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**FOURIER COEFFICIENTS OF A CLASS OF ETA QUOTIENTS OF WEIGHT 10**

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**Abstract.** Recently, Williams [18] and Yao, Xia and Jin[15] discovered explicit formulas for the coefficients of the Fourier series expansions of a class of eta quotients. Williams expressed all coefficients of 126 eta quotients in terms of $\sigma(n), \sigma(\frac{n}{2}), \sigma(\frac{n}{3})$ and $\sigma(\frac{n}{6})$ and Yao, Xia and Jin, following the method of proof of Williams, expressed only even coefficients of 104 eta quotients in terms of $\sigma_3(n), \sigma_3(\frac{n}{2}), \sigma_3(\frac{n}{3})$ and $\sigma_3(\frac{n}{6})$. Here, by using the method of proof of Williams, we will express the even Fourier coefficients of 100 eta quotients i.e., the Fourier coefficients of the sum, f(q)+f(-q), of 100 eta quotients in terms of $\sigma_9(n), \sigma_9\left(\frac{n}{2}\right), \sigma_9\left(\frac{n}{3}\right), \sigma_9\left(\frac{n}{4}\right), \sigma_9\left(\frac{n}{6}\right)$ and $\sigma_9\left(\frac{n}{12}\right)$.

**Keywords:** Dedekind eta function; Eta quotients; Fourier series.

**2010 AMS Subject Classification:** 42A16.

## 1. Introduction

The divisor function $\sigma_i(n)$ is defined for a positive integer $i$ by

$$\sigma_i(n) := \sum_{d \text{ positive integer}, d|n} d^i, \text{if } n \text{ is a positive integer, and} \tag{1}$$

$$\sigma_i(n) := 0 \text{ if n is not a positive integer.}$$

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The Dedekind eta function is defined by

(2) $$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where

(3) $$q := e^{2\pi iz}, z \in H = \{x+iy : y>0\}$$

and an eta quotient of level $n$ is defined by

(4) $$f(z) := \prod_{m|n}\eta(mz)^{a_m}, n \in \mathbb{N}, a_m \in Z, a_n \neq 0.$$

It is interesting and important to determine explicit formulas of the Fourier coefficients of eta quotients since they are the building blocks of modular forms of level n and weight k. The book of Köhler [13] (Chapter 3, pg.39) describes such expansions by means of Hecke Theta series and develops algorithms for the determination of suitable eta quotients. One can find more information in [3], [6], [14], [16], [17]. I have determined the Fourier coefficients of the theta series associated to some quadratic forms, see [7], [8], [9][10], [11] and [12].

Recently, Williams, see [18] discovered explicit formulas for the coefficients of Fourier series expansions of a class of 126 eta quotients in terms of $\sigma(n), \sigma(\frac{n}{2}), \sigma(\frac{n}{3})$ and $\sigma(\frac{n}{6})$. One example is as follows:

$$\frac{\eta^2(2z)\eta^4(4z)\eta^6(6z)}{\eta^2(z)\eta^2(3z)\eta^4(12z)}$$

give the expansion found by Williams.

Then Yao, Xia and Jin [15] expressed the even Fourier coefficients of 104 eta quotients in terms of $\sigma_3(n), \sigma_3(\frac{n}{2}), \sigma_3(\frac{n}{3})$ and $\sigma_3(\frac{n}{6})$.One example is as follows:

$$\frac{\eta^{25}(2z)\eta^4(3z)}{\eta^{12}(z)\eta^5(4z)\eta^3(6z)\eta(12z)},$$

where the even coefficients are obtained. Motivated by these two results, we find that we can express the even Fourier coefficients of 100 eta quotients in terms of

$$\sigma_9(n), \sigma_9\left(\frac{n}{2}\right), \sigma_9\left(\frac{n}{3}\right), \sigma_9\left(\frac{n}{4}\right), \sigma_9\left(\frac{n}{6}\right), \sigma_9\left(\frac{n}{12}\right),$$

see, Table 1. One example is as follows:

$$\frac{\eta^{12}(4z)\,\eta^{10}(6z)\,\eta^{4}(12z)}{\eta^{6}(2z)}.$$

We see that the odd Fourier coefficients of 263 eta quotients are zero and even coefficients can be expressed by simple formula, see Table 2. One example is as follows:

$$\frac{\eta^{16}(4z)\,\eta^{12}(6z)}{\eta^{8}(2z)},$$

the odd coefficients of this eta quotient is zero. Now we can state our main Theorem:

**Theorem 1.** *Let $b_1, b_2, \cdots, b_5$ be non-negative integers satisfying*

$$b_1 + b_2 + \cdots + b_5 \leq 20. \tag{5}$$

*Define the integers $a_1, a_2, a_3, a_4, a_6, a_{12}$ by*

$$a_1 := -b_1 + 2b_2 - 2b_3 - 4b_4 - b_5 + 20, \tag{6}$$

$$a_2 := 3b_1 + b_2 + 3b_3 + 10b_4 + b_5 - 50, \tag{7}$$

$$a_3 := 3b_1 + 2b_2 + 6b_3 + 4b_4 + 3b_5 - 60, \tag{8}$$

$$a_4 := -2b_1 - b_2 - b_3 - 4b_4 + 2b_5 + 20, \tag{9}$$

$$a_6 := -9b_1 - 7b_2 - 9b_3 - 10b_4 - 7b_5 + 150, \tag{10}$$

$$a_{12} := 6b_1 + 3b_2 + 3b_3 + 4b_4 + 2b_5 - 60. \tag{11}$$

*Let*

$$f_1 := \sum_{n=0}^{\infty} f_1(n) = \frac{\eta^{16}(4z)\,\eta^{12}(6z)}{\eta^{8}(2z)},$$
$$f_2 := \sum_{n=0}^{\infty} f_2(n) = \frac{\eta^{11}(4z)\,\eta^{17}(6z)}{\eta^{7}(2z)\,\eta(12z)},$$
$$f_3 := \sum_{n=0}^{\infty} f_3(n) = \frac{\eta^{18}(4z)\,\eta^{10}(12z)}{\eta^{6}(2z)\,\eta^{2}(6z)},$$

$$f_4 := \sum_{n=0}^{\infty} f_4(n) = \frac{\eta^{13}(4z)\,\eta^{3}(6z)\,\eta^{9}(12z)}{\eta^{5}(2z)},$$

$$f_5 := \sum_{n=0}^{\infty} f_5(n) = \frac{\eta^{18}(6z)\,\eta^{6}(12z)}{\eta^{2}(2z)\,\eta^{2}(4z)},$$

$$f_6 := \sum_{n=0}^{\infty} f_6(n) = \frac{\eta^{20}(4z)\,\eta^{8}(6z)}{\eta^{4}(2z)\,\eta^{4}(12z)},$$

$$f_7 := \sum_{n=0}^{\infty} f_7(n) = \frac{\eta^{3}(4z)\,\eta^{13}(6z)\,\eta^{7}(12z)}{\eta^{3}(2z)},$$

$$f_8 := \sum_{n=0}^{\infty} f_8(n) = \frac{\eta^{17}(6z)\,\eta^{11}(12z)}{\eta^{7}(2z)\,\eta(4z)},$$

$$f_9 := \sum_{n=0}^{\infty} f_9(n) = \frac{\eta^{15}(4z)\,\eta^{19}(12z)}{\eta^{9}(2z)\,\eta^{5}(6z)},$$

$$f_{10} := \sum_{n=0}^{\infty} f_{10}(n) = \frac{\eta^{17}(4z)\,\eta^{5}(6z)\,\eta^{5}(12z)}{\eta^{7}(2z)},$$

$$f_{11} := \sum_{n=0}^{\infty} f_{11}(n) = \frac{\eta^{12}(4z)\,\eta^{10}(6z)\,\eta^{4}(12z)}{\eta^{6}(2z)},$$

$$f_{12} := \sum_{n=0}^{\infty} f_{12}(n) = \frac{\eta^{7}(4z)\,\eta^{15}(6z)\,\eta^{3}(12z)}{\eta^{5}(2z)},$$

$$f_{13} := \sum_{n=0}^{\infty} f_{13}(n) = \frac{\eta^{19}(4z)\,\eta^{15}(6z)}{\eta^{5}(2z)\,\eta^{9}(12z)},$$

$$f_{14} := \sum_{n=0}^{\infty} f_{14}(n) = \frac{\eta^{2}(4z)\,\eta^{20}(6z)\,\eta^{2}(12z)}{\eta^{4}(2z)},$$

$$f_{15} := \sum_{n=0}^{\infty} f_{15}(n) = \frac{\eta^{18}(4z)\,\eta^{10}(12z)}{\eta^{8}(6z)}.$$

*They are functions of q by (3). Now define integers*

$$k_0, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, k_{11},$$

$$k_{12}, k_{13}, k_{14}, k_{15}, k_{16}, k_{17}, k_{18}, k_{19} \text{ and } k_{20}$$

*by*

$$\frac{1}{2^{b_1+b_5}}x^{b_1}(1-x)^{b_2}(1+x)^{b_3}(1+2x)^{b_4}(2+x)^{b_5} \tag{12}$$
$$= k_0+k_1x+k_2x^2+k_3x+k_4x^4+k_5x^5+k_6x^6+k_7x^7+k_8x^8$$
$$+k_9x^9+k_{10}x^{10}+k_{11}x^{11}+k_{12}x^{12}+k_{13}x^{13}+k_{14}x^{14}+k_{15}x^{15} \tag{13}$$
$$+k_{16}x^{16}+k_{17}x^{17}+k_{18}x^{18}+k_{19}x^{19}+k_{20}x^{20} \tag{14}$$

