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GENERALIZED TSCHEBYSCHEFF-II WEIGHTED POLYNOMIALS ON SIMPLICIAL DOMAIN

MOHAMMAD A. ALQUDAH

School of Basic Sciences and Humanities, German Jordanian University, Amman 11180, Jordan

Abstract. In this paper, we construct generalized Tschebyscheff-type weighted orthogonal polynomials $\mathbb{U}_{n,r}^{(\gamma,M,N)}(u,v,w)$, $\gamma > -1$, in the Bernstein-Bézer form over the simplicial domain. We show that $\mathbb{U}_{n,r}^{(\gamma,M,N)}(u,v,w)$, $r = 0,1,\ldots,n$; $n = 0,1,2,\ldots$, form an orthogonal system over a triangular domain with respect to the generalized weight function.

Keywords: Generalized Tschebyscheff-type polynomials; Simplicial domain; Bernstein polynomials.

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1. Introduction

Orthogonal polynomials have been studied thoroughly, the Tschebyscheff orthogonal polynomials of the second kind (Tschebyscheff-II), $U_n(x)$, are among these orthogonal polynomials. Although the main definitions and basic properties for univariate case were defined many years ago, the cases of the generalized, bivariate or more variables are limited. For $M,N \geq 0$, the generalized Tschebyscheff-type polynomials of the second type $\left\{\mathscr{U}_n^{(M,N)}(x)\right\}_{n=0}^{\infty}$ (generalized Tschebyscheff-II) were characterized in [3], these polynomials are orthogonal on the interval $[-1,1]$ with respect to the generalized weight function

$$\mathrm{W}^{(\gamma,M,N)}(x) = \frac{2}{\pi}(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} + M\delta(x+1) + N\delta(x-1). \tag{1.1}$$

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In addition, a closed form for the matrix transformation of the generalized Tschebyscheff-II polynomial basis into Bernstein polynomial basis, and for Bernstein polynomial basis into generalized Tschebyscheff-II polynomial basis were provided in [2].

The generalized bivariate Tschebyscheff-II polynomials $\mathscr{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ are orthogonal to each polynomial of degree $\leq n-1$, with respect to the generalized weight function (1.1). However, for $r \neq s$, $d \neq m$, $\mathscr{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ and $\mathscr{U}_{n,s,m}^{(\gamma,M,N)}(u,v,w)$ are not orthogonal with respect to the weight function.

A construction of bivariate orthogonal polynomials $\mathcal{U}_{n,r}^{(\gamma)}(u,v,w)$, $r=0,1,\ldots,n; n=0,1,2,\ldots$, with respect to the weight function $u^{\frac{1}{2}}v^{\frac{1}{2}}(1-w)^{\gamma}$, $\gamma > -1$, on a triangular domain were introduced in [1]. They showed that $\mathcal{U}_{n,r}^{(\gamma)}(u,v,w)$ form an orthogonal system.

In this paper, for $M,N \geq 0$, we construct generalized bivariate orthogonal polynomials $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}$ (u,v,w), $d=0,\ldots,k; k=0,\ldots,n$, $r=0,1,\ldots,n; n=0,1,2,\ldots$, with respect to the generalized Tschebyscheff-II weight function (1.1) on triangular domain. We show that $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ form an orthogonal system over the domain T with respect to (1.1). Worth to mention that these generalized Tschebyscheff-II weighted orthogonal polynomials are given in the Bernstein basis form; which preserve many geometric properties of the Bernstein polynomial basis.

The construction of generalized bivariate orthogonal polynomials on the square $G = \{(x,y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ is straightforward [11]. It can be done by considering the tensor product of the set of orthogonal polynomials over G. Let $\{\mathscr{U}_{n}^{(M,N)}(x)\}$ be the generalized Tschebyscheff-II polynomials over $[-1,1]$ with respect to the weight function $\mathrm{W}_{1}^{(M,N)}(x)$, and $\{\mathscr{Q}_{m}^{(M,N)}(y)\}$ be the generalized Tschebyscheff-II polynomials over $[-1,1]$ with respect to the weight function $\mathrm{W}_{2}^{(M,N)}(y)$. The generalized bivariate polynomials $\{\mathscr{R}_{nm}^{(M,N)}(x,y)\}$ on G formed by the tensor products of the Tschebyscheff-II polynomials defined as $\mathscr{R}_{nm}^{(M,N)}(x,y) = \mathscr{U}_{n-m}^{(M,N)}(x)\mathscr{Q}_{m}^{(M,N)}(y), n=0,1,2,\ldots,m=0,1,\ldots,n.$

The generalized bivariate polynomials $\{\mathscr{R}_{nm}^{(M,N)}(x,y)\}$ are orthogonal on the square G with respect to the weight function $\mathrm{W}^{(M,N)}(x,y) = \mathrm{W}_{1}^{(M,N)}(x)\mathrm{W}_{2}^{(M,N)}(y)$. However, the construction of orthogonal polynomials over a triangular domains are not straightforward like the tensor product over the square G.

