk^2}=\frac{t^{2kI}e^{-\frac{t^k}{k}}}{nk^2}$, where $0\leq t\leq n$ Hence

$$\|\int_{0}^{(nk)^{\frac{1}{k}}}[e^{-\frac{t^k}{k}}-(1-\frac{t^k}{nk})^n]t^{M-I}dt\|\leq\frac{1}{nk^2}\int_{0}^{(nk)^{\frac{1}{k}}}\|t^{M+I}\|e^{-\frac{t^k}{k}}dt. \tag{10}$$

By equation (3) and using $\ln t\leq t$ for $t>0$, we write

$$\begin{aligned}\|t^{M+I}\| &\leq t^{\alpha(M)+1}(\frac{[\sum_{j=0}^{r-1}(\|M\|+1)\sqrt{r}\ln t]^j}{j!})\\ &\leq t^{\alpha(M)+1}\{\frac{[\sum_{j=0}^{r-1}(\|M\|+1)\sqrt{r}t]^j}{j!}\}. \end{aligned}\tag{11}$$

By (10) and (11), we have

$$\frac{1}{nk^2}\int_{0}^{(nk)^{\frac{1}{k}}}\|t^{M+I}\|e^{-\frac{t^k}{k}}dt\leq\frac{1}{nk^2}\{\frac{[\sum_{j=0}^{r-1}(\|M\|+1)\sqrt{r}t]^j}{j!}\}\int_{0}^{(nk)^{\frac{1}{k}}}t^{\alpha(M)+j+1}e^{-\frac{t^k}{k}}dt. \tag{12}$$

Since for $0\leq j\leq r-1$, we have $\int_0^\infty t^{\alpha(M)+j+1}e^{-\frac{t^k}{k}}dt$ is convergent. Thus $\int_0^{(nk)^{\frac{1}{k}}}t^{\alpha(M)+j+1}e^{-\frac{t^k}{k}}dt$ is bounded. Therefore

$$\lim_{n\to\infty}\int_{0}^{(nk)^{\frac{1}{k}}}[e^{-\frac{t^k}{k}}-(1-\frac{t^k}{nk})^n]t^{M-I}dt=0. \tag{13}$$

Hence the following result has been established.

$$\Gamma_k(M)=\int_0^\infty t^{M-I}e^{-\frac{t^k}{k}}dt.$$

Since the reciprocal k-gamma function denoted by $\Gamma_k^{-1}(z)=\frac{1}{\Gamma_k(z)}$ is an entire function of the complex variable z. In case of gamma function, for any matrix M in $C^{r\times r}$ the Riesz-Dunford functional calculus shows that the image of $\Gamma^{-1}(z)$ acting on M, denoted by $\Gamma^{-1}(M)$ is a well defined matrix, see [4]. Similarly the image of $\Gamma_k^{-1}(z)$ acting on M is denoted by $\Gamma_k^{-1}(M)$ is well defined matrix. Furthermore, if M is a matrix such that $M+nkI$ is invertible matrix for every integer $n \geq 0$, then $\Gamma_k(M)$ is invertible, its inverse coincides with $\Gamma_k^{-1}(M)$ and

$$M(M+kI)(M+2kI)\cdots(M+(n-1)kI)\Gamma_k^{-1}(M+nkI) = \Gamma_k^{-1}(M), n \geq 1, k > 0. \tag{14}$$

From equation (14), we can write

$$M(M+kI)(M+2kI)\cdots(M+(n-1)kI) = \Gamma_k(M+nkI)\Gamma_k^{-1}(M), n \geq 1, k > 0. \tag{15}$$

Theorem 3.2. *Let M and N be two matrices in $C^{r\times r}$ such that $\mathbb{R}e(z) > 0$, $\mathbb{R}e(w) > 0$, for all $z \in \sigma(M)$ and $w \in \sigma(N)$, then*

$$B_k(M,N) = \frac{1}{k}\int_0^\infty t^{\frac{M}{k}-I}(1-t)^{\frac{N}{k}-I}dt \tag{16}$$

