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D'ALEMBERT FUNCTIONAL EQUATION FOR MATRIX VALUED FUNCTIONS

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Abstract. In this paper, we solve d'Alembert's functional equation where the function to be determined are defined on the quaternion group Q_8 and take their values in the complex $n\times n-$matrices.

Keywords: D'Alembert's functional equation; Complex $n\times n$-matrice; Quaternion group.

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1. Introduction

The cosine equation, also called classical d'Alembert's equation has the form:

$$f(x+y)+f(x-y)=2f(x)f(y),\ x,\ y\in G, \tag{1.1}$$

where G is an abelian group and the unknown function f is defined on G and assumes values in the complex field $\mathbb{C}$. The theory of d'Alembert's equation is extensively developed (see [1-20]). The basic result for the study of (1.1) in the scalar case is a result obtained by Kannappan [8].

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It says that every solution $f \neq 0$ of d'Alembert's equation (1.1) has the form

$$f(x) = \frac{m(x) + m(-x)}{2}, \ x \in G,$$

where m is a homomorphism of $(G\ ,+)$ into the multiplicative group of non-zero complex numbers.

In the case where G is an arbitrary group, not necessarily abelian, Davison [6] proved the following result

Let G be a topological group and $f : G \to \mathbb{C}$ a continuous function with $f(e) = 1$ satisfying

$$f(xy) + f(xy^{-1}) = 2f(x)f(y), \ x,y \in G. \tag{1.2}$$

Then there is a continuous (group) homomorphism $h : G \longrightarrow SL_2(\mathbb{C})$ such that

$$f(x) = \frac{1}{2} tr(h(x)), \ \ x \in G.$$

Giving solutions of equation (1.2) the theory of representations is introduced by H. Stetkær in [16]. Precisely, he proved that

Let S be a semigroup, the non-zero continuous solutions f of (1.2) *on S are the functions of the form*

$$f = \frac{1}{2} tr\pi$$

where π ranges over the 2-dimensional continuous representations of S for which $\pi(x) \in SL_2(\mathbb{C})$ for all $x \in S$.

The operator valued version of (1.1) was studied by Chojnacki [3], Badora [1] and Stetkær [14,15]. In [17] Székelyhidi determined the matrix valued solution of (1.1), and in [14] the author studied the continuous solutions $f : G \longrightarrow M_2(\mathbb{C})$ of (1.1).

Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group and $M_n(\mathbb{C})$ the algebra of complex $n \times n-$matrices. In the present paper we examine the following functional equation

$$\Phi(xy) + \Phi(xy^{-1}) = 2\Phi(x)\Phi(y), \ x,y \in Q_8, \tag{1.3}$$

where Φ is defined on the quaternion group Q_8 with values in $M_n(\mathbb{C})$. We will here still call (1.3) d'Alembert's functional equation. The main results, Theorem 3.1 and 3.4 are formulated for the quaternion group. Generally, a form of solution of Eq. (1.3) in the non-commutative case is not known.

2. Properties of solution of d'Alembert's equation

Let $n \geq 1$ be an integer and $\Phi : Q_8 \to M_n(\mathbb{C})$ be a solution of the following d'Alembert's equation:

$$\Phi(xy) + \Phi(xy^{-1}) = 2\Phi(x)\Phi(y);\ x, y \in Q_8. \tag{2.1}$$

In this section we stabled some properties the solutions of (2.1).

Proposition 2.1. *Let Φ be a solution of the equation (2.1). Then*

a) $\Phi(1)$ is a projection and satisfies $\Phi(x) = \Phi(x)\Phi(1)$ for all $x \in Q_8$.

b) Φ is even modulo $\Phi(1)$, that is $\Phi(1)\Phi(x) = \Phi(1)\Phi(x^{-1})$ for all $x \in Q_8$.

c) Φ is central modulo $\Phi(1)$, that is $\Phi(1)\Phi(xy) = \Phi(1)\Phi(yx)$ for all $x, y \in Q_8$.

d) For all $x, y \in Q_8$, $\Phi(x)$ and $\Phi(y)$ are commutating modulo $\Phi(1)$, that is

$$\Phi(1)\Phi(x)\Phi(y) = \Phi(1)\Phi(y)\Phi(x)\ .$$

e) For all $P \in GL_n(\mathbb{C})$, the function f defined by

$$f(x) = P^{-1}\Phi(x)P,$$

is a solution of (2.1).