1.1. Bernstein and orthogonal polynomials. Consider a triangle T defined by its three vertices $\mathbf{p}_k = (x_k, y_k)$, $k = 1,2,3$, where $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are not collinear. For each point $\mathbf{p}$ located inside the triangle, there is a sequence of three numbers $u, v, w \geq 0$ such that $\mathbf{p}$ can be written uniquely as a convex combination of the three vertices, $\mathbf{p} = u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3$, where $u + v + w = 1$. The three numbers $u = \frac{area(\mathbf{p},\mathbf{p}_2,\mathbf{p}_3)}{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}$, $v = \frac{area(\mathbf{p}_1,\mathbf{p},\mathbf{p}_3)}{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}$, $w = \frac{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p})}{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}$ indicate the barycentric coordinates of the point $\mathbf{p}$ with respect to the triangle.

Although there are three coordinates, there are only two degrees of freedom, since $u+v+w = 1$. Thus every point is uniquely defined by any two of the barycentric coordinates. That is, the triangular domain defined as $T = \{(u,v,w) : u,v,w \geq 0, u+v+w = 1\}$.

Definition 1.1. The $n+1$ Bernstein polynomials of degree n are defined by

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}, \quad \text{for} \quad i = 0,1,\ldots,n, \tag{1.2}$$

where $\binom{n}{i}$ is the binomial coefficients. For $\zeta = (i,j,k)$ denote triples of non-negative integers such that $|\zeta| = i+j+k$, then the generalized Bernstein polynomials of degree n are defined by the formula $B_\zeta^n(u,v,w) = \binom{n}{\zeta} u^i v^j w^k$, $|\zeta| = n$, where $\binom{n}{\zeta} = \frac{n!}{i!j!k!}$.

The generalized Bernstein polynomials have a number of useful analytical and elegant geometric properties [5]. Note that the generalized Bernstein polynomials are nonnegative over T and form a partition of unity,

$$1 = (u+v+w)^n = \sum_{\substack{0 \leq i,j,k \leq n \\ i+j+k=n}} \frac{n!}{i!j!k!} u^i v^j w^k. \tag{1.3}$$

These polynomials define the Bernstein basis for the space Π_n, the space of all polynomials of degree at most n.

A basis of linearly independent and mutually orthogonal polynomials in the barycentric coordinates (u,v,w) are represented in a triangular table, the kth row of this table contains $k+1$ polynomials. Thus, there are $\frac{(n+1)(n+2)}{2}$ polynomials in a basis of linearly independent polynomials of total degree n. Therefore, the sum (1.3) involves a total of $\frac{(n+1)(n+2)}{2}$ linearly independent polynomials.

Thus, with the revolt of computer graphics, Bernstein polynomials on $[0,1]$ became important in the form of Bézier curves, and the polynomials determined in the Bernstein (Bézier) basis enjoy considerable popularity in Computer Aided Geometric Design applications.

Degree elevation is a common situation in these applications, where polynomials given in the basis of degree n have to be represented in the basis of higher degree. For any polynomial $p(u,v,w)$ of degree n can be written using Bézier coefficients d_ζ in the Bernstein form

$$p(u,v,w) = \sum_{|\zeta|=n} d_\zeta B^n_\zeta(u,v,w). \tag{1.4}$$

With the use of degree elevation algorithm for the Bernstein representation [7],

$$B^r_k(x) = \sum_{i=k}^{n-r+k} \frac{\binom{r}{k}\binom{n-r}{i-k}}{\binom{n}{i}} B^n_i(x), \quad k = 0,1,\ldots,r,$$

the polynomial $p(u,v,w)$ in (1.4) can be written (multiplying both sides by $1 = u+v+w$) as

$$p(u,v,w) = \sum_{|\zeta|=n+1} d^{(1)}_\zeta B^{n+1}_\zeta(u,v,w).$$

The new coefficients defined by Hoschek et al. [8] as $d^{(1)}_\zeta = \frac{1}{n+1}(id_{i-1,j,k} + jd_{i,j-1,k} + kd_{i,j,k-1})$ where $|\zeta| = n+1$. Moreover, the next integration is one of the interesting analytical properties of the Bernstein polynomials $B^n_\zeta(u,v,w)$.

Lemma 1.1. [6] *The Bernstein polynomials $B^n_\zeta(u,v,w), |\zeta| = n$, on T satisfy*

$$\iint_T B^n_\zeta(u,v,w)dA = \frac{\Delta}{\binom{n+2}{2}},$$

where Δ is the double the area of T and $\binom{n+2}{2}$ is the dimension of Bernstein polynomials over the triangle.

This means that the Bernstein polynomials partition the unity with equal integrals over the domain; in other words, they are equally weighted as basis functions.

Definition 1.2. Let $p(u,v,w)$ and $q(u,v,w)$ be two bivariate polynomials over T, then we define their inner product over T by $\langle p,q\rangle = \frac{1}{\Delta}\iint_T pq dA$. With the inner product defined, we say that the two polynomials $p(u,v,w)$ and $q(u,v,w)$ are orthogonal if $\langle p,q\rangle = 0$.

For $m \geq 1$, let $\mathfrak{L}_\mathfrak{m}$ denote the space of polynomials of degree m that are orthogonal to all polynomials of degree $\leq m$ over a triangular domain T, i.e., $\mathfrak{L}_\mathfrak{m} = \{p \in \Pi_m : p \perp \Pi_{m-1}\}$.

