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TWO-STEP LAGUERRE POLYNOMIAL HYBRID BLOCK METHOD FOR STIFF AND OSCILLATORY FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We develop a two-step hybrid block method for the solution of stiff and oscillatory first-order Ordinary Differential Equations (ODEs) using the Laguerre polynomial as our basis function via interpolation and collocation techniques. The paper further investigates the basic properties of the method and found it to be zero-stable, consistent and convergent. The method was also tested on some sampled stiff and oscillatory problems and found to perform better than some existing ones with which we compared our results.

Keywords: Hybrid; Laguerre polynomial; Oscillatory; Stiff; Two-step.

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1. Introduction

This paper considers the numerical solution of stiff and oscillatory first-order initial value problems of the form,

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$$y' = f(x,y),\ \ y(x_0) = y_0,\ \ x \in [a,b]\,, \tag{1}$$

where $f : \Re \times \Re \to \Re,\ \ y, y_0 \in \Re$, f satisfies Lipchitz condition which guarantees the existence and uniqueness of solution of (1).

The development of numerical integration formulas for stiff as well as oscillatory differential equations has attracted considerable attention in the past, Fatunla, [4]. This is because mathematical models of physical situations in kinetic chemical reactions, process control and electrical circuit theory often results to stiff ODEs. Also, differential equations whose solutions are known to be periodic or oscillate with a known fequency can be found in the field of ecology, medical sciences and oscillatory motion in a nonlinear force field, Sanugi and Evans [7].

Scholars have proposed different numerical schemes for the solution of (1) ranging from predictor-corretor methods to hybrid methods. Despite the success recorded by the predictor-corrector methods, its major setbacks are that the predictors are in reducing order of accuracy, high cost of developing separate predictor for the corrector, high cost of human and computer time involved in the execution, Sunday *et al.* [12].Block methods were later proposed to carter for some of the setbacks of the predictor-corrector methods. It is important to state that Milne in 1953 first developed block method to serve as a predictor to a predictor-corrector algorithm before it was later adopted as a full method. Block method has the advantage of generating simultaneous numerical approximations at different grid points within the interval of integration, Sunday [8]. Another advantage of the block method is the fact that it is less expensive in terms of the number of function evaluations compared to the linear multistep and the Runge-Kutta methods. Its major setback however is that the order of interpolation points must not exceed the order of the differential equations, thus when equations of lower order are developed, the accuracy of the developed method is reduced. This led to the development of hybrid methods which permit the incorporation of function evaluation at off-step points which affords the opportunity of circumventing the "Dahlquist Zero-Stabilty Barrier" and it is actually possible to obtain convergent $k-$step methods with order $2k+1$ up to $k = 7$, Awoyemi *et al.* [2].The method is also useful in reducing the step number of a method and still remain zero-stable, Adesanya *et al.* [1].

Definition 1.1 [5] Laguerre polynomial $y_{n+1}(x)$ is defined as,

$$y_{n+1}(x) = (-1)^{n+1} e^{x} \frac{d^{n+1}}{dx^{n+1}} (e^{-x} x^{n+1}) \tag{2}$$

In particular, $y_0(1) = 1, y_1(x) = x-1,\ y_2(x) = x^2 - 4x + 2, \ldots$The Laguerre polynomial $y_n(x)$ are orthogonal with respect to the weight function $w(x) = e^{-x}$ on $[0, \infty)$.

In developing methods for the solution of (1), scholars used different basis functions. For instance, Sunday *et al.* [10], Sunday *et al.* [11] and Sunday *et al.* [12] used basis functions which are the combination of power series and exponential functions to develop block integrators for the solution of (1). Sunday *et al.* [9] and Sunday *et al.* [13] also used Chebyshev and Legendre polynomials as basis functions respectively to develop hybrid methods for the solution of (1).

In this paper, we shall adopt the Laguerre polynomial as a basis function to derive a two-step hybrid block method for the solution of (1).