Proof. a) Putting $y = 1$ in equation (2.1), we obtain

$$\Phi(x) = \Phi(x)\Phi(1),$$

for all $x \in Q_8$. In particular, if $x = 1$, then $\Phi(1) = \Phi(1)^2$, that is, $\Phi(1)$ is a projection.

b) Replacing x by 1 in equation (2.1), we obtain

$$\Phi(y) + \Phi(y^{-1}) = 2\Phi(1)\Phi(y), \tag{2.2}$$

multiplying the two members of (2.2) on the left by $\Phi(1)$, we see that

$$\Phi(1)\Phi(y)+\Phi(1)\Phi(y^{-1})=2\Phi(1)\Phi(y),$$

for all $y\in Q_8$. Then

$$\Phi(1)\Phi(y^{-1})=\Phi(1)\Phi(y),\ for\ all\ y\in Q_8.$$

c) If $x=\pm1$ or $y=\pm1$, then $\Phi(1)\Phi(xy)=\Phi(1)\Phi(yx)$. Assume that $x,y\in\{\pm i,\pm j,\pm k\}$, then $xy=(yx)^{-1}$ which gives $\Phi(1)\Phi(xy)=\Phi(1)\Phi((yx)^{-1})$. Using b), we get that

$$\Phi(1)\Phi(xy)=\Phi(1)\Phi(yx),\quad for\ all\ x,y\in Q_8.$$

d) We multiply equation (2.1) on the left by $\Phi(1)$ yielding that

$$\Phi(1)\Phi(xy)+\Phi(1)\Phi(xy^{-1})=2\Phi(1)\Phi(x)\Phi(y),\ x,y\in Q_8, \tag{2.3}$$

and interchanging x and y in (2.3) we obtain

$$\Phi(1)\Phi(yx)+\Phi(1)\Phi(yx^{-1})=2\Phi(1)\Phi(y)\Phi(x),\ x,y\in Q_8. \tag{2.4}$$

Comparing (2.3) and (2.4) and using b) and c) we infer that

$$\Phi(1)\Phi(x)\Phi(y)=\Phi(1)\Phi(y)\Phi(x),$$

for all $x,y\in Q_8$.

e) If we multiply the both sides of (2.1) on the left by P^{-1} and on the right by P, then we get that

$$P^{-1}\Phi(xy)P+P^{-1}\Phi(xy^{-1})P=2P^{-1}\Phi(x)PP^{-1}\Phi(y)P,$$

then the function f defined by $f(x)=P^{-1}\Phi(x)P,\ x\in Q_8$ is a solution of (2.1).

In particular, if $\Phi(1)=\mathbb{I}_n$ where $\mathbb{I}_n$ is the matrix identity we have the following result.

Corollary 2.2. *Let $\Phi:Q_8\to M_n(\mathbb{C})$ be a solution of (2.1), such that $\Phi(1)=\mathbb{I}_n$. Then*

b) Φ is even.

c) Φ is central.

d) For all $x,y\in Q_8$, we have $\Phi(x)\Phi(y)=\Phi(y)\Phi(x)$.

Proof. The proof of the others assumptions proceeds along the same lines as the one just given, so we leave it out.

3. Matrix solution of d'Alembert's equation

Let n be a non-negative integer. First, we determine the solutions Φ of (2.1) such that $\Phi(1) = \mathbb{I}_n$.

Theorem 3.1. *Let $\Phi : Q_8 \to M_n(\mathbb{C})$ be a function satisfying*

$$\begin{cases} \Phi(xy) + \Phi(xy^{-1}) = 2\Phi(x)\Phi(y);\ x, y \in Q_8, \\ \Phi(1) = \mathbb{I}_n. \end{cases}$$

Then $\Phi(-1) = A$ is a matrix involution, that is $A^2 = \mathbb{I}_n$ and

$$\Phi(\pm i) = \Phi(\pm j) = \Phi(\pm k) = \frac{1}{\sqrt{2}}(A + \mathbb{I}_n)^2.$$