Proof. By equation (3) and using $\ln t \leq t$ and $\ln(1-t) \leq 1-t$ for $0 < t < 1$, it follows that

$$\begin{aligned}
&\frac{1}{k}\|\int_0^\infty t^{\frac{M}{k}-I}(1-t)^{\frac{N}{k}-I}dt\| \\
&\leq \frac{1}{k}\int_0^\infty \|t^{\frac{M}{k}-I}\|\|(1-t)^{\frac{N}{k}-I}\|dt \\
&\leq \sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\frac{(\|M\|+1)^i(\|N\|+j)^j(\sqrt{r})^{i+j}}{i!j!k^{i+j+1}}\int_0^\infty t^{\frac{\alpha(M)}{k}-1}(1-t)^{\frac{\alpha(N)}{k}-1}\ln^i(t)\ln^j(1-t)dt \\
&\leq \sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\frac{(\|M\|+1)^i(\|N\|+j)^j(\sqrt{r})^{i+j}}{i!j!k^{i+j+1}}\int_0^\infty t^{\frac{\alpha(M)}{k}-1}(1-t)^{\frac{\alpha(N)}{k}-1}(t)^i(1-t)^jdt \\
&\leq \sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\frac{(\|M\|+1)^i(\|N\|+j)^j(\sqrt{r})^{i+j}}{i!j!k^{i+j+1}}\int_0^\infty t^{\frac{\alpha(M)}{k}+i-1}(1-t)^{\frac{\alpha(N)}{k}+j-1}dt \\
&\leq \sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\frac{(\|M\|+1)^i(\|N\|+j)^j(\sqrt{r})^{i+j}}{i!j!k^{i+j}}B_k(\alpha(M)+ik,\alpha(N)+jk).
\end{aligned}$$

Since

$$\sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\frac{(\|M\|+1)^i(\|N\|+j)^j(\sqrt{r})^{i+j}}{i!j!k^{i+j}}B_k(\alpha(M)+ik,\alpha(N)+jk) < +\infty,$$

we see that $B_k(M,N) = \frac{1}{k}\int\limits_0^\infty t^{\frac{M}{k}-I}(1-t)^{\frac{N}{k}-I}dt$.

Next we prove the following Lemma related to k-beta matrix function.

Lemma 3.3. *Let M and N be commuting matrices in $C^{r\times r}$ such that $Re(z) > 0$, $Re(w) > 0$, for all $z \in \sigma(M)$ and $w \in \sigma(N)$, then*

$$B_k(M,N) = B_k(N,M). \tag{17}$$

Proof. Since M and N are commutable, therefore $MN = NM$. It follows that

$$(\frac{M}{k}-I)(\ln t)(\frac{N}{k}-I)\ln(1-t) = (\frac{N}{k}-I)\ln(1-t)(\frac{M}{k}-I)(\ln t), \qquad 0 < t < 1.$$

Hence, we write

$$\begin{aligned}
B_k(M,N) &= \frac{1}{k}\int\limits_0^1 t^{\frac{M}{k}-I}(1-t)^{\frac{N}{k}-I}dt \\
&= \frac{1}{k}\int\limits_0^1 e^{(\frac{M}{k}-I)\ln t}e^{(\frac{N}{k}-I)\ln(1-t)}dt; \qquad t^{\frac{M}{k}-I} = e^{(\frac{M}{k}-I)\ln t} \\
&= \frac{1}{k}\int\limits_0^1 e^{(\frac{N}{k}-I)\ln(1-t)}e^{(\frac{M}{k}-I)\ln t}dt \\
&= \frac{1}{k}\int\limits_0^1 e^{(\frac{N}{k}-I)\ln u}e^{(\frac{M}{k}-I)\ln(1-u)}du; \qquad 1-t = u \\
&= \frac{1}{k}\int\limits_0^1 u^{\frac{N}{k}-I}(1-u)^{\frac{M}{k}-I}dt \\
&= B_k(N,M).
\end{aligned}$$

