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J. Math. Comput. Sci. 2 (2012), No. 5, 1269-1292

ISSN: 1927-5307

ON THE SYMPLECTIC GROUP PSp(6, q), WHERE q = 2^K

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Abstract. In this paper, we investigate the canonical forms of the conjugacy classes of PSp(6, q), where q = 2^k, and the maximal subgroups of PSp(6, q), where q = 2^k.

Keywords: Finite groups; linear groups, matrix groups, maximal subgroups.

2000AMS Subject Classification: 20B05; 20G40, 20H30, 20E28.

1. Introduction:

In a matrix form**,** *the symplectic group* Sp(2n, q) = {g $\in$ GL(2n, q): $g^t P g = P$, where g^t is the transpose matrix of the matrix g and $P = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ or $P = \text{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, ..., \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$}. Since, the determinant of any skew-symmetric matrix {$A^t = -A$} of odd size is zero, thus in the symplectic case, the dimension must be even. If $g = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, then g $\in$ Sp(2n, q) if and only if $X_1^t X_3 - X_3^t X_1 = 0 = X_2^t X_4 - X_4^t X_2$ and $X_1^t X_4 - X_3^t X_2 = I_n$. Thus, $\begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$, $\begin{pmatrix} A & 0 \\ 0 & \text{inv}(A^t) \end{pmatrix}$, $\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}$, $\begin{pmatrix} Q & I_n\text{-}Q \\ Q\text{-}I_n & Q \end{pmatrix}$ are in Sp(2n, q), where A is an invertible $n \times n$ matrix, B is $n \times n$ symmetric

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Received March 4, 2012

matrix, Q is a diagonal matrix of 0's and 1's, so that $Q^2 = Q$ and $(Q - I_3)^2 = I_3 - Q$ {see [3] and [7]}.

The projective symplectic group PSp(2n, q) is the quotient group PSp(2n, q) $\cong$ Sp(2n, q) / (Sp(2n, q) $\cap$ Z), where Z is the group of non-zero scalar matrices. The group PSp(2m, q) (= Sp(2m, q)) is simple, except for PSp(2, 2), PSp(2, 3) and PSp(4, 2).

Through this article, G will denote PSp(6, q), $q = 2^k$, unless otherwise stated. G is a simple group of order $q^9(q^6-1)(q^4-1)(q^2-1)$ and there are two sets which can generate the group Sp(6, q), q even by the two elements:

$$\left\{ \begin{pmatrix} 1 & . & . & . & . & . \\ \alpha & 1 & . & . & . & . \\ . & . & 1 & . & . & \alpha^{-1} \\ . & . & . & 1 & \alpha & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \end{pmatrix} \text{ and } \begin{pmatrix} . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \\ 1 & . & . & . & . & . \end{pmatrix} \right\},$$

where α is a generator element of the multiplicative group of GF(q) {see [17]}.

Or by the two elements:

$$\left\{ \begin{pmatrix} \alpha & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & \alpha & . & . & . \\ . & . & . & \alpha^{-1} & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & \alpha \end{pmatrix} \text{ and } \begin{pmatrix} . & 1 & 1 & 1 & . & . \\ 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & 1 & . \\ . & . & . & . & . & 1 \\ . & . & 1 & . & . & . \end{pmatrix} \right\},$$

where α is a generator element of the multiplicative group of GF(q) {see [18]}.

Inside G, there exist

$$\left\{ \begin{pmatrix} 1 & . & . & x_1 & x_2 & x_3 \\ . & 1 & . & x_4 & x_5 & x_2 \\ . & . & 1 & x_6 & x_4 & x_1 \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \end{pmatrix} \right\}$$

is an elementary group of order q^6 {see [21]}. G acts primitively on the points of the projective space PG(5, q) which is a rank 3 permutation group on PG(5, q) and Psp(2n, q) does not 2-transtive on the points of PG(2n-1, q) for all $n \geq 2$ {see [2]}.