2. Preliminaries

2.1. Derivation of the two-step hybrid block method

We consider the first six terms of the Laguerre polynomial as our basis function. This is given by,

$$y(x) = \sum_{n=-1}^{4} (-1)^{n+1} e^{x} \frac{d^{n+1}}{dx^{n+1}} (e^{-x} x^{n+1}). \tag{3}$$

Interpolating (3) at point $x_{n+s},\ s = 0$ and collocating its first derivatives at points $x_{n+r},\ r = 0(\frac{1}{2})2$, where s and r are the numbers of interpolation and collocation points respectively, leads to the following system of equations,

$$XA = U \tag{4}$$

where

$$A = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5]^T$$

$$U = \left[y_n \quad f_n \quad f_{n+\frac{1}{2}} \quad f_{n+1} \quad f_{n+\frac{3}{2}} \quad f_{n+2}\right]^T$$

and

$$X=\begin{bmatrix} 96 & -504x_n & 528x_n^2 & -184x_n^3 & 24x_n^4 & -x_n^5\\ 0 & -504 & 1056x_n & -552x_n^2 & 96x_n^3 & -5x_n^4\\ 0 & -504 & 1056x_{n+\frac{1}{2}} & -552x_{n+\frac{1}{2}}^2 & 96x_{n+\frac{1}{2}}^3 & -5x_{n+\frac{1}{2}}^4\\ 0 & -504 & 1056x_{n+1} & -552x_{n+1}^2 & 96x_{n+1}^3 & -5x_{n+1}^4\\ 0 & -504 & 1056x_{n+\frac{3}{2}} & -552x_{n+\frac{3}{2}}^2 & 96x_{n+\frac{3}{2}}^3 & -5x_{n+\frac{3}{2}}^4\\ 0 & -504 & 1056x_{n+2} & -552x_{n+2}^2 & 96x_{n+2}^3 & -5x_{n+2}^4 \end{bmatrix}$$

Solving (4), for $a_j's$, $j=0(1)5$ and substituting back into (3) gives a continuous linear multistep method of the form,

(5) $$y(x)=\alpha_0(x)y_n+h\sum_{j=0}^{2}\beta_j(x)f_{n+j},\ j=0\left(\frac{1}{2}\right)2,$$

where

(6) $$\begin{bmatrix} \alpha_0(t)=1\\ \beta_0(t)=\frac{1}{180}(24t^5-150t^4+350t^3-375t^2+180t)\\ \beta_{\frac{1}{2}}(t)=-\frac{1}{45}(24t^5-135t^4+260t^3-180t^2)\\ \beta_1(t)=\frac{1}{15}(12t^5-60t^4+95t^3-45t^2)\\ \beta_{\frac{3}{2}}(t)=-\frac{1}{45}(24t^5-105t^4+140t^3-60t^2)\\ \beta_2(t)=\frac{1}{180}(24t^5-90t^4+110t^3-45t^2) \end{bmatrix}$$

and $t=\frac{x-x_n}{h}$. Evaluating (5) at $t=\frac{1}{2}\left(\frac{1}{2}\right)2$ gives a discrete block scheme of the form,

(7) $$A^{(0)}Y_m=Ey_n+hdf(y_n)+hbF(Y_m),$$

where

$$Y_m=\begin{bmatrix} y_{n+\frac{1}{2}} & y_{n+1} & y_{n+\frac{3}{2}} & y_{n+2}\end{bmatrix}^T,\quad y_n=\begin{bmatrix} y_{n-\frac{3}{2}} & y_{n-1} & y_{n-\frac{1}{2}} & y_n\end{bmatrix}^T$$

$$F(Y_m)=\begin{bmatrix} f_{n+\frac{1}{2}} & f_{n+1} & f_{n+\frac{3}{2}} & f_{n+2}\end{bmatrix}^T,\quad f(y_n)=\begin{bmatrix} f_{n-\frac{3}{2}} & f_{n-1} & f_{n-\frac{1}{2}} & f_n\end{bmatrix}^T$$

$$A^{(0)}=\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix},\quad E=\begin{bmatrix} 0&0&0&1\\ 0&0&0&1\\ 0&0&0&1\\ 0&0&0&1\end{bmatrix}$$

$$d=\begin{bmatrix} 0 & 0 & 0 & \frac{251}{1440} \\ 0 & 0 & 0 & \frac{29}{180} \\ 0 & 0 & 0 & \frac{27}{160} \\ 0 & 0 & 0 & \frac{7}{45} \end{bmatrix},\ b=\begin{bmatrix} \frac{323}{720} & \frac{-11}{60} & \frac{53}{720} & \frac{-19}{1440} \\ \frac{31}{45} & \frac{2}{15} & \frac{1}{45} & \frac{-1}{180} \\ \frac{51}{80} & \frac{9}{20} & \frac{21}{80} & \frac{-3}{160} \\ \frac{32}{45} & \frac{4}{15} & \frac{32}{45} & \frac{7}{45} \end{bmatrix}$$