For an integrable function $f(u,v,w)$ over T, consider the operator $S_n(f)$ defined in [4] as $S_n(f)=(n+1)(n+2)\sum_{|\zeta|=n}\left\langle f,B_\zeta^n\right\rangle B_\zeta^n$. For $n\geq m$, $\lambda_{m,n}=\frac{(n+2)!n!}{(n+m+2)!(n-m)!}$ is an eigenvalue of S_n, and $\mathfrak{L}_\mathfrak{m}$ is the corresponding eigenspace. The following lemmas will be used in the proof of the main results.

Lemma 1.2. [6] *Let* $p=\sum_{|\zeta|=n}c_\zeta B_\zeta^n\in\mathfrak{L}_\mathfrak{m}$ *and let* $q=\sum_{|\zeta|=n}d_\zeta B_\zeta^n\in\Pi_n$ *with* $m\leq n$. *Then,* $\langle p,q\rangle=\frac{(n!)^2}{(n+m+2)!(n-m)!}\sum_{|\zeta|=n}c_\zeta d_\zeta$.

Lemma 1.3. [6] *Let* $p=\sum_{|\zeta|=n}c_\zeta B_\zeta^n\in\Pi_n$. *Then we have* $p\in\mathfrak{L}_\mathfrak{n}\iff\sum_{|\zeta|=n}c_\zeta d_\zeta=0\ \ \forall q=\sum_{|\zeta|=n}d_\zeta B_\zeta^n\in\Pi_{n-1}$.

1.2. Factorial Minus Half. For the main results simplifications, we present some results concerning factorials, double factorials and some combinatorial identities. The double factorial of an integer n is given by

$$\begin{aligned}(2n-1)!!&=(2n-1)(2n-3)(2n-5)\dots(3)(1)\quad\text{if } n \text{ is odd}\\ n!!&=(n)(n-2)(n-4)\dots(4)(2)\qquad\qquad\text{if } n \text{ is even,}\end{aligned}\tag{1.5}$$

where $0!!=(-1)!!=1$. From (1.5), we can derive the following relation for factorials

$$n!!=\begin{cases}2^{\frac{n}{2}}(\frac{n}{2})! & \text{if } n \text{ is even}\\ \frac{n!}{2^{\frac{n-1}{2}}(\frac{n-1}{2})!} & \text{if } n \text{ is odd}\end{cases}.\tag{1.6}$$

From (1.6) we obtain

$$(2n)!!=[2(n)][2(n-1)]\dots[2.1]=2^n n!,\tag{1.7}$$

and

$$(2n)!=[(2n-1)(2n-3)\dots(1)]\left([2(n)][2(n-1)][2(n-2)]\dots[2(1)]\right)=(2n-1)!!2^n n!.\tag{1.8}$$

It is easy to derive the factorial of an integer plus half as

$$\left(n+\frac{1}{2}\right)!=\frac{\sqrt{\pi}}{2^{n+1}}(2n+1)!!.$$

By combining (1.7) and (1.8), we get $\binom{2n}{n}=\frac{2^{2n}(2n-1)!!}{(2n)!!}$. In addition, using (1.8) with some simplifications we obtain

$$\frac{\binom{2n}{2k}}{\binom{n}{k}}=\frac{(2n-1)!!}{(2k-1)!!(2n-2k-1)!!}.$$

2. The generalized Tschebyscheff-II polynomials

For $M,N \geq 0$, the generalized Tschebyscheff-II polynomials $\left\{\mathscr{U}_n^{(M,N)}(x)\right\}_{n=0}^{\infty}$ are orthogonal on the interval $[-1,1]$ with respect to the weight function (1.1) defined in [9], and been characterized in [3],

$$\mathscr{U}_n^{(M,N)}(x) = \frac{(2n+1)!!}{2^n(n+1)!}U_n(x) + \sum_{k=0}^{n} \lambda_k \frac{(2k+1)!!}{2^k(k+1)!}U_k(x), \tag{2.1}$$

where

$$\lambda_k = \frac{k(k+1)(2k+1)(M+N)}{6} + \frac{(k+2)(k+1)^2k^2(k-1)MN}{9}, \tag{2.2}$$

and $U_n(x)$ is the Tschebyscheff-II polynomial of degree n in x, Szegö [12]:

$$U_n(x) := \frac{(n+1)(2n)!!}{(2n+1)!!} \sum_{k=0}^{n} \binom{n+\frac{1}{2}}{n-k}\binom{n+\frac{1}{2}}{k}\left(\frac{x+1}{2}\right)^{n-k}\left(\frac{x-1}{2}\right)^{k}.$$

The univariate Tschebyscheff-II polynomials of degree n in x can be transformed in terms of Bernstein basis on $x \in [0,1]$, as

$$U_n(2x-1) := \frac{(n+1)(2n)!!}{(2n+1)!!} \sum_{k=0}^{n} (-1)^{n+1} \frac{\binom{n+\frac{1}{2}}{k}\binom{n+\frac{1}{2}}{n-k}}{\binom{n}{k}} B_k^n(x).$$

and can be represented in terms of Gauss hypergeometric series as follows [10],

$$U_n(x) := (n+1)_2F_1\left(-n, n+2, \frac{3}{2}; \frac{1-x}{2}\right).$$

The next theorem, see [3] for the proof, provides a closed form for generalized Tschebyscheff-II polynomial $\mathscr{U}_r^{(M,N)}(x)$ of degree r as a linear combination of the Bernstein polynomials $B_i^r(x), i = 0,1,\ldots,r$ of degree r.