2. The canonical forms for the conjugacy classes of PSp(6, q), q even:

Each element of Sp(6, q) is an element of GL(6, q) and so there correspond to its characteristic polynomial $f_1^{n1} f_2^{n2} \ldots$, where f_1 , f_2, … are distinct irreducible polynomial over F_q, thus by using the method of Bhama Srinivasan {see [16]} that are used for calculating the conjugacy classes for Sp(4, q), q is odd, so, set $q = 2^k$ and let α, β and γ be the primitive roots of GF(q), GF(q^2) and GF(q^3) respectively such that $\alpha = \beta^{q+1} = \gamma^{q2+q+1}$. Then, according to the possible factorizations of the characteristic polynomial in GF(2^k), G has the following canonical forms of the conjugacy classes:

Class representative	Number of conjugacy classes
$\begin{pmatrix} \alpha_a &&&&& \\ & \alpha_a^{-1} &&&& \\ && \alpha_a &&& \\ &&& \alpha_a^{-1} && \\ &&&& \alpha_a & \\ &&&&& \alpha_a^{-1} \end{pmatrix}$ *where* $\alpha_a \neq 0$	$\frac{1}{2}(q-1)$
$\begin{pmatrix} \alpha_a && 1 &&& \\ & \alpha_a^{-1} && 1 && \\ && \alpha_a &&& \\ &&& \alpha_a^{-1} && \\ &&&& \alpha_a & \\ &&&&& \alpha_a^{-1} \end{pmatrix}$ *where* $\alpha_a \neq 0$	$\frac{1}{2}(q-1)$
$\begin{pmatrix} \alpha_a && 1 &&& \\ & \alpha_a^{-1} && 1 && \\ && \alpha_a && 1 & \\ &&& \alpha_a^{-1} && 1 \\ &&&& \alpha_a & \\ &&&&& \alpha_a^{-1} \end{pmatrix}$ *where* $\alpha_a \neq 0$	$\frac{1}{2}(q-1)$

$\begin{pmatrix} \alpha_a &&&&& \\ & \alpha_a^{-1} &&&& \\ && \alpha_a &&& \\ &&& \alpha_a^{-1} && \\ &&&& \alpha_b & \\ &&&&& \alpha_b^{-1} \end{pmatrix}$ *where* $\alpha_a \neq 0,\ \alpha_b \neq 0,\ \alpha_a \neq \alpha_b,\ \alpha_a^{-1} \neq \alpha_b^{-1}$	$\frac{1}{4}(q-1)(q-3)$
$\begin{pmatrix} \alpha_a && 1 &&& \\ & \alpha_a^{-1} && 1 && \\ && \alpha_a &&& \\ &&& \alpha_a^{-1} && \\ &&&& \alpha_b & \\ &&&&& \alpha_b^{-1} \end{pmatrix}$ *where* $\alpha_a \neq 0,\ \alpha_b \neq 0,\ \alpha_a \neq \alpha_b,\ \alpha_a^{-1} \neq \alpha_b^{-1}$	$\frac{1}{4}(q-1)(q-3)$
$\begin{pmatrix} \alpha_a &&&&& \\ & \alpha_a^{-1} &&&& \\ && \alpha_b &&& \\ &&& \alpha_b^{-1} && \\ &&&& \alpha_c & \\ &&&&& \alpha_c^{-1} \end{pmatrix}$ *where* $\alpha_a \neq 0,\ \alpha_b \neq 0,\ \alpha_a \neq \alpha_b,\ \alpha_a^{-1} \neq \alpha_b^{-1}$ $\alpha_c \neq 0,\ \alpha_a \neq \alpha_c,\ \alpha_a^{-1} \neq \alpha_c^{-1}, \alpha_c \neq \alpha_b,\ \alpha_c^{-1} \neq \alpha_b^{-1}$	$\frac{1}{8}(q-1)(q-3)(q-5)$
$\begin{pmatrix} \alpha_a &&&&& \\ & \alpha_a^{-1} &&&& \\ && \beta_a &&& \\ &&& \beta_a^{-1} && \\ &&&& \beta_a^{q} & \\ &&&&& \beta_a^{-q} \end{pmatrix}$ *where* $\beta_a \in F_{q^2} \setminus \mathrm{F_q}$	$\frac{1}{8}(q-1)(q^2-q)$

$\begin{pmatrix} \gamma_a & & & & & \\ & \gamma_a^{-1} & & & & \\ & & \gamma_a^{q} & & & \\ & & & \gamma_a^{-q} & & \\ & & & & \gamma_a^{q^2} & \\ & & & & & \gamma_a^{-q^2} \end{pmatrix}$ where $\gamma_a \in F_{q^3} \backslash \mathrm{F_q}$	$\frac{1}{6}(q^3 - q)$