*Define the rational numbers*

$$c_1,c_2,c_3,c_4,c_6,c_{12},r_1,r_2,r_3,r_4,r_5,r_6,r_7,r_8,r_9,r_{10},r_{11},r_{12},r_{13},r_{14}$$

*and* $r_{15}$ *by*

$$c_1 = -20k_0+2k_1,$$

$$\begin{aligned} c_2 = {} & \frac{4642947536}{451737}k_0-\frac{28421116216}{27555957}k_1+\frac{37828292}{27555957}k_2-\frac{7391728}{27555957}k_3 \\ & +\frac{869600}{27555957}k_4+\frac{32}{27555957}k_5-\frac{64}{27555957}k_6+\frac{128}{27555957}k_7 \\ & -\frac{256}{27555957}k_8+\frac{512}{27555957}k_9-\frac{1024}{27555957}k_{10}+\frac{2048}{27555957}k_{11} \\ & -\frac{4096}{27555957}k_{12}+\frac{8192}{27555957}k_{13}-\frac{16384}{27555957}k_{14}+\frac{32768}{27555957}k_{15} \\ & -\frac{65536}{27555957}k_{16}+\frac{131072}{27555957}k_{17}-\frac{262144}{27555957}k_{18}+\frac{524288}{27555957}k_{19} \\ & -\frac{1048576}{27555957}k_{20}, \end{aligned}$$

$$\begin{aligned} c_3 = {} & \frac{612}{31}k_0-\frac{54}{31}k_1-\frac{8}{31}k_2+\frac{8}{31}k_3-\frac{8}{31}k_4+\frac{8}{31}k_5-\frac{8}{31}k_6 \\ & +\frac{8}{31}k_7-\frac{8}{31}k_8+\frac{8}{31}k_9-\frac{8}{31}k_{10}+\frac{8}{31}k_{11}-\frac{8}{31}k_{12}+\frac{8}{31}k_{13}-\frac{8}{31}k_{14} \\ & +\frac{8}{31}k_{15}-\frac{8}{31}k_{16}+\frac{8}{31}k_{17}-\frac{8}{31}k_{18}+\frac{8}{31}k_{19}, \end{aligned}$$

$$\begin{aligned}
c_4 &= -\frac{4633910768}{451737}k_0+\frac{28366003288}{27555957}k_1-\frac{37826264}{27555957}k_2+\frac{7387672}{27555957}k_3\\
&-\frac{861488}{27555957}k_4-\frac{16256}{27555957}k_5+\frac{32512}{27555957}k_6-\frac{65024}{27555957}k_7\\
&+\frac{130048}{27555957}k_8-\frac{260096}{27555957}k_9+\frac{520192}{27555957}k_{10}-\frac{1040384}{27555957}k_{11}\\
&+\frac{2080768}{27555957}k_{12}-\frac{4161536}{27555957}k_{13}+\frac{8323072}{27555957}k_{14}-\frac{16646144}{27555957}k_{15}\\
&+\frac{33292288}{27555957}k_{16}-\frac{66584576}{27555957}k_{17}+\frac{133169152}{27555957}k_{18}\\
&-\frac{266338304}{27555957}k_{19}+\frac{532676608}{27555957}k_{20},
\end{aligned}$$

$$\begin{aligned}
c_6 &= -\frac{20639200765024}{2263955265}k_0+\frac{114605218713824}{138101271165}k_1\\
&-\frac{1446950657356}{138101271165}k_2+\frac{1601269436504}{138101271165}k_3\\
&+\frac{2567146092344}{138101271165}k_4-\frac{5083004890168}{138101271165}k_5\\
&+\frac{5587733809496}{138101271165}k_6-\frac{4824727176088}{138101271165}k_7\\
&+\frac{3426911025176}{138101271165}k_8-\frac{1712593527064}{138101271165}k_9\\
&-\frac{160954715368}{138101271165}k_{10}+\frac{2113291577576}{138101271165}/k_{11}\\
&-\frac{4106156138728}{138101271165}k_{12}+\frac{1223752886728}{27620254233}k_{13}\\
&-\frac{8142991257832}{138101271165}k_{14}+\frac{2034746755528}{27620254233}k_{15}\\
&-\frac{12211933901032}{138101271165}k_{16}+\frac{14259475443944}{138101271165}k_{17}\\
&-\frac{16325699825896}{138101271165}k_{18}+\frac{4091629717960}{27620254233}k_{19}\\
&-\frac{15496389976064}{27620254233}k_{20},
\end{aligned}$$

$$
\begin{aligned}
c_{12} \quad = \quad & \frac{19996811552624}{2263955265}k_0 - \frac{114364650127384}{138101271165}k_1 \\
& + \frac{1482579531416}{138101271165}k_2 - \frac{1636888146904}{138101271165}k_3 \\
& - \frac{2531547709264}{138101271165}k_4 + \frac{5047447161728}{138101271165}k_5 \\
& - \frac{5552257390336}{138101271165}k_6 + \frac{4789413375488}{138101271165}k_7 \\
& - \frac{3391922461696}{138101271165}k_8 + \frac{1678255437824}{138101271165}k_9 \\
& + \frac{193991856128}{138101271165}k_{10} - \frac{2143726821376}{138101271165}k_{11} \\
& + \frac{4131387588608}{138101271165}k_{12} - \frac{1226717659136}{27620254233}k_{13} \\
& + \frac{8136999944192}{138101271165}k_{14} - \frac{2025222422528}{27620254233}k_{15} \\
& + \frac{12081051533312}{138101271165}k_{16} - \frac{13962071670784}{138101271165}k_{17} \\
& + \frac{15695253241856}{138101271165}k_{18} - \frac{3832323276800}{27620254233}k_{19} \\
& + \frac{14963521478656}{27620254233}k_{20},
\end{aligned}
$$

$$
\begin{aligned}
r_1 \quad = \quad & -\frac{952339568}{31}k_0 + \frac{93141584}{31}k_1 + \frac{2126128}{31}k_2 - \frac{1909872}{31}k_3 \\
& + \frac{1697336}{31}k_4 - \frac{1467192}{31}k_5 + \frac{1189432}{31}k_6 - \frac{832312}{31}k_7 \\
& + \frac{364088}{31}k_8 + \frac{246984}{31}k_9 - \frac{1032648}{31}k_{10} + \frac{2024648}{31}k_{11} \\
& - \frac{3254728}{31}k_{12} + \frac{4754632}{31}k_{13} - \frac{6556104}{31}k_{14} + \frac{8690888}{31}k_{15} \\
& - \frac{11190728}{31}k_{16} + \frac{14055624}{31}k_{17} - \frac{16920520}{31}k_{18} + \frac{16920520}{31}k_{19},
\end{aligned}
$$

$$\begin{aligned} r_2 \;=\;& \frac{2113111472}{31}k_0-\frac{208885136}{31}k_1-\frac{2271984}{31}k_2+\frac{1948592}{31}k_3\\ &-\frac{1705304}{31}k_4+\frac{1468216}{31}k_5-\frac{1189464}{31}k_6+\frac{832344}{31}k_7\\ &-\frac{364120}{31}k_8-\frac{246952}{31}k_9+\frac{1032616}{31}k_{10}-\frac{2024616}{31}k_{11}\\ &+\frac{3254696}{31}k_{12}-\frac{4754600}{31}k_{13}+\frac{6556072}{31}k_{14}-\frac{8690856}{31}k_{15}\\ &+\frac{11190696}{31}k_{16}-\frac{14055592}{31}k_{17}+\frac{16920488}{31}k_{18}-\frac{16920488}{31}k_{19}, \end{aligned}$$

$$\begin{aligned} r_3 \;=\;& -131072k_1+131072k_2-114688k_3+98304k_4-81920k_5\\ &+65536k_6-49152k_7+32768k_8-16384k_9+16384k_{11}-32768k_{12}\\ &+49152k_{13}-65536k_{14}+81920k_{15}-98304k_{16}+114688k_{17}-131072k_{18}\\ &+131072k_{19}, \end{aligned}$$

$$\begin{aligned} r_4 \;=\;& -\frac{221503616}{31}k_0+\frac{26270080}{31}k_1-\frac{4461952}{31}k_2+\frac{4255616}{31}k_3\\ &-\frac{3936192}{31}k_4+\frac{3491776}{31}k_5-\frac{2920384}{31}k_6+\frac{2222016}{31}k_7\\ &-\frac{1396672}{31}k_8+\frac{444352}{31}k_9+\frac{634944}{31}k_{10}-\frac{1841216}{31}k_{11}\\ &+\frac{3174464}{31}k_{12}-\frac{4634688}{31}k_{13}+\frac{6221888}{31}k_{14}-\frac{7936064}{31}k_{15}\\ &+\frac{9777216}{31}k_{16}-\frac{11745344}{31}k_{17}+\frac{13713472}{31}k_{18}-\frac{13713472}{31}k_{19}, \end{aligned}$$

$$\begin{aligned} r_5 \;=\;& \frac{20408982256}{31}k_0-\frac{2033197072}{31}k_1-\frac{4811376}{31}k_2+\frac{2403792}{31}k_3\\ &-\frac{1200248}{31}k_4+\frac{599096}{31}k_5-\frac{299512}{31}k_6+\frac{151704}{31}k_7\\ &-\frac{81272}{31}k_8+\frac{52504}{31}k_9-\frac{49528}{31}k_{10}+\frac{68376}{31}k_{11}-\frac{113016}{31}k_{12}\\ &+\frac{195352}{31}k_{13}-\frac{335224}{31}k_{14}+\frac{560408}{31}k_{15}-\frac{898680}{31}k_{16}\\ &+\frac{1350040}{31}k_{17}-\frac{1801400}{31}k_{18}+\frac{1801400}{31}k_{19}, \end{aligned}$$