Theorem 2.1. [3] *For $M,N \geq 0$, the generalized Tschebyscheff-II polynomials $\mathscr{U}_r^{(M,N)}(x)$ of degree r have the following Bernstein representation,*

$$\mathscr{U}_r^{(M,N)}(x) = \frac{(2r+1)!!}{2^r(r+1)!} \sum_{i=0}^{r} (-1)^{r-i} \vartheta_{i,r} B_i^r(x) + \sum_{k=0}^{r} \lambda_k \frac{(2k+1)!!}{2^k(k+1)!} \sum_{i=0}^{k} (-1)^{k-i} \vartheta_{i,k} B_i^k(x)$$

where λ_k defined by (2.2), $\vartheta_{0,r} = \frac{(2r+1)}{2^{2r}}\binom{2r}{r}$, and

$$\vartheta_{i,r} = \frac{(2r+1)^2}{2^{2r}(2r-2i+1)(2i+1)} \frac{\binom{2r}{r}\binom{2r}{2i}}{\binom{r}{i}}, \ i = 0,1,\ldots,r. \tag{2.3}$$

The coefficients $\vartheta_{i,r}$ satisfy the recurrence relation $\vartheta_{i,r} = \frac{(2r-2i+3)}{(2i+1)} \vartheta_{i-1,r}, \quad i = 1,\ldots,r.$

The following theorem used to combine the superior performance of the least-squares of the generalized Tschebyscheff-II polynomials with the geometric insights of the Bernstein polynomials basis.

Theorem 2.2. [2] *The entries $M^n_{i,r}, i,r = 0,1,\ldots,n$ of the matrix transformation of the generalized Tschebyscheff-II polynomial basis into Bernstein polynomial basis of degree n are given by*

$$M^n_{i,r} = \Phi^r_{i,n} + \sum_{k=0}^{r} \lambda_k \Phi^k_{i,n}, \tag{2.4}$$

where λ_k defined in (2.2) and

$$\Phi^r_{i,n} = \frac{(2r+1)!!}{2^r(r+1)!} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} (-1)^{r-k} \frac{\binom{n-r}{i-k}\binom{r+\frac{1}{2}}{k}\binom{r+\frac{1}{2}}{r-k}}{\binom{n}{i}}.$$

Now, we have the following corollary which enables us to write Tschebyscheff-II polynomials of degree r where $r \leq n$ in terms of Bernstein polynomials of degree n.

Corollary 2.3. [2] *The generalized Tschebyscheff-II polynomials of degree less than or equal to n, $\mathscr{U}_0^{(M,N)}(x),\ldots,\mathscr{U}_n^{(M,N)}(x)$, can be expressed in the Bernstein basis of fixed degree n by the following formula $\mathscr{U}_r^{(M,N)}(x) = \sum\limits_{i=0}^{n} N^n_{r,i} B^n_i(x),\ r = 0,1,\ldots,n$ where*

$$\begin{aligned} N^n_{r,i} =& \frac{(2r+1)!!}{2^r(r+1)!} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} \frac{(-1)^{r-k}(2r+1)^2}{2^{2r}(2r-2k+1)(2k+1)} \frac{\binom{n-r}{i-k}\binom{2r}{r}\binom{2r}{2k}}{\binom{n}{i}} \\ &+ \sum_{k=0}^{r} \lambda_k \frac{(2k+1)!!}{2^k(k+1)!} \sum_{j=\max(0,i+k-n)}^{\min(i,k)} \frac{(-1)^{k-j}(2k+1)^2}{2^{2k}(2k-2j+1)(2j+1)} \frac{\binom{n-k}{i-j}\binom{2k}{k}\binom{2k}{2j}}{\binom{n}{i}}. \end{aligned}$$

3. Generalized bivariate Tschebyscheff-II-weighted polynomials

In this section, we generalize the construction in [1] to formulate a simple closed-form representation of degree-ordered system of generalized orthogonal polynomials $\mathbb{U}^{(\gamma,M,N)}_{n,r,d}(u,v,w)$ on a triangular domain T.

The basic idea in this construction is to make $\mathbb{U}^{(\gamma,M,N)}_{n,r,d}(u,v,w)$ coincide with the univeriate Tschebyscheff-II polynomial along one edge of T, and to make its variation along each chord parallel to that edge a scaled version of this Tschebyscheff-II polynomial. The variation of $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ with w can then be arranged so as to ensure its orthogonality on T with every polynomial of degree $< n$, and with other basis polynomials $\mathbb{U}_{n,s,d}^{(\gamma,M,N)}(u,v,w)$ of degree n for $r \neq s$, $d \neq m$.