3. The maximal subgroups of the symplectic group PSp(6, q), q = 2^k:

The main theorem of this section is the following theorem:

Theorem 3.1: Let G = PSp(6, q), q= 2^k. If H is a maximal subgroup of G, then H isomorphic to one of the following subgroups:

1. A group $G_{(p)}$, stabilizing a point. This is isomorphic to a group of form q^5**:**(PGL(1, q)×PSp(4, q));
2. A group $G_{(l)}$, stabilizing a line. This are isomorphic to a group of form q^7**:**(PGL(2, q)×PSp(2, q));
3. A group $G_{(2\text{-}\pi)}$, stabilizing a plane. This are isomorphic to a group of form q^6**:**PGL(3, q);
4. PSp(2, q) × PSp(4, q);
5. H_1 = PSp(2, q)**:**S_3;
6. H_2 = PSp(2, q^3)**.**3;
7. H_3 = PSp(6, q′), where q′ = $2^{k'}$ and k′ is a prime number divides k;
8. PSGO$^+$(6, q) ≅ PSL(4, q), where q = 2^k;
9. PSGO$^-$(6, q) ≅ PGSU(4, q), where q = 2^k;
10. PGSU(3, 3).

We will prove this theorem by Aschbacher's theorem (Result 3.1.9) {see [1]}:

3.1 Aschbacher's theorem:

A classification of the maximal subgroups of GL(n, q) by Aschbacher's theorem {see [1]}, is a very strong tool in the finite groups for finding the maximal subgroups of finite linear groups. There are many good works in finite groups which simplify this theorem, see for example {[12] and [19]}. But before starting a brief description of this theorem, we will give the following definitions:

Definition 3.1.1: *A split extension* (*a semidirect product*) A**:**B is a group G with a normal subgroup A and a subgroup B such that G = AB and $A \cap B = 1$. *A non-split extension* A**.**B is a group G with a normal subgroup A and $G/A \cong B$, but with no subgroup B satisfying G = AB and $A \cap B = 1$. A group $G = A \circ B$ is *a central product* of its subgroups A and B if G = AB and [A, B], the commutator of A and B = {1}, in this case A and B are normal subgroups of G and $A \cap B \leq Z(G)$. If $A \cap B = \{1\}$, then $A \circ B = AB$.

Definition 3.1.2: Let V be a vector space of dimensional n over a finite field q, a subgroup H of GL(n, q) is called *reducible* if it stabilizes a proper nontrivial subspace of V. If H is not reducible, then it is called *irreducible*. If H is irreducible for all field extension F of F_q, then H is *absolutely irreducible*. An irreducible subgroup H of GL(n, q) is called *imprimitive* if there are subspaces $V_1, V_2, \ldots, V_k$, $k \geq 2$, of V such that $V = V_1 \oplus \ldots \oplus V_k$ and H permutes the elements of the set $\{V_1, V_2, \ldots, V_k\}$ among themselves. When H is not imprimitive then it is called *primitive*.

Definition 3.1.3: A group $H \leq GL(n, q)$ is a *superfield group* of degree s if for some s divides n with $s > 1$, the group H may be embedded in $GL(n/s, q^s)$.

Definition 3.1.4: If the group $H \leq GL(n, q)$ preserves a decomposition $V = V_1 \otimes V_2$ with $\dim(V_1) \neq \dim(V_2)$, then H is a *tensor product group.*

Definition 3.1.5: Suppose that $n = r^m$ and $m > 1$. If the group $H \leq GL(n, q)$ preserves a decomposition $V = V_1 \otimes \ldots \otimes V_m$ with $\dim(V_i) = r$ for $1 \leq i \leq m$, then H is a *tensor induced* group.

Definition 3.1.6: A group $H \leq GL(n, q)$ is a *subfield group* if there exists a subfield $F_{q_o} \subset F_q$ such that H can be embedded in $GL(n, q_o).Z$, where Z is the centre group of H.

Definition 3.1.7: A p-group H is called a *special group* if $Z(H) = H'$ and is called *an extraspecial group* if also $|Z(H)| = p$.