$$\begin{aligned}
r_6 &= \frac{12150768}{31}k_0 - \frac{1208016}{31}k_1 - \frac{2224}{31}k_2 + \frac{240}{31}k_3 + \frac{8}{31}k_4 - \frac{8}{31}k_5 + \frac{8}{31}k_6 \\
&\quad - \frac{8}{31}k_7 + \frac{8}{31}k_8 - \frac{8}{31}k_9 + \frac{8}{31}k_{10} - \frac{8}{31}k_{11} + \frac{8}{31}k_{12} - \frac{8}{31}k_{13} + \frac{8}{31}k_{14} \\
&\quad - \frac{8}{31}k_{15} + \frac{8}{31}k_{16} - \frac{8}{31}k_{17} + \frac{8}{31}k_{18} - \frac{8}{31}k_{19},
\end{aligned}$$

$$\begin{aligned}
r_7 &= -\frac{2292213792}{31}k_0 + \frac{228500640}{31}k_1 + \frac{389472}{31}k_2 - \frac{130560}{31}k_3 + \frac{11024}{31}k_4 \\
&\quad + \frac{32624}{31}k_5 - \frac{28656}{31}k_6 - \frac{14000}{31}k_7 + \frac{98320}{31}k_8 - \frac{236208}{31}k_9 \\
&\quad + \frac{447504}{31}k_{10} - \frac{759984}{31}k_{11} + \frac{1209360}{31}k_{12} - \frac{1839280}{31}k_{13} + \frac{2701328}{31}k_{14} \\
&\quad - \frac{3855024}{31}k_{15} + \frac{5359888}{31}k_{16} - \frac{7215920}{31}k_{17} + \frac{9071952}{31}k_{18} - \frac{9071952}{31}k_{19},
\end{aligned}$$

$$\begin{aligned}
r_8 &= 5925190016k_0 - 590283200k_1 - 1396800k_2 + 697760k_3 - 348224k_4 \\
&\quad + 173472k_5 - 86080k_6 + 42400k_7 - 20544k_8 + 9632k_9 - 4160k_{10} \\
&\quad + 1440k_{11} - 64k_{12} - 608k_{13} + 960k_{14} - 1120k_{15} + 1216k_{16} - 1248k_{17} \\
&\quad + 1280k_{18} - 1280k_{19},
\end{aligned}$$

$$
\begin{aligned}
r_9 \;=\;& -\frac{1984390709172224}{219092445}k_0+\frac{8259667471227904}{13364639145}k_1\\
&+\frac{15128649567560704}{13364639145}k_2-\frac{13643933395567616}{13364639145}k_3\\
&+\frac{8319338878142464}{13364639145}k_4-\frac{4397726961287168}{13364639145}k_5\\
&+\frac{2209912340709376}{13364639145}k_6-\frac{1117422385430528}{13364639145}k_7\\
&+\frac{573410148745216}{13364639145}k_8-\frac{302859666980864}{13364639145}k_9\\
&+\frac{169893730189312}{13364639145}k_{10}-\frac{105019524644864}{13364639145}k_{11}\\
&+\frac{75197978509312}{13364639145}k_{12}-\frac{12501694676992}{2672927829}k_{13}\\
&+\frac{60004287643648}{13364639145}k_{14}-\frac{12684696616960}{2672927829}k_{15}\\
&+\frac{73873688363008}{13364639145}k_{16}-\frac{93570160787456}{13364639145}k_{17}\\
&+\frac{131759167504384}{13364639145}k_{18}-\frac{41627436187648}{2672927829}k_{19}\\
&+\frac{83254872375296}{2672927829}k_{20},
\end{aligned}
$$

$$
\begin{aligned}
r_{10} \quad = \quad & -\frac{48856130407424}{43818489}k_0+\frac{307703233879552}{2672927829}k_1 \\
& +\frac{281634772042240}{2672927829}k_2-\frac{284920099860992}{2672927829}k_3 \\
& +\frac{189596109546496}{2672927829}k_4-\frac{116758751264768}{2672927829}k_5 \\
& +\frac{73694715510784}{2672927829}k_6-\frac{45355674337280}{2672927829}k_7 \\
& +\frac{18926537555968}{2672927829}k_8+\frac{12035111231488}{2672927829}k_9 \\
& -\frac{50707353632768}{2672927829}k_{10}+\frac{98705846468608}{2672927829}k_{11} \\
& -\frac{156778780983296}{2672927829}k_{12}+\frac{225293847666688}{2672927829}k_{13} \\
& -\frac{304309212299264}{2672927829}k_{14}+\frac{393715468042240}{2672927829}k_{15} \\
& -\frac{493068062818304}{2672927829}k_{16}+\frac{601477892472832}{2672927829}k_{17} \\
& -\frac{728002191884288}{2672927829}k_{18}+\frac{981050790707200}{2672927829}k_{19} \\
& -\frac{1962101581414400}{2672927829}k_{20},
\end{aligned}
$$

$$\begin{aligned}
r_{11} = {} & \frac{7305941762925824}{6791865795}k_0 - \frac{35917903496692864}{414303813495}k_1 \\
& - \frac{52999832470920064}{414303813495}k_2 + \frac{51812959134321536}{414303813495}k_3 \\
& - \frac{35636448534609664}{414303813495}k_4 + \frac{23040901841708288}{414303813495}k_5 \\
& - \frac{14951161551387136}{414303813495}k_6 + \frac{8760943767016448}{414303813495}k_7 \\
& - \frac{2147127300468736}{414303813495}k_8 - \frac{6266897619552256}{414303813495}k_9 \\
& + \frac{17370673118885888}{414303813495}k_{10} - \frac{31816058750089216}{414303813495}k_{11} \\
& + \frac{50114300154441728}{414303813495}k_{12} - \frac{14539253178368000}{82860762699}k_{13} \\
& + \frac{99899020606472192}{414303813495}k_{14} - \frac{26387794271633408}{82860762699}k_{15} \\
& + \frac{168791210038034432}{414303813495}k_{16} - \frac{6787287755751424}{13364639145}k_{17} \\
& + \frac{261545574239338496}{414303813495}k_{18} - \frac{72764976372285440}{82860762699}k_{19} \\
& + \frac{145529952744570880}{82860762699}k_{20},
\end{aligned}$$

$$\begin{aligned}
r_{12} \quad = \quad & -\frac{135743150965088}{754651755}k_0+\frac{569127797062768}{46033757055}k_1 \\
& +\frac{1087382418152848}{46033757055}k_2-\frac{1049017995949712}{46033757055}k_3 \\
& +\frac{728786998224928}{46033757055}k_4-\frac{466908315967136}{46033757055}k_5 \\
& +\frac{266305048635712}{46033757055}k_6-\frac{58037437759616}{46033757055}k_7 \\
& -\frac{228442680812288}{46033757055}k_8+\frac{654361000195072}{46033757055}k_9 \\
& -\frac{1281244620732416}{46033757055}k_{10}+\frac{2175536001236992}{46033757055}k_{11} \\
& -\frac{3409040578297856}{46033757055}k_{12}+\frac{1011379048767488}{9206751411}k_{13} \\
& -\frac{7190578539167744}{46033757055}k_{14}+\frac{1972786208620544}{9206751411}k_{15} \\
& -\frac{13095517409460224}{46033757055}k_{16}+\frac{545886030512128}{1484959905}k_{17} \\
& -\frac{21940142822408192}{46033757055}k_{18}+\frac{6395098915094528}{9206751411}k_{19} \\
& -\frac{12790197830189056}{9206751411}k_{20},
\end{aligned}$$

$$\begin{aligned}
r_{13} \quad = \quad & \frac{91256224}{451737}k_0-\frac{898422032}{27555957}k_1+\frac{72395536}{27555957}k_2 \\
& +\frac{7391728}{27555957}k_3-\frac{869600}{27555957}k_4-\frac{32}{27555957}k_5 \\
& +\frac{64}{27555957}k_6-\frac{128}{27555957}k_7+\frac{256}{27555957}k_8-\frac{512}{27555957}k_9 \\
& +\frac{1024}{27555957}k_{10}-\frac{2048}{27555957}k_{11}+\frac{4096}{27555957}k_{12}-\frac{8192}{27555957}k_{13} \\
& +\frac{16384}{27555957}k_{14}-\frac{32768}{27555957}k_{15}+\frac{65536}{27555957}k_{16} \\
& -\frac{131072}{27555957}k_{17}+\frac{262144}{27555957}k_{18}-\frac{524288}{27555957}k_{19} \\
& +\frac{1048576}{27555957}k_{20},
\end{aligned}$$

$$\begin{aligned}
r_{14} \quad = \quad & \frac{256163647424}{83850195}k_0 - \frac{1176360110944}{5114861895}k_1 \\
& -\frac{1836809263264}{5114861895}k_2 + \frac{1575858858656}{5114861895}k_3 \\
& -\frac{624705865024}{5114861895}k_4 - \frac{503522197312}{5114861895}k_5 \\
& +\frac{2020559297024}{5114861895}k_6 - \frac{4421224784512}{5114861895}k_7 \\
& +\frac{8436608155904}{5114861895}k_8 - \frac{15132626937856}{5114861895}k_9 \\
& +\frac{26078642628608}{5114861895}k_{10} - \frac{43514430785536}{5114861895}k_{11} \\
& +\frac{70451023659008}{5114861895}k_{12} - \frac{22115292434432}{1022972379}k_{13} \\
& +\frac{167771366592512}{5114861895}k_{14} - \frac{49028015575040}{1022972379}k_{15} \\
& +\frac{344234378350592}{5114861895}k_{16} - \frac{15101865570304}{164995545}k_{17} \\
& +\frac{641739594715136}{5114861895}k_{18} - \frac{197780623757312}{1022972379}k_{19} \\
& +\frac{395561247514624}{1022972379}k_{20},
\end{aligned}$$