Proof. In (2.1), we replace x and y by -1 into equation (2.1) yielding that

$$\Phi(1) + \Phi(1) = 2\Phi(-1)^2.$$

Then $\Phi(-1)^2 = \mathbb{I}_n$, that is, $\Phi(-1)$ is a matrix involution. Put $\Phi(-1) = A$. Replacing x and y by $\pm i$ in (2.1), we obtain $\Phi(-1) + \Phi(1) = 2\Phi(\pm i)^2$. Then

$$\Phi(\pm i)^2 = \frac{1}{2}(A + \mathbb{I}_n).$$

Changing x and y by $\pm j$ in (2.1), we get $\Phi(-1) + \Phi(1) = 2\Phi(\pm j)^2$. Then

$$\Phi(\pm j)^2 = \frac{1}{2}(A + \mathbb{I}_n)$$

and if $x = y = \pm k$, (2.1) implies that $\Phi(-1) + \Phi(1) = 2\Phi(\pm k)^2$. Then

$$\Phi(\pm k)^2 = \frac{1}{2}(A + \mathbb{I}_n).$$

We conclude that $\Phi(\pm i), \Phi(\pm j)$ and $\Phi(\pm k)$ are square root of $\frac{1}{2}(A + \mathbb{I}_n)$, i.e. $\Phi(\pm i)^2 = \Phi(\pm j)^2 = \Phi(\pm k)^2 = \frac{1}{2}(A + \mathbb{I}_n)$

In the following result, we give the explicit form of solutions of (2.1).

Theorem 3.2. *Let $g : Q_8 \to M_n(\mathbb{C})$ be a function satisfying*

$$\begin{cases} \Phi(xy)+\Phi(xy^{-1}) = 2\Phi(x)\Phi(y),\ x,y \in Q_8, \\ \Phi(1) = \mathbb{I}_n. \end{cases}$$

Then there is $P \in GL_n(\mathbb{C})$ such that

$$\Phi(-1) = P\begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix} P^{-1},\ p+q=n,$$

and

$$\Phi(\pm i) = \Phi(\pm j) = \Phi(\pm k) = P\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} P^{-1},$$

where 0 is a zero matrix and $A \in M_p(\mathbb{C})$.

Proof. By Theorem 3.1, $\Phi(-1)$ is a matrix involution which implies that there exists $P \in GL_n(\mathbb{C})$ such that $\Phi(-1) = P\begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix} P^{-1}$, where $p+q=n$. For all $x \in \{\pm i, \pm j, \pm k\}$ we find from Theorem 3.1 $\Phi(x)^2 = \frac{1}{2}(\Phi(-1)+\mathbb{I}_n)$. Then

$$\begin{aligned} \Phi(x)^2 &= \frac{1}{2}(\Phi(-1)+\mathbb{I}_n) \\ &= \frac{1}{2}(P\begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix} P^{-1} + PP^{-1}) \\ &= \frac{1}{2}P\begin{pmatrix} 2\mathbb{I}_p & 0 \\ 0 & 0 \end{pmatrix} P^{-1} \\ &= P\begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & 0 \end{pmatrix} P^{-1}, \end{aligned}$$

which shows that for all $x \in \{\pm i, \pm j, \pm k\}$, $\Phi(x)$ is the square root of $P\begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$. Consequently, for all $x \in \{\pm i, \pm j, \pm k\}$ $\Phi(x) = P\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$, where $A \in M_p(\mathbb{C})$.

Remark 3.3. *Let $1 \leq p \leq n$ be an integer and $A \in M_p(\mathbb{C})$ be a matrix involution, Theorem 3.2 and Proposition 2.1 e) implies that the function f defined by $f(x) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ for all $x \in \{\pm i, \pm j, \pm k\}$, $f(1) = \mathbb{I}_n$ and $f(-1) = \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix}$ is solution of (2.1).*

In the next theorem, we determine the solutions of the d'Alembert's functional equation (2.1).