Definition 3.1.8: Let Z denote the centre group of H. Then H is *almost simple modulo scalars* if there is a non-abelian simple group T such that $T \leq H/Z \leq Aut(T)$, the automorphism group of T.

A classification of the maximal subgroups of GL(n, q) by Aschbacher's theorem {see [1]}, can be summarized as follows:

Result 3.1.9. (Aschbacher's theorem):

Let H be a subgroup of GL(n, q), $q = p^e$ with the centre Z and let V be the underlying n-dimensional vector space over a field q. If H is a maximal subgroup of GL(n, q), then one of the following holds:

C_1:- H is a reducible group.

C_2:- H is an imprimitive group.

C_3:- H is a superfield group.

C_4:- H is a tensor product group.

C_5:- H is a subfield group.

C_6:- H normalizes an irreducible extraspecial or symplectic-type group.

C_7:- H is a tensor induced group.

C_8:- H normalizes a classical group in its natural representation.

C_9:- H is absolutely irreducible and $H/(H\cap Z)$ is almost simple.

3.2. Classes $C_1 - C_8$ of Result 3.1.9:

In this section, we will find the maximal subgroups in the classes $C_1 - C_8$ of Result 3.1.9:

Lemma 3.2.1: There are four reducible maximal subgroups of C_1 in G which are:

1. A group $G_{(p)}$, stabilizing a point. This is isomorphic to a group of form q^5**:**(PGL(1, q)$\times$PSp(4, q)).

2. A group $G_{(l)}$, stabilizing a line. This are isomorphic to a group of form q^7**:**(PGL(2, q)$\times$PSp(2, q)).

3. A group $G_{(2-\pi)}$, stabilizing a plane. This are isomorphic to a group of form q^6**:**PGL(3, q).

4. PSp(2, q)$\times$PSp(4, q).

Proof:

Let H be a reducible subgroup of the symplectic group Sp(2n, q) and W be an invariant subspace of H. Let r = dim (W), $1 \le r \le n/2$ and let $G_r = G_{(W)}$ denote the subgroup of Sp(2n, q) containing all elements fixing W as a whole and $H \subseteq G_{(W)}$. with a suitable choice of a basis, $G_{(W)}$ consists of all matrices of the form $\begin{pmatrix} A & C & D \\ & B & C \\ & & A \end{pmatrix}$ where n = r + m, C is elementary abelian groups of order q^{2rm} , A is a p- group of upper triangular matrix of order $q^{\frac{r(r+1)}{2}}$, $D\in$GL(r, q), $B\in$Sp(2m, q) such that $A^t P A = P$ with $P = \begin{pmatrix} 0_r & I_r \\ -I_r & 0_r \end{pmatrix}$. Thus the maximal parabolic subgroups are the stabilizers of totally isotopic subspaces <e_1, e_2, …, e_r> is isomorphic to a group of the form $q^{\frac{r(r+1)}{2}+2rm}$**:**(GL(r, q)$\times$Sp(2m, q)).

Thus we have the following reducible maximal subgroups of PSp(6, q):

1. If r = 1 and m = 2, then we get a group $G_{(p)}$ stabilizing a point is isomorphic to a group of the form q^5**:**(PGL(1, q)$\times$PSp(4, q)).

2. If r = 2 and m = 1, then we get a group $G_{(l)}$ stabilizing a line is isomorphic to a group of the form q^7**:**(PGL(2, q)$\times$PSp(2, q)).

3. If r = 3 and m = 0, then we get a group $G_{(2-\pi)}$ stabilizing a plane is isomorphic to a group of the form q^6**:**PGL(3, q).

Also, H is a maximal reducible subgroup of the unitary group Sp(2n, q) which stabilizers of non-singular subspaces of dimension d have the shape H = Sp(2d, q) $\times$Sp(2b, q) where n = d + b and d < b. Thus, we have the following reducible maximal subgroups of PSp(6, q):

4. If d = 1 and b = 2, then we get a group PSp(2, q) $\times$PSp(4, q).

Which prove the points (1), (2), (3), and (4) of the main theorem 1.1.