$$\begin{aligned} r_{15} &= \frac{15771216859136}{219092445}k_0 - \frac{166777818105856}{13364639145}k_1 \\ &\quad - \frac{7758041430016}{13364639145}k_2 + \frac{9599181584384}{13364639145}k_3 \\ &\quad + \frac{15215191199744}{13364639145}k_4 - \frac{30284194152448}{13364639145}k_5 \\ &\quad + \frac{33312566706176}{13364639145}k_6 - \frac{28734524981248}{13364639145}k_7 \\ &\quad + \frac{20347624226816}{13364639145}k_8 - \frac{10061711540224}{13364639145}k_9 \\ &\quad - \frac{1179593310208}{13364639145}k_{10} + \frac{12893645275136}{13364639145}k_{11} \\ &\quad - \frac{24850894225408}{13364639145}k_{12} + \frac{7385333432320}{2672927829}k_{13} \\ &\quad - \frac{49072274440192}{13364639145}k_{14} + \frac{12251444445184}{2672927829}k_{15} \\ &\quad - \frac{73487408300032}{13364639145}k_{16} + \frac{85774628225024}{13364639145}k_{17} \\ &\quad - \frac{98175915851776}{13364639145}k_{18} + \frac{24595698221056}{2672927829}k_{19} \\ &\quad - \frac{49191396442112}{2672927829}k_{20}. \end{aligned}$$

*Here* $\{f_1, \ldots f_{15}\} \setminus \{f_8, f_9, f_{15}\} \in S_{10}(\Gamma_0(12))$, *while* $f_8, f_9, f_{15} \in M_{10}(\Gamma_0(12)) \setminus S_{10}(\Gamma_0(12))$

*and*

$$\eta^{a_1}(z)\,\eta^{a_2}(2z)\,\eta^{a_3}(3z)\,\eta^{a_4}(4z)\,\eta^{a_6}(6z)\,\eta^{a_{12}}(12z) = \delta(b_1) + \sum_{n=1}^{\infty} c(n)q^n,$$

*where for* $n \in \mathbb{N}$,

$$\begin{aligned} c(n) &= -c_1\sigma_9(n) - c_2\sigma_9\left(\frac{n}{2}\right) - c_3\sigma_9\left(\frac{n}{3}\right) - c_4\sigma_9\left(\frac{n}{4}\right) - c_6\sigma_9\left(\frac{n}{6}\right) - c_{12}\sigma_9\left(\frac{n}{12}\right) \\ &\quad + r_1 f_1(n) + \cdots + r_{15} f_{15}(n). \end{aligned}$$

*In particular,*

$$\begin{aligned} c(2n) &= -c_1\sigma_9(2n) - c_2\sigma_9(n) - c_4\sigma_9\left(\frac{n}{2}\right) - (513c_3 + c_6)\,\sigma_9\left(\frac{n}{3}\right) \\ &\quad - (c_{12} - 512c_3)\sigma_9\left(\frac{n}{6}\right) + r_9 f_9(2n) + \cdots + r_{15} f_{15}(2n), \end{aligned}$$

$$\begin{aligned} c(2n-1) &= -c_1\sigma_9(2n-1) - c_3\sigma_9\left(\frac{2n-1}{3}\right) \\ &\quad + r_1 f_1(2n-1) + \cdots + r_8 f_8(2n-1), \end{aligned}$$

*for* $n \in \mathbb{N}$.

*Proof.* It follows from (6-11) that

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 12a_{12} = 24b_1, \tag{15}$$

$$a_1 + a_2 + a_3 + a_4 + a_6 + a_{12} = 20, \tag{16}$$

$$-\frac{a_1}{6} - \frac{a_2}{3} - \frac{a_3}{6} - 2\frac{a_4}{3} - \frac{a_6}{3} - 2\frac{a_{12}}{3} = -b_1 - b_5.$$

Now we will use $p-k$ parametrization of Alaca, Alaca and Williams, see [1]:

$$p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}, \tag{17}$$

where the theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{-\infty}^{\infty} q^{n^2}.$$

Setting $x = p$ in (12), and multiplying both sides by $k^{10}$,we obtain

$$\begin{aligned} &\frac{k^{10}}{2^{b_1+b_5}} p^{b_1}(1-p)^{b_2}(1+p)^{b_3}(1+2p)^{b_4}(2+p)^{b_5} \\ &= (k_0 + k_1 p + k_2 p^2 + k_3 p^3 + k_4 p^4 + k_5 p^5 + k_6 p^6 + k_7 p^7 \\ &\quad + k_8 p^8 + k_9 p^9 + k_{10} p^{10} + k_{11} p^{11} + +k_{12} p^{12} + k_{13} p^{13} + k_{14} p^{14} \\ &\quad + k_{15} p^{15} + k_{16} p^{16} + k_{17} p^{17} + k_{18} p^{18} + k_{19} p^{19} + k_{20} p^{20}) k^{10}. \end{aligned}$$

Alaca, Alaca and Williams [2] have established the following representations in terms of $p$ and

$k$:

$$\eta(q) = 2^{-1/6} p^{1/24} (1-p)^{1/2} (1+p)^{1/6} (1+2p)^{1/8} (2+p)^{1/8} k^{1/2}, \tag{18}$$

(19) $$\eta\left(q^2\right)=2^{-1/3}p^{1/12}(1-p)^{1/4}(1+p)^{1/12}(1+2p)^{1/4}(2+p)^{1/4}k^{1/2},$$

(20) $$\eta\left(q^3\right)=2^{-1/6}p^{1/8}(1-p)^{1/6}(1+p)^{1/2}(1+2p)^{1/24}(2+p)^{1/24}k^{1/2},$$

(21) $$\eta\left(q^4\right)=2^{-2/3}p^{1/6}(1-p)^{1/8}(1+p)^{1/24}(1+2p)^{1/8}(2+p)^{1/2}k^{1/2},$$

(22) $$\eta\left(q^6\right)=2^{-1/3}p^{1/4}(1-p)^{1/12}(1+p)^{1/4}(1+2p)^{1/12}(2+p)^{1/12}k^{1/2},$$

(23) $$\eta\left(q^{12}\right)=2^{-2/3}p^{1/2}(1-p)^{1/24}(1+p)^{1/8}(1+2p)^{1/24}(2+p)^{1/6}k^{1/2},$$

$$E_6(q):=1-504\sum_{n=1}^{\infty}\sigma_5(n)\,q^n$$

$$\begin{aligned}
&= (1-246p-5532p^2-38614p^3-135369p^4-276084p^5\\
&\quad -348024p^6-276084p^7-135369p^8-38614p^9-5532p^{10}\\
&\quad -246p^{11}+p^{12})k^6,
\end{aligned}$$

$$E_4(q):=1+240\sum_{n=1}^{\infty}\sigma_3(n)\,q^n$$

$$\begin{aligned}
&= (1+124p+964p^2+2788p^3+3910p^4+2788p^5\\
&\quad +964p^6+124p^7+p^8)k^4.
\end{aligned}$$

Therefore, since

$$E_{10}(q)=E_6(q)*E_4(q),$$

we immediately obtain:

$$\begin{aligned}
E_{10}(q) &= (p^{20}-122p^{19}-35072p^{18}-958938p^{17}-10938291p^{16}\\
&\quad -70668024p^{15}-295048992p^{14}-853625784p^{13}\\
&\quad -1781899086p^{12}-2748067580p^{11}-3170796992p^{10}\\
&\quad -2748067580p^9-1781899086p^8-853625784p^7-\\
&\quad 295048992p^6-70668024p^5-10938291p^4-958938p^3\\
&\quad -35072p^2-122p+1)k^{10}
\end{aligned}$$

$$\begin{aligned} E_{10}\left(q^{2}\right) &= (p^{20}+10p^{19}-26p^{18}-519p^{17}-\frac{21045}{2}p^{16}-71124p^{15} \\ &\quad -289560p^{14}-829884p^{13}-1736031p^{12}-2685710p^{11} \\ &\quad -3102176p^{10}-2685710p^{9}-1736031p^{8}-829884p^{7} \\ &\quad -289560p^{6}-71124p^{5}-\frac{21045}{2}p^{4}-519p^{3} \\ &\quad -26p^{2}+10p+1)k^{10} \end{aligned}$$

$$\begin{aligned} E_{10}\left(q^{3}\right) &= (p^{20}+10p^{19}+40p^{18}+42p^{17}-243p^{16}-1032p^{15}-3648 \\ &\quad p^{14}-15048p^{13}-36366p^{12}-46436p^{11}-43472p^{10} \\ &\quad -46436p^{9}-36366p^{8}-15048p^{7}-3648p^{6} \\ &\quad -1032p^{5}-243p^{4}+42p^{3}+40p^{2}+10p+1)k^{10} \end{aligned}$$

$$\begin{aligned} E_{10}\left(q^{4}\right) &= (\frac{1}{1024}p^{20}+\frac{71}{512}p^{19}-\frac{8141}{256}p^{18}+\frac{87411}{256}p^{17} \\ &\quad +\frac{60351}{512}p^{16}-\frac{128715}{64}p^{15}-\frac{88041}{64}p^{14} \\ &\quad +\frac{43791}{16}p^{13}+\frac{71511}{64}p^{12}-\frac{164731}{32}p^{11} \\ &\quad -\frac{23507}{4}p^{10}-\frac{34367}{16}p^{9}-\frac{12903}{32}p^{8} \\ &\quad -\frac{4983}{8}p^{7}-\frac{4929}{8}p^{6}-240p^{5}+21p^{4} \\ &\quad +75p^{3}+40p^{2}+10p+1)k^{10} \end{aligned}$$