Lemma 3.4. *Let D, E be diagonal matrices in $C^{r\times r}$ such that $Re(z) > 0$, $Re(w) > 0$, for all $z \in \sigma(M)$ and $w \in \sigma(N)$, then*

$$B_k(D,E) = \Gamma_k(D)\Gamma_k(E)\Gamma_k^{-1}(D+E).$$

Theorem 3.3. *Let M and E be diagonalizable matrices in $C^{r\times r}$ such that $MN = NM$ and $Re(z) > 0$, $Re(w) > 0$, for all $z \in \sigma(M)$ and $w \in \sigma(N)$, then*

$$B_k(M,N) = \Gamma_k(M)\Gamma_k(N)\Gamma_k^{-1}(M+N). \tag{18}$$

Proof. Since M,N are diagonalizable and commute by [8], they are simultaneously diagonalizable. Let S be an invertible matrix in $C^{r\times r}$ such that

$$S^{-1}MS = D, \qquad S^{-1}NS = E, \tag{19}$$

where D and E are diagonal matrices. In order to prove (18), by [8, p. 54], if $\sigma(M) = \{\lambda_1,\cdots,\lambda_r\}$ and $\sigma(N) = \{\mu_1,\cdots,\mu_r\}$, Then $\sigma(M+N) = \{\lambda_1 + \mu_{i_j}\}_{j=1}^{r}$, for some permutation $i_1,i_2,\cdots,i_r$ of $1,2,\cdots,r$. Since matrices M and N satisfy $Re(z) > 0$, $Re(w) > 0$, for all $z \in \sigma(M)$ and $w \in \sigma(N)$, it follows that $\mathbb{R}e(w) > 0$, for all $w \in \sigma(M+N)$. By Lemmas 3.3 and 3.4 and by equation (19), it follows $M+N = S(D+E)S^{-1}$ and

$$\Gamma_k(M+N) = S[\int_0^\infty e^{-\frac{t^k}{k}} t^{D+E-I} dt]S^{-1} = S\Gamma_k(D+E)S^{-1}, \tag{20}$$

$$\Gamma_k(M) = S[\int_0^\infty e^{-\frac{t^k}{k}} t^{D-I} dt]S^{-1} = S\Gamma_k(D)S^{-1}, \tag{21}$$

$$\Gamma_k(N) = S[\int_0^\infty e^{-\frac{t^k}{k}} t^{E-I} dt]S^{-1} = S\Gamma_k(E)S^{-1}, \tag{22}$$

and

$$B_k(M,N) = S[\int_0^\infty t^{\frac{D}{k}-I}(1-t)^{\frac{E}{k}-I} dt]S^{-1} \tag{23}$$

$$= SB_k(D,E)S^{-1} \tag{24}$$

$$= S[\Gamma_k(D)\Gamma_k(E)\Gamma_k^{-1}(D+E)]S^{-1}. \tag{25}$$

By (20), we get $\Gamma_k^{-1}(D+E) = S^{-1}\Gamma_k^{-1}(M+N)$ and by (21), (22) and (23), it follows that

$$\begin{aligned} B_k(M,N) &= S\Gamma_k(D)\Gamma_k(E)[S\Gamma_k^{-1}(D+E)S^{-1}]S^{-1} \\ &= (S\Gamma_k(D)S^{-1})(S\Gamma_k(E)S^{-1})(S\Gamma_k^{-1}(D+E)S^{-1}) \\ &= \Gamma_k(M)\Gamma_k(N)\Gamma_k^{-1}(M+N). \end{aligned}$$

This completes the proof.

Remark Apart from the commutativity hypothesis, the diagnalizability condition of Theorem 2.2 guarantees that every eigenvalue z of the matrix $M+N$ lies in the right half plane.

Conflict of Interests

The authors declare that there is no conflict of interests.

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