Now, for $M,N \geq 0$, $\gamma > -1$, $n = 0,1,2,\ldots$, $k = 0,\ldots,n$, $r = 0,1,\ldots,n$ and $d = 0,1,\ldots,k$, we define the generalized bivariate polynomials

$$\begin{aligned}\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w) &= \sum_{i=0}^{r}(-1)^{r-i}\vartheta_{i,r}B_i^r(u,v)\sum_{j=0}^{n-r}(-1)^j\binom{n+r+1}{j}B_j^{n-r}(w,u+v)\\ &\quad+\sum_{k=0}^{n}\lambda_k\sum_{i=0}^{d}(-1)^{d-i}\vartheta_{i,d}B_i^d(u,v)\sum_{j=0}^{k-d}(-1)^j\binom{k+d+1}{j}B_j^{k-d}(w,u+v),\end{aligned} \tag{3.1}$$

where $B_i^r(u,v)$ defined in (1.2), λ_k defined in (2.2), and $\vartheta_{i,r}$ defined in (2.3).

Furthermore, the next theorem, Farouki [6], will be used in the simplification and the construction of the generalized orthogonal bivariate polynomials and the proof of the main results.

Lemma 3.1. [6] *For $r = 0,\ldots,n$; $n = 0,1,\ldots$ define the polynomials*

$$\mathcal{G}_{n,r}(w) = \sum_{j=0}^{n-r}(-1)^j\binom{n+r+1}{j}B_j^{n-r}(w), \tag{3.2}$$

then $\mathcal{G}_{n,r}(w)$ are scalar multiple of ${}_2F_1\left(-n+r,n+2r+2,2r+2;1-w\right)$, and for $i = 0,1,\ldots,n-r-1$, $\mathcal{G}_{n,r}(w)$ is orthogonal to $(1-w)^{2r+i+1}$on $[0,1]$, and hence $\int_0^1 \mathcal{G}_{n,r}(w)p(w)(1-w)^{2r+1}dw = 0$ for every polynomial $p(w)$ of degree less than or equal $n-r-1$.

Using Theorem 2.1, and $B_i^r(u,v) = (u+v)^r B_i^r(\frac{u}{1-w})$, the polynomials in (3.1) can be written using the univariate Tschebyscheff-II polynomials form as

$$\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w) = \frac{\binom{r+\frac{1}{2}}{r}}{(r+1)}U_r(\frac{u}{1-w})(1-w)^r\mathcal{G}_{n,r}(w)+\frac{\binom{d+\frac{1}{2}}{d}}{(d+1)}U_d(\frac{u}{1-w})(1-w)^d\sum_{k=0}^{n}\lambda_k\mathcal{G}_{k,d}(w),$$

where $U_r(t)$ is the univariate Tschebyscheff-II polynomial of degree r in t and $\mathcal{G}_{n,r}(w)$ defined by (3.2).

To show that the generalized bivariate polynomials $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ form an orthogonal system over the triangular domain T with respect to the generalized weight function (1.1), we prove $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w) \in \mathscr{L}_n$, $r = 0,1,\ldots,n$; $n = 1,2\ldots$, and $\mathbb{U}_{n,r,d}^{(\gamma,M,N)} \perp \mathbb{U}_{n,s,m}^{(\gamma,M,N)}$ for $r \neq s$, $d \neq m$.

Choose $\mathbb{U}_{0,0,0}^{(\gamma,M,N)}=1$, then the polynomials $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ for $r=0,1,\ldots,n$; $n=0,1,2,\ldots$ form a degree-ordered orthogonal sequence over the triangular domain T. First, we show that the polynomials $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$, $r=0,\ldots,n$, are orthogonal to all polynomials of degree less than n over T.

Theorem 3.2. *Let $M,N\geq 0$, $\gamma>-1$, and the generalized weight function $\mathrm{W}^{(\gamma,M,N)}(u,v,w)$, defined in (1.1), then for each $n=1,2,\ldots$, $r=0,1,\ldots,n$, $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)\in\mathscr{L}_n$.*

Proof. For each $l=0,\ldots,m$; $m=0,\ldots,n-1$; we define the bivariate polynomials

$$H_{l,m}(u,v,w)=(1-w)^m w^{n-m-1}\sum_{j=0}^{l}(-1)^{l-j}\vartheta_{j,l}B_j^l(\frac{u}{1-w}). \tag{3.3}$$

The span of $H_{l,m}(u,v,w)$ in (3.3) includes the set of Bernstein polynomials $B_j^m(u,v)w^{n-m-1}$, $j=0,\ldots,m$; $m=0,\ldots,n-1$, which span Π_{n-1}. So, it is sufficient to show that for each $l=0,\ldots,m$; $m=0,\ldots,n-1$,

$$I=\iint\limits_T \mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)H_{l,m}(u,v,w)\mathrm{W}^{(\gamma,M,N)}(u,v,w)dA=0.$$

The integral I can be simplified and rearranged as

$$\begin{aligned}\frac{I}{\Delta}=&\frac{\binom{r+\frac{1}{2}}{r}}{(r+1)}\int_0^1\int_0^{1-w}U_r(\frac{u}{1-w})U_l(\frac{u}{1-w})(1-w)^{r+m}w^{n-m-1}\mathcal{G}_{n,r}(w)\\&\qquad\times\left[\frac{2}{\pi}u^{\frac{1}{2}}v^{\frac{1}{2}}(1-w)^{\gamma}+M\delta v-N\delta u\right]dudw\\&+\frac{\binom{d+\frac{1}{2}}{d}}{(d+1)}\sum_{k=0}^{n}\lambda_k\int_0^1\int_0^{1-w}U_d(\frac{u}{1-w})U_l(\frac{u}{1-w})(1-w)^{d+m}w^{n-m-1}\mathcal{G}_{k,d}(w)\\&\qquad\times\left[\frac{2}{\pi}u^{\frac{1}{2}}v^{\frac{1}{2}}(1-w)^{\gamma}+M\delta v-N\delta u\right]dudw.\end{aligned}$$