$$\begin{aligned} E_{10}\left(q^{6}\right) &= (p^{20}+10p^{19}+40p^{18}+75p^{17}+\frac{75}{2}p^{16}-108p^{15} \\ &\quad -216p^{14}-132p^{13}+33p^{12}+94p^{11}+88p^{10}+94p^{9} \\ &\quad +33p^{8}-132p^{7}-216p^{6}-108p^{5}+\frac{75}{2}p^{4}+75p^{3} \\ &\quad +40p^{2}+10p+1)k^{10} \end{aligned}$$

$$\begin{aligned} E_{10}\left(q^{12}\right) &= (\frac{1}{1024}p^{20}+\frac{5}{512}p^{19}+\frac{5}{128}p^{18}+\frac{27}{256}p^{17}+\frac{159}{512}p^{16} \\ &\quad +\frac{51}{64}p^{15}-\frac{63}{64}p^{14}-\frac{231}{16}p^{13}-\frac{2541}{64}p^{12}-\frac{985}{32}p^{11} \\ &\quad +\frac{1045}{16}p^{10}+\frac{2725}{16}p^{9}+\frac{3465}{32}p^{8}-\frac{825}{8}p^{7}-\frac{1695}{8}p^{6} \\ &\quad -108p^{5}+\frac{75}{2}p^{4}+75p^{3}+40p^{2}+10p+1)k^{10}. \end{aligned}$$

It is easy to check the following expressions by (18-23)

$$\begin{aligned} f_1 &: =\sum_{n=0}^{\infty} f_1(n)=\frac{\eta^{16}(4z)\,\eta^{12}(6z)}{\eta^{8}(2z)} \\ &= (-\frac{1}{2048}p^{17}-\frac{33}{4096}p^{16}-\frac{15}{256}p^{15}-\frac{251}{1024}p^{14} \\ &\quad -\frac{657}{1024}p^{13}-\frac{4311}{4096}p^{12}-\frac{1953}{2048}p^{11}-\frac{99}{1024}p^{10} \\ &\quad +\frac{459}{512}p^{9}+\frac{299}{256}p^{8}+\frac{93}{128}p^{7}+\frac{15}{64}p^{6}+\frac{1}{32}p^{5})k^{10}, \end{aligned}$$

$$\begin{aligned} f_2 &: =\sum_{n=0}^{\infty} f_2(n)=\frac{\eta^{11}(4z)\,\eta^{17}(6z)}{\eta^{7}(2z)\,\eta(12z)} \\ &= (-\frac{1}{512}p^{16}-\frac{27}{1024}p^{15}-\frac{157}{1024}p^{14}-\frac{127}{256}p^{13} \\ &\quad -\frac{243}{256}p^{12}-\frac{1019}{1024}p^{11}-\frac{253}{1024}p^{10}+\frac{393}{512}p^{9} \\ &\quad +\frac{143}{128}p^{8}+\frac{23}{32}p^{7}+\frac{15}{64}p^{6}+\frac{1}{32}p^{5})k^{10}, \end{aligned}$$

$$\begin{aligned} f_3 &: =\sum_{n=0}^{\infty} f_3(n)=\frac{\eta^{18}(4z)\,\eta^{10}(12z)}{\eta^{6}(2z)\,\eta^{2}(6z)} \\ &= (-\frac{1}{32768}p^{19}-\frac{37}{65536}p^{18}-\frac{19}{4096}p^{17}-\frac{1451}{65536}p^{16} \\ &\quad -\frac{2199}{32768}p^{15}-\frac{537}{4096}p^{14}-\frac{315}{2048}p^{13}-\frac{141}{2048}p^{12} \\ &\quad +\frac{87}{1024}p^{11}+\frac{11}{64}p^{10}+\frac{17}{128}p^{9}+\frac{13}{256}p^{8}+\frac{1}{128}p^{7})k^{10}, \end{aligned}$$

$$\begin{aligned} f_4 \quad : \quad &= \sum_{n=0}^{\infty} f_4(n) = \frac{\eta^{13}(4z)\,\eta^3(6z)\,\eta^9(12z)}{\eta^5(2z)} \\ &= (-\frac{1}{8192}p^{18} - \frac{31}{16384}p^{17} - \frac{209}{16384}p^{16} - \frac{795}{16384}p^{15} \\ &\quad -\frac{1833}{16384}p^{14} - \frac{1239}{8192}p^{13} - \frac{357}{4096}p^{12} + \frac{129}{2048}p^{11} \\ &\quad +\frac{165}{1024}p^{10} + \frac{67}{512}p^9 + \frac{13}{256}p^8 + \frac{1}{128}p^7)k^{10}, \end{aligned}$$

$$\begin{aligned} f_5 \quad : \quad &= \sum_{n=0}^{\infty} f_5(n) = \frac{\eta^{18}(6z)\,\eta^6(12z)}{\eta^2(2z)\,\eta^2(4z)} \\ &= (-\frac{1}{128}p^{15} - \frac{13}{256}p^{14} - \frac{1}{8}p^{13} - \frac{33}{256}p^{12} + \frac{33}{256}p^{10} \\ &\quad +\frac{1}{8}p^9 + \frac{13}{256}p^8 + \frac{1}{128}p^7)k^{10}, \end{aligned}$$

$$\begin{aligned} f_6 \quad : \quad &= \sum_{n=0}^{\infty} f_6(n) = \frac{\eta^{20}(4z)\,\eta^8(6z)}{\eta^4(2z)\,\eta^4(12z)} \\ &= (\frac{1}{1024}p^{18} + \frac{19}{1024}p^{17} + \frac{641}{4096}p^{16} + \frac{1565}{2048}p^{15} + \frac{4801}{2048}p^{14} \\ &\quad +\frac{2297}{512}p^{13} + \frac{18505}{4096}p^{12} - \frac{1239}{2048}p^{11} - \frac{2211}{256}p^{10} - \frac{1475}{128}p^9 \\ &\quad -\frac{689}{128}p^8 + \frac{199}{64}p^7 + \frac{95}{16}p^6 + \frac{29}{8}p^5 + \frac{17}{16}p^4 + \frac{1}{8}p^3)k^{10}, \end{aligned}$$

$$\begin{aligned} f_7 \quad : \quad &= \sum_{n=0}^{\infty} f_7(n) = \frac{\eta^3(4z)\,\eta^{13}(6z)\,\eta^7(12z)}{\eta^3(2z)} \\ &= (-\frac{1}{512}p^{16} - \frac{19}{1024}p^{15} - \frac{73}{1024}p^{14} - \frac{35}{256}p^{13} - \frac{15}{128}p^{12} \\ &\quad +\frac{21}{1024}p^{11} + \frac{143}{1024}p^{10} + \frac{65}{512}p^9 + \frac{13}{256}p^8 + \frac{1}{128}p^7)k^{10}, \end{aligned}$$

$$\begin{aligned} f_8 \quad : \quad &= \sum_{n=0}^{\infty} f_8(n) = \frac{\eta^{17}(6z)\,\eta^{11}(12z)}{\eta^7(2z)\,\eta(4z)} \\ &= (\frac{1}{1024}p^{15} + \frac{7}{1024}p^{14} + \frac{5}{256}p^{13} + \frac{15}{512}p^{12} + \frac{25}{1024}p^{11} \\ &\quad +\frac{11}{1024}p^{10} + \frac{1}{512}p^9)k^{10}, \end{aligned}$$

$$f_9 \quad : \quad = \sum_{n=0}^{\infty} f_9(n) = \frac{\eta^{15}(4z)\,\eta^{19}(12z)}{\eta^{9}(2z)\,\eta^{5}(6z)}$$
$$= \Big(\frac{1}{262144}p^{19} + \frac{17}{262144}p^{18} + \frac{1}{2048}p^{17} + \frac{35}{16384}p^{16} + \frac{49}{8192}p^{15}$$
$$+\frac{91}{8192}p^{14} + \frac{7}{512}p^{13} + \frac{11}{1024}p^{12} + \frac{5}{1024}p^{11} + \frac{1}{1024}p^{10}\Big)k^{10},$$

$$f_{10} \quad : \quad = \sum_{n=0}^{\infty} f_{10}(n) = \frac{\eta^{17}(4z)\,\eta^{5}(6z)\,\eta^{5}(12z)}{\eta^{7}(2z)}$$
$$= \Big(-\frac{1}{8192}p^{18} - \frac{35}{16384}p^{17} - \frac{271}{16384}p^{16} - \frac{1213}{16384}p^{15}$$
$$-\frac{3423}{16384}p^{14} - \frac{3}{8}p^{13} - \frac{399}{1024}p^{12} - \frac{57}{512}p^{11}$$
$$+\frac{147}{512}p^{10} + \frac{29}{64}p^{9} + \frac{5}{16}p^{8} + \frac{7}{64}p^{7} + \frac{1}{64}p^{6}\Big)k^{10},$$

$$f_{11} \quad : \quad = \sum_{n=0}^{\infty} f_{11}(n) = \frac{\eta^{12}(4z)\,\eta^{10}(6z)\,\eta^{4}(12z)}{\eta^{6}(2z)}$$
$$= \Big(-\frac{1}{2048}p^{17} - \frac{29}{4096}p^{16} - \frac{91}{2048}p^{15} - \frac{5}{32}p^{14}$$
$$-\frac{337}{1024}p^{13} - \frac{1615}{4096}p^{12} - \frac{169}{1024}p^{11} + \frac{239}{1024}p^{10}$$
$$+\frac{55}{128}p^{9} + \frac{79}{256}p^{8} + \frac{7}{64}p^{7} + \frac{1}{64}p^{6}\Big)k^{10},$$