2.2. Analysis of basic properties of the two-step hybrid block method

Order

We define the linear operator $L\{y(x);h\}$ associated with the hybrid block method (7) as,

$$L\{y(x);h\}=A^{(0)}Y_m-Ey_n-h(df(y_n)+bF(Y_m)). \tag{8}$$

Expanding (8) using Taylor series and comparing the coefficients of h gives,

$$L\{y(x);h\}=c_0y(x)+c_1hy'(x)+c_2h^2y''(x)+...+c_ph^py^p(x)+c_{p+1}h^{p+1}y^{p+1}(x)+... \tag{9}$$

Definition 2.1 (Fatunla [4]): The linear operator L and the associated continuous linear multi-step method (5) are said to be of order p if $c_0=c_1=c_2=...=c_p=0$ and $c_{p+1}\neq 0$. c_{p+1} is called the error constant and the local truncation error is given by,

$$t_{n+k}=c_{p+1}h^{(p+1)}y^{(p+1)}(x_n)+\bigcirc(h^{p+2}). \tag{10}$$

It is important to state that the order is also defined as the largest positive real number p that quantifies the rate of convergence of a numerical approximation of a differential equation to that of the exact solution while the error constant is the accumulated error when the order of a method has been computed. For our hybrid block method,

$$L\{y(x);h\}=\left[\begin{array}{c}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{bmatrix}-\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-\frac{3}{2}}\\y_{n-1}\\y_{n-\frac{1}{2}}\\y_{n}\end{bmatrix}\\ \\ -h\begin{bmatrix}\frac{251}{1440}&\frac{323}{720}&\frac{-11}{60}&\frac{53}{720}&\frac{-19}{1440}\\\frac{29}{180}&\frac{31}{45}&\frac{2}{15}&\frac{1}{45}&\frac{-1}{180}\\\frac{27}{160}&\frac{51}{80}&\frac{9}{20}&\frac{21}{80}&\frac{-3}{160}\\\frac{7}{45}&\frac{32}{45}&\frac{4}{15}&\frac{32}{45}&\frac{7}{45}\end{bmatrix}\begin{bmatrix}f_n\\f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{bmatrix}\end{array}\right] \tag{11}$$

Expanding (11) in Taylor series gives,

$$\begin{bmatrix}\sum_{j-0}^{\infty}\frac{(\frac{1}{2}h)^j}{j!}y_n^j-y_n-\frac{251h}{1440}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{array}{c}\frac{323}{720}(\frac{1}{2})^j-\frac{11}{60}(1)^j\\+\frac{53}{720}(\frac{3}{2})^j-\frac{19}{1440}(2)^j\end{array}\right\}\\ \sum_{j-0}^{\infty}\frac{(h)^j}{j!}y_n^j-y_n-\frac{29h}{180}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{array}{c}\frac{31}{45}(\frac{1}{2})^j+\frac{2}{15}(1)^j\\+\frac{1}{45}(\frac{3}{2})^j-\frac{1}{180}(2)^j\end{array}\right\}\\ \sum_{j-0}^{\infty}\frac{(\frac{3}{2}h)^j}{j!}y_n^j-y_n-\frac{27h}{160}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{array}{c}\frac{51}{80}(\frac{1}{2})^j+\frac{9}{20}(1)^j\\+\frac{21}{80}(\frac{3}{2})^j-\frac{3}{160}(2)^j\end{array}\right\}\\ \sum_{j-0}^{\infty}\frac{(2h)^j}{j!}y_n^j-y_n-\frac{7h}{45}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{array}{c}\frac{32}{45}(\frac{1}{2})^j+\frac{4}{15}(1)^j\\+\frac{32}{45}(\frac{3}{2})^j+\frac{7}{45}(2)^j\end{array}\right\}\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{12}$$

Hence, $\bar{c}_0=\bar{c}_1=\bar{c}_2=\bar{c}_3=\bar{c}_4=\bar{c}_5=0$,

$\bar{c}_6=[2.9297(-04)\quad 1.7361(-04)\quad 2.9297(-04)\quad -6.6138(-05)]^T$. Therefore, the two-step hybrid block method is of accurate order 5.

Zero stability

Definition 2.2 (Fatunla [4]): The block method (7) is said to be zero-stable, if the roots $z_s, s=1,2,...,k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z)=\det(zA^{(0)}-E)$ satisfies

$|z_s| \leq 1$ and every root satisfying $|z_s| = 1$ have multiplicity not exceeding the order of the differential equation. Moreover, as $h \to 0, \rho(z) = z^{r-\mu}(z-1)^{\mu}$ where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ and E (see Awoyemi *et al.* [2] for details). The main consequence of zero-stability is to control the propagation of the error as the integration proceeds.