By making the substitution $t(1-w) = u$, we get

$$
\begin{aligned}
\frac{I}{\Delta} =& \frac{2}{\pi}\frac{\binom{r+\frac{1}{2}}{r}}{(r+1)}\int_0^1 U_r(t)U_l(t)t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}dt\int_0^1 \mathcal{G}_{n,r}(w)(1-w)^{\gamma+r+m+2}w^{n-m-1}dw \\
& -N\delta\frac{\binom{r+\frac{1}{2}}{r}}{(r+1)}\int_0^1 U_r(t)U_l(t)t\,dt\int_0^1 \mathcal{G}_{n,r}(w)(1-w)^{r+m+2}w^{n-m-1}dw \\
& +M\delta\frac{\binom{r+\frac{1}{2}}{r}}{(r+1)}\int_0^1 U_r(t)U_l(t)(1-t)dt\int_0^1 \mathcal{G}_{n,r}(w)(1-w)^{r+m+2}w^{n-m-1}dw \\
& +\frac{2}{\pi}\frac{\binom{d+\frac{1}{2}}{d}}{(d+1)}\sum_{k=0}^{n}\lambda_k\int_0^1 U_d(t)U_l(t)t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}dt\int_0^1 \mathcal{G}_{k,d}(w)(1-w)^{\gamma+d+m+2}w^{n-m-1}dw \\
& -N\delta\frac{\binom{d+\frac{1}{2}}{d}}{(d+1)}\sum_{k=0}^{n}\lambda_k\int_0^1 U_d(t)U_l(t)t\,dt\int_0^1 \mathcal{G}_{k,d}(w)(1-w)^{d+m+2}w^{n-m-1}dw \\
& +M\delta\frac{\binom{d+\frac{1}{2}}{d}}{(d+1)}\sum_{k=0}^{n}\lambda_k\int_0^1 U_d(t)U_l(t)(1-t)dt\int_0^1 \mathcal{G}_{k,d}(w)(1-w)^{d+m+2}w^{n-m-1}dw.
\end{aligned}
$$

If $m < d < r$, then we have $l < d < r$, and the first integral of the of each term is zero by the orthogonality property of the Tschebyscheff-II polynomials. If $r \le d \le m \le n-1$, we have by Lemma 3.1 the second integral of each term equals zero. Thus the theorem follows. This completes the proof.

Note that taking $\mathrm{W}^{(\gamma,M,N)}(u,v,w) = u^{\frac{1}{2}}v^{\frac{1}{2}}(1-w)^{\gamma} + M\delta v - N\delta u$ enables us to separate the integrand in the proof of Theorem 3.2. Also note that taking $\gamma > -1$ enables us to use Lemma 3.1 in the proof of Theorem 3.2.

Using equation (2.1) with simple calculations, and for $\gamma > -1$, $n = 0,1,2,\ldots$, $k = 0,\ldots,n$, $r = 0,1,\ldots,n$ and $d = 0,1,\ldots,k$, we can rewrite the generalized bivariate polynomials (3.1) as

$$
\begin{aligned}
\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w) =& (1+\lambda_n)\frac{\binom{r+\frac{1}{2}}{r}}{(r+1)}U_r\Big(\frac{u}{1-w}\Big)(1-w)^r\mathcal{G}_{n,r}(w) \\
& + \frac{\binom{d+\frac{1}{2}}{d}}{(d+1)}U_d\Big(\frac{u}{1-w}\Big)(1-w)^d\sum_{k=0}^{n-1}\lambda_k\mathcal{G}_{k,d}(w),
\end{aligned} \tag{3.4}
$$

where λ_k defined in equations (1.2), $\mathcal{G}_{n,r}(w)$ defined in (3.2), and $U_d(t)$is the univariate Tschebyscheff-II polynomials of degree d in t.

In the following theorem, we show that $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ is orthogonal to each polynomial $\mathbb{U}_{n,s,m}^{(\gamma,M,N)}(u,v,w)$ where $r \ne s$, $d \ne m$.

Theorem 3.3. *Let $M,N \geq 0$, $\gamma > -1$, for $r \neq s$ and $d \neq m$, we have $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w) \perp \mathbb{U}_{n,s,m}^{(\gamma,M,N)}(u,v,w)$ with respect to the generalized weight function $\mathrm{W}^{(\gamma,M,N)}(u,v,w)$.*

Proof. For $r \neq s$ and $d \neq m$, and using the equation (3.4), we have

$$I = \iint_T \mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)\mathbb{U}_{n,s,m}^{(\gamma,M,N)}(u,v,w)\mathrm{W}^{(\gamma,M,N)}(u,v,w)dA.$$