$$f_{12} \quad : \quad = \sum_{n=0}^{\infty} f_{12}(n) = \frac{\eta^{7}(4z)\,\eta^{15}(6z)\,\eta^{3}(12z)}{\eta^{5}(2z)}$$
$$= \Big(-\frac{1}{512}p^{16} - \frac{23}{1024}p^{15} - \frac{111}{1024}p^{14} - \frac{143}{512}p^{13}$$
$$-\frac{25}{64}p^{12} - \frac{219}{1024}p^{11} + \frac{185}{1024}p^{10} + \frac{13}{32}p^{9}$$
$$+\frac{39}{128}p^{8} + \frac{7}{64}p^{7} + \frac{1}{64}p^{6}\Big)k^{10},$$

$$\begin{aligned} f_{13} \quad &: \quad =\sum_{n=0}^{\infty} f_{13}(n)=\frac{\eta^{19}(4z)\,\eta^{15}(6z)}{\eta^{5}(2z)\,\eta^{9}(12z)} \\ &= \quad (\frac{1}{256}p^{17}+\frac{9}{128}p^{16}+\frac{573}{1024}p^{15}+\frac{2625}{1024}p^{14} \\ &\quad +\frac{3741}{512}p^{13}+\frac{6507}{512}p^{12}+\frac{10853}{1024}p^{11}-\frac{5679}{1024}p^{10} \\ &\quad -\frac{207}{8}p^{9}-\frac{1849}{64}p^{8}-\frac{315}{32}p^{7}+\frac{339}{32}p^{6} \\ &\quad +15p^{5}+\frac{33}{4}p^{4}+\frac{9}{4}p^{3}+\frac{1}{4}p^{2})k^{10}, \end{aligned}$$

$$\begin{aligned} f_{14} \quad &: \quad =\sum_{n=0}^{\infty} f_{14}(n)=\frac{\eta^{2}(4z)\,\eta^{20}(6z)\,\eta^{2}(12z)}{\eta^{4}(2z)} \\ &= \quad (-\frac{1}{128}p^{15}-\frac{17}{256}p^{14}-\frac{29}{128}p^{13}-\frac{97}{256}p^{12}-\frac{33}{128}p^{11} \\ &\quad +\frac{33}{256}p^{10}+\frac{49}{128}p^{9}+\frac{77}{256}p^{8}+\frac{7}{64}p^{7}+\frac{1}{64}p^{6})k^{10}, \end{aligned}$$

$$\begin{aligned} f_{15} \quad &: \quad =\sum_{n=0}^{\infty} f_{15}(n)=\frac{\eta^{18}(4z)\,\eta^{10}(12z)}{\eta^{8}(6z)} \\ &= \quad (\frac{1}{16384}p^{20}+\frac{19}{16384}p^{19}+\frac{637}{65536}p^{18}+\frac{1531}{32768}p^{17} \\ &\quad +\frac{9101}{65536}p^{16}+\frac{4079}{16384}p^{15}+\frac{3381}{16384}p^{14}-\frac{69}{512}p^{13}-\frac{1131}{2048}p^{12} \\ &\quad -\frac{287}{512}p^{11}-\frac{53}{512}p^{10}+\frac{37}{128}p^{9}+\frac{73}{256}p^{8}+\frac{7}{64}p^{7}+\frac{1}{64}p^{6})k^{10}. \end{aligned}$$

Obviously, $f_1,\dots f_{15}$ are functions of $q$, see (3),( 17). We see that $\{f_1,\dots f_{15}\}\setminus\{f_8,f_9,f_{15}\}\in S_{10}(\Gamma_0(12))$, $f_8,f_9,f_{15}\in M_{10}(\Gamma_0(12))\setminus S_{10}(\Gamma_0(12))$ and

$$ord_{1/1}f_8=ord_{1/2}f_8=ord_{1/1}f_9=ord_{1/2}f_9=0, ord_{1/3}f_{15}=ord_{1/6}f_{15}=0$$

by [4]. Now

$$
\begin{aligned}
&\eta^{a_1}(z)\,\eta^{a_2}(2z)\,\eta^{a_3}(3z)\,\eta^{a_4}(4z)\,\eta^{a_6}(6z)\,\eta^{a_{12}}(12z)\\
=\;& q^{b_1}\prod_{n=1}^{\infty}(1-q^n)^{a_1}\left(1-q^{2n}\right)^{a_2}\left(1-q^{3n}\right)^{a_3}\left(1-q^{4n}\right)^{a_4}\left(1-q^{6n}\right)^{a_6}\left(1-q^{12n}\right)^{a_{12}}\\
=\;& 2^{-\frac{a_1}{6}-\frac{a_2}{3}-\frac{a_3}{6}-2\frac{a_4}{3}-\frac{a_6}{3}-2\frac{a_{12}}{3}}p^{\frac{a_1}{24}+\frac{a_2}{12}+\frac{a_3}{8}+\frac{a_4}{6}+\frac{a_6}{4}+\frac{a_{12}}{2}}(1-p)^{\frac{a_1}{2}+\frac{a_2}{4}+\frac{a_3}{6}+\frac{a_4}{8}+\frac{a_6}{12}+\frac{a_{12}}{24}}\\
&(1+p)^{\frac{a_1}{6}+\frac{a_2}{12}+\frac{a_3}{2}+\frac{a_4}{24}+\frac{a_6}{4}+\frac{a_{12}}{8}}(1+2p)^{\frac{a_1}{8}+\frac{a_2}{4}+\frac{a_3}{24}+\frac{a_4}{8}+\frac{a_6}{12}+\frac{a_{12}}{24}}(2+p)^{\frac{a_1}{8}+\frac{a_2}{4}+\frac{a_3}{24}+\frac{a_4}{2}+\frac{a_6}{12}+\frac{a_{12}}{6}}\\
&k^{\frac{a_1+a_2+a_3+a_4+a_6+a_{12}}{2}}=\frac{k^{10}}{2^{b_1+b_5}}p^{b_1}(1-p)^{b_2}(1+p)^{b_3}(1+2p)^{b_4}(2+p)^{b_5}
\end{aligned}
$$

$$
\begin{aligned}
=\;& k^{10}(k_0+k_1p+k_2p^2+k_3p^3+k_4p^4+k_5p^5+k_6p^6\\
&+k_7p^7+k_8p^8+k_9p^9+k_{10}p^{10}+k_{11}p^{11}\\
&+k_{12}p^{12}+k_{13}p^{13}+k_{14}p^{14}+k_{15}p^{15}+k_{16}p^{16}\\
&k_{17}p^{17}+k_{18}p^{18}+k_{19}p^{19}+k_{20}p^{20})
\end{aligned}
$$

$$\begin{aligned}
&= -\frac{c_1}{264}\left(1-264\sum_{n=1}^{\infty}\sigma_9(n)q^n\right)-\frac{c_2}{264}\left(1-264\sum_{n=1}^{\infty}\sigma_9(n)q^{2n}\right)\\
&\quad-\frac{c_3}{264}\left(1-264\sum_{n=1}^{\infty}\sigma_9(n)q^{3n}\right)-\frac{c_4}{264}\left(1-264\sum_{n=1}^{\infty}\sigma_9(n)q^{4n}\right)\\
&\quad-\frac{c_5}{264}\left(1-264\sum_{n=1}^{\infty}\sigma_9(n)q^{6n}\right)-\frac{c_6}{264}\left(1-264\sum_{n=1}^{\infty}\sigma_9(n)q^{12n}\right)\\
&\quad+r_1q^5\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{16}\left(1-q^{6n}\right)^{12}}{\left(1-q^{2n}\right)^8}+r_2q^5\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{11}\left(1-q^{6n}\right)^{17}}{\left(1-q^{2n}\right)^7\left(1-q^{12n}\right)}\\
&\quad+r_3q^7\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{18}\left(1-q^{12n}\right)^{10}}{\left(1-q^{2n}\right)^6\left(1-q^{6n}\right)^2}+r_4q^7\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{13}\left(1-q^{6n}\right)^3\left(1-q^{12n}\right)^9}{\left(1-q^{2n}\right)^5}\\
&\quad+r_5.q^7\prod_{n=1}^{\infty}\frac{\left(1-q^{6n}\right)^{18}\left(1-q^{12n}\right)^6}{\left(1-q^{2n}\right)^2\left(1-q^{4n}\right)^2}+r_6q^3\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{20}\left(1-q^{6n}\right)^8}{\left(1-q^{2n}\right)^4\left(1-q^{12n}\right)^4}\\
&\quad+r_7q^7\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^3\left(1-q^{6n}\right)^{13}\left(1-q^{12n}\right)^7}{\left(1-q^{2n}\right)^3}+r_8q^9\prod_{n=1}^{\infty}\frac{\left(1-q^{6n}\right)^{17}\left(1-q^{12n}\right)^{11}}{\left(1-q^{2n}\right)^7\left(1-q^{4n}\right)}\\
&\quad+r_9q^{10}\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{15}\left(1-q^{12n}\right)^{19}}{\left(1-q^{2n}\right)^9\left(1-q^{6n}\right)^5}+r_{10}q^6\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{17}\left(1-q^{6n}\right)^5\left(1-q^{12n}\right)^5}{\left(1-q^{2n}\right)^7}\\
&\quad+r_{11}q^6\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{12}\left(1-q^{6n}\right)^{10}\left(1-q^{12n}\right)^4}{\left(1-q^{2n}\right)^6}+r_{12}q^6\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^7\left(1-q^{6n}\right)^{15}\left(1-q^{12n}\right)^3}{\left(1-q^{2n}\right)^5}\\
&\quad+r_{13}q^2\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{19}\left(1-q^{6n}\right)^{15}}{\left(1-q^{2n}\right)^5\left(1-q^{12n}\right)^{19}}+r_{14}q^6\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^2\left(1-q^{6n}\right)^{20}\left(1-q^{12n}\right)^2}{\left(1-q^{2n}\right)^4}\\
&\quad+r_{15}q^6\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{18}\left(1-q^{12n}\right)^{10}}{\left(1-q^{6n}\right)^8}\\
&= \delta(b_1)-\sum_{n=1}^{\infty}\left(c_1\sigma_9(n)+c_2\sigma_9\left(\frac{n}{2}\right)+c_3\sigma_9\left(\frac{n}{3}\right)+c_4\sigma_9\left(\frac{n}{4}\right)\right.\\
&\quad\left.+c_6\sigma_9\left(\frac{n}{6}\right)+c_{12}\sigma_9\left(\frac{n}{12}\right)\right)+r_1f_1(n)+\ldots.+r_{15}f_{15}(n),
\end{aligned}$$