For our hybrid block method,

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0 \tag{13}$$

$\rho(z) = z^3(z-1) = 0 \Longrightarrow z_1 = z_2 = z_3 = 0,\ z_4 = 1$. Hence, the hybrid block method is zero-stable.

Consistency

The hybrid block method (7) is consistent since it has order $p = 5 \geq 1$. It is important to note that consistency controls the magnitude of the local truncation error committed at each stage of the computation.

Convergence

Theorem 2.1. *(Dahlquist [3]): The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.*

Region of Absolute Stability

Definition 2.3 (Yan [15]]): Region of absolute stability is a region in the complex z plane, where $z = \lambda h$. It is defined as those values of z such that the numerical solutions of $y' = -\lambda y$ satisfy $y_j \to 0$ as $j \to \infty$ for any initial condition.

In ploting the stability region shown in Figure 1 below, we shall adopt the boundary locus method. This gives the stability polynomial below,

(14)

$$\bar{h}(w) = -h^4\left(\frac{1}{80}w^3 - \frac{1}{80}w^4\right) - h^3\left(\frac{5}{48}w^4 + \frac{5}{48}w^3\right) - h^2\left(\frac{7}{16}w^3 - \frac{7}{16}w^4\right) - h\left(w^4 + w^3\right) + w^4 - w^3$$

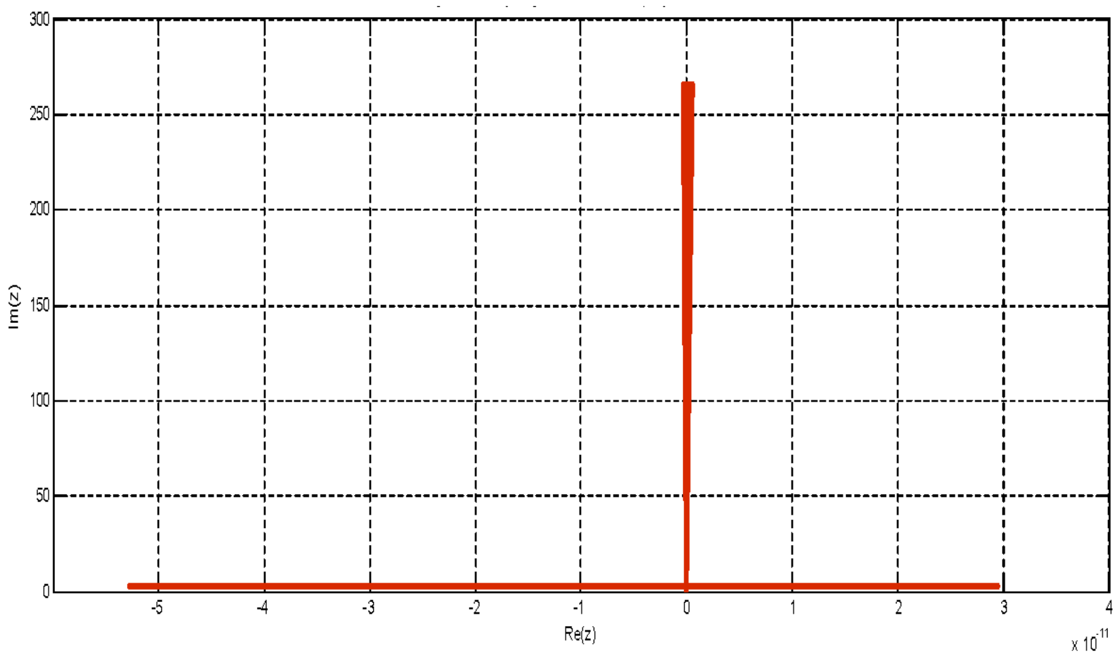

Figure 1: Stability Rgion of the Two-Step Laguerre Hybrid Method

According to Fatunla [4], stiff algorithms have unbounded RAS. Also, Lambert [16] showed that the stability region for L-stable schemes must encroach into the positive half of the complex plane.

3. Main results

We shall evaluate the performance of the two-step hybrid block method developed on some challeging stiff and oscillatory problems which have appeared in literatures and compare our results with existing ones. The following notations shall be used in the tables below.

ERR- Exact Solution-Computed Solution

ERYS- Error in Yahaya and Sokoto [14]

ERSOAJ- Error in Sunday *et al.* [11].