Substitute the polynomials, where $r,s = 0,\ldots,n$, $m = 0,\ldots,j$, and $d = 0,\ldots,i$, and with the substitution $t(1-w) = u$, we obtain

$$\begin{aligned}
\frac{I}{\Delta} =& \frac{2}{\pi}(1+\lambda_n)^2\Lambda_{r,s}\int_0^1 U_r(t)U_s(t)t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}dt\int_0^1 \mathcal{G}_{n,r}(w)\mathcal{G}_{n,s}(w)(1-w)^{\gamma+r+s+2}dw \\
&-(1+\lambda_n)^2N\delta\Lambda_{r,s}\int_0^1 U_r(t)U_s(t)tdt\int_0^1 \mathcal{G}_{n,r}(w)\mathcal{G}_{n,s}(w)(1-w)^{r+s+2}dw \\
&+(1+\lambda_n)^2M\delta\Lambda_{r,s}\int_0^1 U_r(t)U_s(t)(1-t)dt\int_0^1 \mathcal{G}_{n,r}(w)\mathcal{G}_{n,s}(w)(1-w)^{r+s+2}dw \\
&+\frac{2}{\pi}(1+\lambda_n)\Lambda_{s,d}\sum_{i=0}^{n-1}\lambda_i\int_0^1 U_s(t)U_d(t)t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}dt\int_0^1 \mathcal{G}_{n,s}(w)\mathcal{G}_{i,d}(w)(1-w)^{\gamma+s+d+2}dw \\
&-(1+\lambda_n)N\delta\Lambda_{s,d}\sum_{i=0}^{n-1}\lambda_i\int_0^1 U_s(t)U_d(t)tdt\int_0^1 \mathcal{G}_{n,s}(w)\mathcal{G}_{i,d}(w)(1-w)^{s+d+2}dw \\
&+(1+\lambda_n)M\delta\Lambda_{s,d}\sum_{i=0}^{n-1}\lambda_i\int_0^1 U_s(t)U_d(t)(1-t)dt\int_0^1 \mathcal{G}_{n,s}(w)\mathcal{G}_{i,d}(w)(1-w)^{s+d+2}dw \\
&+\frac{2}{\pi}(1+\lambda_n)\Lambda_{r,m}\sum_{j=0}^{n-1}\lambda_j\int_0^1 U_r(t)U_m(t)t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}dt\int_0^1 \mathcal{G}_{n,r}(w)\mathcal{G}_{j,m}(w)(1-w)^{\gamma+r+m+2}dw \\
&-(1+\lambda_n)N\delta\Lambda_{r,m}\sum_{j=0}^{n-1}\lambda_j\int_0^1 U_r(t)U_m(t)tdt\int_0^1 \mathcal{G}_{n,r}(w)\mathcal{G}_{j,m}(w)(1-w)^{r+m+2}dw \\
&+(1+\lambda_n)M\delta\Lambda_{r,m}\sum_{j=0}^{n-1}\lambda_j\int_0^1 U_r(t)U_m(t)(1-t)dt\int_0^1 \mathcal{G}_{n,r}(w)\mathcal{G}_{j,m}(w)(1-w)^{r+m+2}dw \\
&+\frac{2}{\pi}\Lambda_{d,m}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\lambda_i\lambda_j\int_0^1 U_d(t)U_m(t)t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}dt\int_0^1 \mathcal{G}_{i,d}(w)\mathcal{G}_{j,m}(w)(1-w)^{\gamma+m+d+3}dw \\
&-N\delta\Lambda_{d,m}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\lambda_i\lambda_j\int_0^1 U_d(t)U_m(t)dt\int_0^1 \mathcal{G}_{i,d}(w)\mathcal{G}_{j,m}(w)t(1-w)^{m+d+2}dw \\
&+M\delta\Lambda_{d,m}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\lambda_i\lambda_j\int_0^1 U_d(t)U_m(t)dt\int_0^1 \mathcal{G}_{i,d}(w)\mathcal{G}_{j,m}(w)(1-t)(1-w)^{m+d+2}dw,
\end{aligned}$$

where $\Lambda_{i,j} = \frac{\binom{i+\frac{1}{2}}{i}}{(i+1)} \frac{\binom{j+\frac{1}{2}}{j}}{(j+1)}$. Since $d \neq r$, and $m \neq s$, the first integral in all terms equals zero by Tschebyscheff-II polynomials orthogonality property, and this completes the proof.

4. Applications: Recurrence relation and recursion

The Bernstein-Bézier form of curves and surfaces exhibits some interesting geometric properties, see [5, 8]. So, for computational purposes, we are interested in finding a closed form of the Bernstein coefficients $a_{\zeta}^{n,r,d}$, and the recursion relation that allow us to compute the coefficients efficiently.

We write the orthogonal polynomials $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$, $r = 0,1,\ldots,n$, $d = 0,\ldots,k$, and $n = 0,1,2,\ldots$ in the following Bernstein-Bézier form,

$$\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w) = \sum_{|\zeta|=n} a_{\zeta}^{n,r,d} B_{\zeta}^{n}(u,v,w). \tag{4.1}$$

The following theorem provides a closed explicit form of the Bernstein coefficients $a_{\zeta}^{n,r,d}$.

Theorem 4.1. *The Bernstein coefficients $a_{\zeta}^{n,r,d}$ of equation (4.1) are given explicitly by*

$$a_{ijk}^{n,r,d} = \begin{cases} (-1)^k \binom{n+r+1}{k}\binom{n-r}{k} M_{i,r}^{n-k} + \lambda_k (-1)^j \binom{k+d+1}{j}\binom{k-d}{j} M_{i,d}^{k-j} & \text{if } 0 \leq k \leq n-r \\ 0 & \text{if } k > n-r. \end{cases}$$

where $M_{i,r}^{n}$ given in (2.4).