where

$$\delta(b_1)=\begin{cases}0 \text{ if } b_1\neq 0\\ 1 \text{ if } b_1=0\end{cases}.$$

So

$$\begin{aligned} c(n) &= -(c_1\sigma_9(n)+c_2\sigma_9\left(\frac{n}{2}\right)+c_3\sigma_9\left(\frac{n}{3}\right)+c_4\sigma_9\left(\frac{n}{4}\right) \\ &\quad +c_6\sigma_9\left(\frac{n}{6}\right)+c_{12}\sigma_9\left(\frac{n}{12}\right))+r_1f_1(n)+....+r_{15}f_{15}(n). \end{aligned}$$

Therefore, for n=1,2,..., 

$$\begin{aligned} c(2n) &= -c_1\sigma_9(2n)-c_2\sigma_9(n)-c_4\sigma_9\left(\frac{n}{2}\right)-(513c_3+c_6)\sigma_9\left(\frac{n}{3}\right) \\ &\quad -(c_{12}-512c_3)\sigma_9\left(\frac{n}{6}\right)+r_9f_9(2n)+\cdots+r_{15}f_{15}(2n), \end{aligned}$$

$$\begin{aligned} c(2n-1) &= -c_1\sigma_9(2n-1)-c_3\sigma_9\left(\frac{2n-1}{3}\right) \\ &\quad +r_1f_1(2n-1)+\cdots+r_8f_8(2n-1), \end{aligned}$$

since it is easy to see that

$$\sigma_k\left(\frac{2n}{3}\right)=\left(2^k+1\right)\sigma_k\left(\frac{n}{3}\right)-2^k\sigma_k\left(\frac{n}{6}\right)$$

hence,

$$\sigma_9\left(\frac{2n}{3}\right)=513\sigma_9\left(\frac{n}{3}\right)-512\sigma_9\left(\frac{n}{6}\right).$$

and, for n=1,2,..., 

$$f_1(2n)=\cdots=f_8(2n)=0,$$

$$f_9(2n-1)=\cdots=f_{15}(2n-1)=0.$$

∎

**Remark 2.** *These formulas are valid for 51661 nontrivial eta quotients, Among them, we have found 100 eta quotients, see Table 1, such that, for n=1,2,...,*

$$c(2n)=-c_1\sigma_9(2n)-c_2\sigma_9(n)-c_4\sigma_9\left(\frac{n}{2}\right)-(513c_3+c_6)\sigma_9\left(\frac{n}{3}\right)-(c_{12}-512c_3)\sigma_9\left(\frac{n}{6}\right)$$

$$c(2n-1)=-c_1\sigma_9(2n-1)-c_3\sigma_9\left(\frac{2n-1}{3}\right)+r_1f_1(2n-1)+\cdots+r_8f_8(2n-1).$$

*and 263 eta quotients, see Table 2, such that, for n=1,2,..., ,*

$$\begin{aligned} c(2n) &= -c_1\sigma_9(2n) - c_2\sigma_9(n) - c_4\sigma_9\left(\frac{n}{2}\right) - c_6\sigma_9\left(\frac{n}{3}\right) \\ &\quad -c_{12}\sigma_9\left(\frac{n}{6}\right) + r_9 f_9(2n) + \cdots + r_{15} f_{15}(2n), \\ c(2n-1) &= 0 \end{aligned}$$

**Remark 3.** *If $f$ is an eta quotient, then $f(-q)$ is also an eta quotient, so the coefficients of $\frac{1}{2}(f(q)+f(-q))$ are exactly the even coefficients of $f$. In particular, it means that we have obtained all coefficients of some sum of* 100 *eta quotients.*

**Remark 4.** *$S_{10}(\Gamma_0(12))$ is* 15 *dimensional, $M_{10}(\Gamma_0(12))$ is* 21 *dimensional,see* [5] *(Chapter 3, pg.87 and Chapter 5, pg.197), and generated by*

$$\begin{aligned} &\Delta_{2,10}, \Delta_{2,10}(2z), \Delta_{2,10}(3z), \Delta_{2,10}(6z), \Delta_{3,10,1}, \Delta_{3,10,1}(2z), \\ &\Delta_{3,10,1}(4z), \Delta_{3,10,2}, \Delta_{3,10,2}(2z), \Delta_{3,10,2}(4z), \\ &\Delta_{4,10}, \Delta_{4,10}(3z), \\ &\Delta_{6,10}, \Delta_{6,10}(2z), \Delta_{12,10}, \end{aligned}$$

*where $\Delta_{2,10}$ is the unique newform in $S_{10}(\Gamma_0(2))$,$\Delta_{3,10,1}$,$\Delta_{3,10,2}$ are the newforms in $S_{10}(\Gamma_0(3))$, $\Delta_{4,10}$ is the unique newform in $S_{10}(\Gamma_0(4))$ and $\Delta_{6,10}$ is the unique newform in $S_{10}(\Gamma_0(6))$, $\Delta_{12,10}$ is the unique newform in $S_{10}(\Gamma_0(12))$ . By simple calculation, we see that*

$$\begin{aligned} f_1 &= -\frac{1}{10080}\Delta_{2,10}(z) + \frac{1}{630}\Delta_{2,10}(2z) - \frac{297}{4480}\Delta_{2,10}(3z) + \frac{297}{280}\Delta_{2,10}(6z) \\ &\quad + \frac{1}{5280}\Delta_{3,10,1}(z) - \frac{3}{880}\Delta_{3,10,1}(2z) + \frac{16}{165}\Delta_{3,10,1}(4z) \\ &\quad - \frac{1}{4032}\Delta_{3,10,2}(z) - \frac{1}{112}\Delta_{3,10,2}(2z) - \frac{8}{63}\Delta_{3,10,2}(4z) - \frac{5}{13248}\Delta_{4,10}(z) \\ &\quad + \frac{351}{2944}\Delta_{4,10}(3z) + \frac{1}{6336}\Delta_{6,10}(z) + \frac{1}{396}\Delta_{6,10}(2z) + \frac{5}{13248}\Delta_{12,10}(z), \end{aligned}$$

$$\begin{aligned}
f_2 &= -\frac{1}{13440}\Delta_{2,10}(z)+\frac{1}{840}\Delta_{2,10}(2z)-\frac{249}{4480}\Delta_{2,10}(3z)\\
&\quad+\frac{249}{280}\Delta_{2,10}(6z)+\frac{43}{285120}\Delta_{3,10,1}(z)-\frac{43}{15840}\Delta_{3,10,1}(2z)\\
&\quad+\frac{344}{4455}\Delta_{3,10,1}(4z)-\frac{17}{72576}\Delta_{3,10,2}(z)-\frac{17}{2016}\Delta_{3,10,2}(2z)\\
&\quad-\frac{68}{567}\Delta_{3,10,2}(4z)-\frac{1}{2944}\Delta_{4,10}(z)+\frac{309}{2944}\Delta_{4,10}(3z)\\
&\quad+\frac{1}{6336}\Delta_{6,10}(z)+\frac{1}{396}\Delta_{6,10}(2z)+\frac{1}{2944}\Delta_{12,10}(z),
\end{aligned}$$

$$\begin{aligned}
f_3 &= -\frac{53}{860160}\Delta_{2,10}(z)+\frac{53}{53760}\Delta_{2,10}(2z)-\frac{5427}{286720}\Delta_{2,10}(3z)\\
&\quad+\frac{5427}{17920}\Delta_{2,10}(6z)+\frac{13}{450560}\Delta_{3,10,1}(z)-\frac{117}{225280}\Delta_{3,10,1}(2z)\\
&\quad+\frac{13}{880}\Delta_{3,10,1}(4z)-\frac{1}{86016}\Delta_{3,10,2}(z)-\frac{3}{7168}\Delta_{3,10,2}(2z)\\
&\quad-\frac{1}{168}\Delta_{3,10,2}(4z)-\frac{11}{565248}\Delta_{4,10}(z)+\frac{2835}{188416}\Delta_{4,10}(3z)\\
&\quad+\frac{1}{270336}\Delta_{6,10}(z)+\frac{1}{16896}\Delta_{6,10}(2z)+\frac{17}{282624}\Delta_{12,10}(z),
\end{aligned}$$