Numerical experiments

Problem 3.1 : *Consider the highly stiff ODE*

$$y'(t) = -9t, \;\; y(0) = e, \;\; 0 \leq t \leq 1, \tag{15}$$

which has the exact solution

$$y(t) = e^{1-9t} \tag{16}$$

This problem was solved by Yahaya and Sokoto [14] where they applied a block hybrid method with step number $k = 4$ of order 6.We applied our newly developed method on this problem and obtained the results presented in Table 3.1 below.

Problem 3.2 : *Consider the highly oscillatory ODE*

$$y'(t) = -\sin t - 200(y(t) - \cos t), \quad y(0) = 0 \tag{17}$$

with the exact solution

$$y(t) = \cos t - e^{-200t} \tag{18}$$

Sunday *et al.* [11] solved this problem using an extended block integrator of order 6. We applied our newly developed method on this problem and obtained the results presented in Table 3.2 below.

Table 3.1 : Showing the result for stiff problem 3.1

t	Exact Solution	Computed Solution	ERR	ERYS	t/sec
0.1000	1.1051709180756477	1.1051709259345621	$7.858914e-009$	$1.18e-03$	0.0170
0.2000	0.4493289641172217	0.4493289705076141	$6.390392e-009$	$1.62e-04$	0.0368
0.3000	0.1826835240527347	0.1826835279499442	$3.897209e-009$	$2.22e-04$	0.0391
0.4000	0.0742735782143338	0.0742735803269834	$2.112650e-009$	$1.37e-04$	0.0430
0.5000	0.0301973834223185	0.0301973844959925	$1.073674e-009$	$8.79e-05$	0.0452
0.6000	0.0122773399030684	0.0122773404268964	$5.238280e-010$	$5.23e-04$	0.0473
0.7000	0.0049915939069102	0.0049915941553782	$2.484680e-010$	$1.53e-05$	0.0498
0.8000	0.0020294306362957	0.0020294307517466	$1.154509e-010$	$1.64e-06$	0.0519
0.9000	0.0008251049232659	0.0008251049760721	$5.280619e-011$	$3.92e-06$	0.0539
1.0000	0.0003354626279025	0.0003354626517574	$2.385488e-011$	$1.35e-06$	0.0562

Table 3.2: Showing the result for oscillatory problem 3.2

t	Exact Solution	Computed Solution	ERR	ESOAJ	$t/\sec$
0.0010	0.1730404148066717	0.1730404236249725	$8.818301e-009$	$1.827076e-07$	0.0172
0.0020	0.3229412242524378	0.3229412421033776	$1.785094e-008$	$1.408505e-07$	0.0195
0.0030	0.4511838639093487	0.4511838908541543	$2.694481e-008$	$5.560940e-07$	0.0222
0.0040	0.5461474034777876	0.5461474394379181	$3.596013e-008$	$3.927080e-07$	0.0243
0.0050	0.5506630358934452	0.5506630718490411	$3.595560e-008$	$2.258932e-07$	0.0244
0.0060	0.6987877881417977	0.6987878421480225	$5.400622e-008$	$1.856178e-07$	0.0289
0.0070	0.7533785361584349	0.7533785992010643	$6.304263e-008$	$1.519389e-07$	0.0311
0.0080	0.7980714821760108	0.7980715542606559	$7.208465e-008$	$1.260184e-07$	0.0333
0.0090	0.8346606120517877	0.8346606931830203	$8.113123e-008$	$1.159464e-07$	0.1124
0.0100	0.8646147171800526	0.8646148073615927	$9.018154e-008$	$1.661978e-07$	0.1409

Discussion of results

We considered two numerical examples in this paper. The first being a stiff problem was earlier solved by Yahaya and Sokoto [14] while the second being an oscillatory problem was solved by Sunday *et al.* [11]. The two methods these authors employed are both of oders 6. We applied the newly developed two-step method (which is of order 5) on the two problems and from the results obtained, it is obvious that the method performs better than the existing methods. It was also observed that the evaluation time per seconds of the results are very small, implying that the method generates results very fast.

4. Conclusion

We have derived a two-step hybrid method for the solution of stiff and oscillatory problems of the form (1) using Laguerre polynomial as our basis function. The method developed was found to be L-stable and that explains why it performed well on this class of problems. The method was also found to be zero-stable, consistent and convergent. The numerical results obtained shows that the method developed perform better than the existing ones with which we compared our results. One may simply conclude that the newly developed method is computationally reliable.

Conflict of Interests

The authors declare that there is no conflict of interests.

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