Theorem 3.4. *Let $\Phi : Q_8 \to M_n(\mathbb{C})$ be a solution of the d'Alembert's equation*

$$\Phi(xy) + \Phi(xy^{-1}) = 2\Phi(x)\Phi(y),\ x, y \in Q_8.$$

Then $\Phi(1) = \mathbb{P}$ is a matrix projection, $\Phi(-1)$ is a square root of $\mathbb{P}$ and

$$\Phi(\pm i)^2 = \Phi(\pm j)^2 = \Phi(\pm k)^2 = \frac{1}{2}(\Phi(-1) + \mathbb{P}).$$

Proof. According to Proposition 2.1, a) $\Phi(1) = \mathbb{P}$ is a matrix projection. Substitute $x = -1, y = -1$ into equation (2.1), we get $\Phi(1) + \Phi(1) = 2\Phi(-1)^2$. Then $\Phi(-1)^2 = \mathbb{P}$. Taking $y = x$ in (2.1), we find that

$$\Phi(x^2) + \Phi(1) = 2\Phi(x)^2,$$

which implies that $\Phi(x)^2 = \frac{1}{2}(\Phi(x^2) + \mathbb{P})$. Then

$$\Phi(x)^2 = \frac{1}{2}(\Phi(-1) + \mathbb{P}),$$

for all $x \in \{\pm i, \pm j, \pm k\}$.

In the following result, we give the explicit form of solutions of (2.1) such that $\Phi(1) = \mathbb{P}$ is a matrix projection.

Theorem 3.5. *Let $\Phi : Q_8 \to M_n(\mathbb{C})$ be a function satisfying*

$$\Phi(xy) + \Phi(xy^{-1}) = 2\Phi(x)\Phi(y),\ x, y \in Q_8.$$

Then there exist $P \in GL_n(\mathbb{C})$ *such that*

$$\Phi(-1) = P. \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} .P^{-1},\ A \in GL_p(\mathbb{C}) \text{ and } p \leq n$$

and for all $x \in \{\pm i, \pm j, \pm k\}$

$$\Phi(x) = P \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} P^{-1} \text{ where } B^2 = \frac{1}{2}(A + \mathbb{I}_p).$$

Proof. By Proposition 2.1, $\Phi(1) = \mathbb{P}$ is a matrix projection of rank $1 \leq p \leq n$. Then there exists $P \in GL_n(\mathbb{C})$ such that $\Phi(1) = P \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$. Or $\Phi(-1)^2 = \mathbb{P}$. Then there exists $p \times p$-matrix involution A such that $\Phi(-1) = P \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$. For all $x \in \{\pm i, \pm j, \pm k\}$, we from Theorem 3.4 that $\Phi(x)^2 = \frac{1}{2}(\Phi(-1) + \mathbb{P})$. Then

$$\begin{aligned} \Phi(x)^2 &= \frac{1}{2}(\Phi(-1) + \mathbb{P}) \\ &= \frac{1}{2}(P \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} P^{-1} + P \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & 0 \end{pmatrix} P^{-1}) \\ &= \frac{1}{2} P \begin{pmatrix} A + \mathbb{I}_p & 0 \\ 0 & 0 \end{pmatrix} P^{-1}. \end{aligned}$$

The matrix $\begin{pmatrix} \frac{1}{2}(A + \mathbb{I}_p) & 0 \\ 0 & 0 \end{pmatrix}$ is a projection. Indeed,

$$\begin{aligned} \begin{pmatrix} \frac{1}{2}(A + \mathbb{I}_p) & 0 \\ 0 & 0 \end{pmatrix}^2 &= \begin{pmatrix} \frac{1}{4}(A^2 + 2A + \mathbb{I}_p) & 0 \\ 0 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} \frac{1}{4}(2A + 2\mathbb{I}_p) & 0 \\ 0 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} \frac{1}{2}(A + \mathbb{I}_p) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Then $\Phi(x) = P \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$ where $B^2 = \frac{1}{2}(A + \mathbb{I}_p)$ for all $x \in \{\pm i, \pm j, \pm k\}$.

Remark 3.6. *Let $1 \leq p \leq n$ be an integer and $B \in M_p(\mathbb{C})$ be a matrix involution, From Theorem 3.5 and Proposition 2.1, e) the function f defined by $f(x) = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ where $B^2 = \frac{1}{2}(A + \mathbb{I}_p)$ for all $x \in \{\pm i, \pm j, \pm k\}$, $f(1) = \mathbb{I}_n$ and $f(-1) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ is solution of (2.1).*

Conflict of Interests

The authors declare that there is no conflict of interests.

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