Proof. From equation (3.1), it is clear that $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ has degree $\leq n-r$ in the variable w, and thus $a_{ijk}^{n,r} = 0$ for $k > n-r$. For $0 \leq k \leq n-r$, the remaining coefficients are determined by equating (3.1) and (4.1) as follows

$$\begin{aligned}\sum_{i+j=n-k} a_{ijk}^{n,r} B_{ijk}^{n}(u,v,w) &= (-1)^k \binom{n+r+1}{k} B_k^{n-r}(w,u+v) \sum_{i=0}^{r} (-1)^{r-i} \vartheta_{i,r} B_i^r(u,v) \\ &\quad + \lambda_k \sum_{j=0}^{k-d} (-1)^j \binom{k+d+1}{j} B_j^{k-d}(w,u+v) \sum_{i=0}^{d} (-1)^{d-i} \vartheta_{i,d} B_i^d(u,v),\end{aligned}$$

where $\gamma > -1$, $B_i^r(u,v)$, $i = 0,1,\ldots,r$, defined in equation (1.2), and λ_k defined in (2.2). Comparing powers of w on both sides, we have

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!}{i!j!k!} u^i v^j = (-1)^k \binom{n+r+1}{k} \binom{n-r}{k} (u+v)^{n-r-k} \sum_{i=0}^{r} (-1)^{r-i} \vartheta_{i,r} B_i^r(u,v)$$
$$+ \lambda_k \sum_{j=0}^{k-d} (-1)^j \binom{k+d+1}{j} \binom{k-d}{j} (u+v)^{k-d-j} \sum_{i=0}^{d} (-1)^{d-i} \vartheta_{i,d} B_i^d(u,v).$$

The left hand side of the last equation can be written in the form

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!(n-k)!}{i!j!k!(n-k)!} u^i v^j = \sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!(n-k)!}{i!(n-k-i)!k!(n-k)!} u^i v^j = \sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} B_i^{n-k}(u,v).$$

Now, we have

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} B_i^{n-k}(u,v) = (-1)^k \binom{n+r+1}{k} \binom{n-r}{k} (u+v)^{n-r-k} \sum_{i=0}^{r} (-1)^{r-i} \vartheta_{i,r} B_i^r(u,v)$$
$$+ \lambda_k \sum_{j=0}^{k-d} (-1)^j \binom{k+d+1}{j} \binom{k-d}{j} (u+v)^{k-d-j} \sum_{i=0}^{d} (-1)^{d-i} \vartheta_{i,d} B_i^d(u,v),$$

With some binomial simplifications, and using Corollary 2.3, we get

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} B_i^{n-k}(u,v) = (-1)^k \binom{n+r+1}{k} \binom{n-r}{k} \sum_{i=0}^{r} M_{i,r}^{n-k} B_i^{n-k}(u,v)$$
$$+ \lambda_k \sum_{j=0}^{k-d} (-1)^j \binom{k+d+1}{j} \binom{k-d}{j} \sum_{i=0}^{d} M_{i,d}^{k-j} B_i^{k-j}(u,v),$$

where $M_{i,r}^{n-k}$ are the coefficients resulting from writing Tschebyscheff-II polynomial of degree r in the Bernstein basis of degree $n-k$, as defined by expression (2.4). Thus, the required Bernstein-Bézier coefficients are given by

$$a_{ijk}^{n,r,d} = \begin{cases} (-1)^k \binom{n+r+1}{k} \binom{n-r}{k} M_{i,r}^{n-k} + \lambda_k (-1)^j \binom{k+d+1}{j} \binom{k-d}{j} M_{i,d}^{k-j} & \text{if } 0 \le k \le n-r \\ 0 & \text{if } k > n-r, \end{cases}$$

which completes the proof.

4.1. Recurrence relation. To derive a recurrence relation for the coefficients $a_{ijk}^{n,r}$ of $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$, we consider the generalized Bernstein polynomial of degree $n-1$;

$$\begin{aligned}B_{ijk}^{n-1}(u,v,w) &= \frac{(n-1)!}{i!j!k!}u^i v^j w^k (u+v+w)\\ &= \frac{(i+1)}{n}B_{i+1,j,k}^n(u,v,w)+\frac{(j+1)}{n}B_{i,j+1,k}^n(u,v,w)+\frac{(k+1)}{n}B_{i,j,k+1}^n(u,v,w).\end{aligned}$$

From the construction of $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$, we have $\langle B_{ijk}^{n-1}(u,v,w), \mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)\rangle = 0$, $i+j+k=n-1$. By using Lemma 1.3, we have

$$(i+1)a_{i+1,j,k}^{n,r}+(j+1)a_{i,j+1,k}^{n,r}+(k+1)a_{i,j,k+1}^{n,r}=0. \tag{4.2}$$

But, by Theorem 4.1 we have $a_{i,n-i,0}^{n,r} = M_{i,r}^n$ for $i=0,1,\ldots,n$; and thus we can use (4.2) to generate $a_{i,j,k}^{n,r}$ recursively on k.

Conflict of Interests

The author declares that there is no conflict of interests.

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