$$\begin{aligned}
f_4 &= -\frac{23}{645120}\Delta_{2,10}(z)+\frac{23}{40320}\Delta_{2,10}(2z)-\frac{999}{71680}\Delta_{2,10}(3z)\\
&\quad+\frac{999}{4480}\Delta_{2,10}(6z)+\frac{13}{337920}\Delta_{3,10,1}(z)-\frac{39}{56320}\Delta_{3,10,1}(2z)\\
&\quad+\frac{13}{660}\Delta_{3,10,1}(4z)-\frac{1}{129024}\Delta_{3,10,2}(z)-\frac{1}{3584}\Delta_{3,10,2}(2z)\\
&\quad-\frac{1}{252}\Delta_{3,10,2}(4z)-\frac{17}{423936}\Delta_{4,10}(z)+\frac{783}{47104}\Delta_{4,10}(3z)\\
&\quad+\frac{1}{202752}\Delta_{6,10}(z)+\frac{1}{12672}\Delta_{6,10}(2z)+\frac{17}{423936}\Delta_{12,10}(z),
\end{aligned}$$

$$\begin{aligned} f_5 &= \frac{1}{60480}\Delta_{2,10}(z) - \frac{1}{3780}\Delta_{2,10}(2z) + \frac{319}{20160}\Delta_{2,10}(3z) \\ &\quad - \frac{319}{1260}\Delta_{2,10}(6z) - \frac{1}{26730}\Delta_{3,10,1}(z) + \frac{1}{1485}\Delta_{3,10,1}(2z) \\ &\quad - \frac{256}{13365}\Delta_{3,10,1}(4z) + \frac{1}{13608}\Delta_{3,10,2}(z) + \frac{1}{378}\Delta_{3,10,2}(2z) \\ &\quad + \frac{64}{1701}\Delta_{3,10,2}(4z) - \frac{1}{9936}\Delta_{4,10}(z) + \frac{103}{3312}\Delta_{4,10}(3z) \\ &\quad - \frac{1}{19008}\Delta_{6,10}(z) - \frac{1}{1188}\Delta_{6,10}(2z) + \frac{1}{9936}\Delta_{12,10}(z), \end{aligned}$$

$$\begin{aligned} f_6 &= \frac{1}{3840}\Delta_{2,10}(z) - \frac{1}{240}\Delta_{2,10}(2z) + \frac{729}{1280}\Delta_{2,10}(3z) \\ &\quad - \frac{729}{80}\Delta_{2,10}(6z) - \frac{1}{42240}\Delta_{3,10,1}(z) + \frac{3}{7040}\Delta_{3,10,1}(2z) \\ &\quad - \frac{2}{165}\Delta_{3,10,1}(4z) - \frac{1}{768}\Delta_{3,10,2}(z) - \frac{3}{64}\Delta_{3,10,2}(2z) \\ &\quad - \frac{2}{3}\Delta_{3,10,2}(4z) + \frac{3}{5888}\Delta_{4,10}(z) + \frac{729}{5888}\Delta_{4,10}(3z) \\ &\quad + \frac{3}{2816}\Delta_{6,10}(z) + \frac{3}{176}\Delta_{6,10}(2z) - \frac{3}{5888}\Delta_{12,10}(z), \end{aligned}$$

$$\begin{aligned} f_7 &= -\frac{1}{40320}\Delta_{2,10}(z) + \frac{1}{2520}\Delta_{2,10}(2z) - \frac{109}{13440}\Delta_{2,10}(3z) \\ &\quad + \frac{109}{840}\Delta_{2,10}(6z) + \frac{1}{25920}\Delta_{3,10,1}(z) - \frac{1}{1440}\Delta_{3,10,1}(2z) \\ &\quad + \frac{8}{405}\Delta_{3,10,1}(4z) - \frac{1}{72576}\Delta_{3,10,2}(z) - \frac{1}{2016}\Delta_{3,10,2}(2z) \\ &\quad - \frac{4}{567}\Delta_{3,10,2}(4z) - \frac{1}{26496}\Delta_{4,10}(z) + \frac{103}{8832}\Delta_{4,10}(3z) \\ &\quad + \frac{1}{26496}\Delta_{12,10}(z), \end{aligned}$$

$$\begin{aligned} f_8 \;=\; & -\frac{1}{535680}\Delta_{2,10}(z)+\frac{1}{33480}\Delta_{2,10}(2z)-\frac{289}{178560}\Delta_{2,10}(3z) \\ & +\frac{289}{1160}\Delta_{2,10}(6z)+\frac{37}{9408960}\Delta_{3,10,1}(z)-\frac{37}{522720}\Delta_{3,10,1}(2z) \\ & +\frac{296}{147015}\Delta_{3,10,1}(4z)-\frac{13}{1897344}\Delta_{3,10,2}(z)-\frac{13}{52704}\Delta_{3,10,2}(2z) \\ & -\frac{52}{14823}\Delta_{3,10,2}(4z)+\frac{1}{79488}\Delta_{4,10}(z)-\frac{103}{26496}\Delta_{4,10}(3z) \\ & +\frac{1}{209088}\Delta_{6,10}(z)+\frac{1}{13068}\Delta_{6,10}(2z)-\frac{1}{79488}\Delta_{12,10}(z) \\ & -\frac{1}{1123254}E_{10}(z)+\frac{19}{41602}E_{10}(2z)+\frac{1}{1123254}E_{10}(3z) \\ & -\frac{256}{561627}E_{10}(4z)-\frac{19}{41602}E_{10}(6z)+\frac{256}{561627}E_{10}(12z), \end{aligned}$$

$$\begin{aligned} f_9 \;=\; & -\frac{1}{27776}\Delta_{2,10}(2z)-\frac{495}{27776}\Delta_{2,10}(6z)+\frac{1}{6534}\Delta_{3,10,1}(2z) \\ & -\frac{79}{13068}\Delta_{3,10,1}(4z)+\frac{5}{92232}\Delta_{3,10,2}(2z)+\frac{239}{92232}\Delta_{3,10,2}(4z) \\ & -\frac{1}{5808}\Delta_{6,10}(2z)-\frac{81}{332816}E_{10}(2z)+\frac{81}{332816}E_{10}(4z) \\ & 4z)+\frac{81}{332816}E_{10}(6z)-\frac{81}{332816}E_{10}(12z), \end{aligned}$$

$$\begin{aligned} f_{10} \;=\; & \frac{13}{6720}\Delta_{2,10}(2z)+\frac{2187}{2240}\Delta_{2,10}(6z)-\frac{1}{440}\Delta_{3,10,1}(2z) \\ & +\frac{73}{660}\Delta_{3,10,1}(4z)-\frac{1}{336}\Delta_{3,10,2}(2z)-\frac{13}{84}\Delta_{3,10,2}(4z) \\ & +\frac{7}{2112}\Delta_{6,10}(2z), \end{aligned}$$

$$\begin{aligned} f_{11} \;=\; & \frac{1}{672}\Delta_{2,10}(2z)+\frac{207}{224}\Delta_{2,10}(6z)-\frac{1}{594}\Delta_{3,10,1}(2z) \\ & +\frac{28}{297}\Delta_{3,10,1}(4z)-\frac{1}{378}\Delta_{3,10,2}(2z)-\frac{26}{189}\Delta_{3,10,2}(4z) \\ & +\frac{1}{352}\Delta_{6,10}(2z), \end{aligned}$$

$$\begin{aligned} f_{12} = &\frac{1}{840}\Delta_{2,10}(2z)+\frac{249}{280}\Delta_{2,10}(6z)-\frac{23}{17820}\Delta_{3,10,1}(2z)\\ &+\frac{344}{4455}\Delta_{3,10,1}(4z)-\frac{11}{4536}\Delta_{3,10,2}(2z)-\frac{68}{567}\Delta_{3,10,2}(4z)\\ &+\frac{1}{396}\Delta_{6,10}(2z), \end{aligned}$$

$$\begin{aligned} f_{13} = &\frac{55}{168}\Delta_{2,10}(2z)+\frac{2187}{56}\Delta_{2,10}(6z)+\frac{37}{132}\Delta_{3,10,1}(2z)\\ &-\frac{56}{33}\Delta_{3,10,1}(4z)+\frac{43}{168}\Delta_{3,10,2}(2z)+\frac{164}{21}\Delta_{3,10,2}(4z)\\ &+\frac{3}{22}\Delta_{6,10}(2z), \end{aligned}$$

$$\begin{aligned} f_{14} = &\frac{1}{1260}\Delta_{2,10}(2z)+\frac{319}{420}\Delta_{2,10}(6z)-\frac{1}{1485}\Delta_{3,10,1}(2z)\\ &+\frac{256}{4455}\Delta_{3,10,1}(4z)-\frac{1}{378}\Delta_{3,10,2}(2z)-\frac{64}{567}\Delta_{3,10,2}(4z)\\ &+\frac{1}{396}\Delta_{6,10}(2z), \end{aligned}$$

$$\begin{aligned} f_{15} = &\frac{501}{69440}\Delta_{2,10}(2z)+\frac{111537}{69440}\Delta_{2,10}(6z)+\frac{151}{19360}\Delta_{3,10,1}(2z)\\ &-\frac{2831}{9680}\Delta_{3,10,1}(4z)-\frac{47}{6832}\Delta_{3,10,2}(2z)-\frac{583}{1708}\Delta_{3,10,2}(4z)\\ &-\frac{63}{7744}\Delta_{6,10}(2z)-\frac{3}{41602}E_{10}(2z)+\frac{3}{41602}E_{10}(4z)\\ &+\frac{177147}{41602}E_{10}(6z)-\frac{177147}{41602}E_{10}(12z). \end{aligned}$$

## Conflict of Interests

The authors declare that there is no conflict of interests.

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