Note: To find the Sylow's q-subgroup of the group Sp(6, q), substitute the Sylow's q-subgroup for GL(3, q) in place of GL(3, q) in the group $G_{(2-\pi)}$ which stabilizing a plane of Sp(6, q), then we have

$$\left\{\begin{pmatrix} y_1 & y_2 & y_3 & 1 & x_1 & x_2 \\ 0 & y_4 & y_5 & 0 & 1 & x_3 \\ 0 & 0 & y_6 & 0 & 0 & 1 \\ 0 & 0 & 0 & y_1 & y_2 & y_3 \\ 0 & 0 & 0 & 0 & y_4 & y_5 \\ 0 & 0 & 0 & 0 & 0 & y_6 \end{pmatrix}\right\}$$ is the Sylow's q-subgroup of order q^9 for the symplectic group Sp(6, q).

Lemma 3.2.2: There is one imprimitive group of C_2 in G which is H_1 = PSp(2, q)**:**S_3

Proof:

If H is imprimitive of the symplectic group Sp(2n, q), then H preserves a decomposition of V as a direct sum $V = V_1 \oplus \dots \oplus V_t$, $t \geq 2$, into subspaces of V, each of dimension m = n/t, which are permuted transitively by H, thus H are isomorphic to $Sp(2m, q){:}S_t$ with $0 < m < n = mt$, $t \geq 2$. Consequently, there are one imprimitive group of C_2 in PSp(6, q) which is $H_1 = PSp(2, q){:}S_3$, a group preserving three mutually skew lines of projective plane PG(5, q) and H_1 interchanges them. This proves the point (5) of the main theorem 3.1.

Lemma 3.2.3: There is one semilinear group of C_3 in G which is $H_2 = PSp(2, q^3).3$

Proof:

Let H is (superfield group) a semilinear groups of PSp(2n, q) over extension field F_r of GF(q) of prime degree r > 1 where r prime number divide n. Thus V is an F_r-vector space in a natural way, so there is an F-vector space isomorphism between 2n-dimensional vector space over F and the m-dimensional vector space over F_r, where m=n/r, thus H embeds in $PSp(2m, q^r).r$. Consequently, there is one C_3 group in PSp(6, q) which is $H_2 = PSp(2, q^3).3$.

This proves the point (6) of the main theorem 3.1.

Note: From [14], $PSp(2, q^n)$ with n odd and q even is a maximal subgroup of PSp(2n, q) and $PSp(4, q^n)$ with q even, is a maximal subgroup of PSp(4n, q) which also prove that H_2 is maximal subgroup of G.

Lemma 3.2.4: There is no a tensor product group of C_4 in G.

Proof:

If H is a tensor product group of Sp(2n, q), then H preserves a decomposition of V as a tensor product $V_1 \otimes V_2$, where $\dim(V_1) \neq \dim(V_2)$ of spaces of dimensions 2k and 2m over GF(q) and 2n = 4km, k ≠ m. So, H stabilize the tensor product decomposition $F^{2k} \otimes F^{2m}$. Thus, H is a subgroup of the central product of $Sp(2k, q) \circ Sp(2m, q)$. Consequently, there are no C_4 groups in PSp(6, q) since n = 3 is an odd number.

Lemma 3.2.5: There are subfield groups of C_5 in G which are $H_3 = PSp(6, q')$, where $q' = 2^{k'}$ and k' is a prime number divides k..

Proof:

If H is a subfield group of the symplectic group Sp(2n, q) and $q = p^k$, then H is the symplectic group over subfield of GF(q) of prime index. Thus H can be embedded in $Sp(2n, p^f)$, where f is prime number divides k. Consequently, since $q = 2^k$, then there are subfield groups in PSp(6, q) which are $H_3 = PSp(6, q')$, where $q' = 2^{k'}$ and k' is a prime number divides k. This proves the point (7) of the main theorem 3.1.

Lemma 3.2.6: There are no C_6 groups in G.

Proof:

For the dimension $2n = r^m$, if r = 2 and 4 divides q-1, then $H = 2^{2m+1}.O^-(2m, 2)$ normalizes a 2-group of symplectic type of order 2^{2m+2} {see [12]}, consequently, there are no C_6 groups in PSp(6, q) since 6 is not prime power.

Lemma 3.2.7: There is no tensor induced group of C_7 in G.

Proof:

If H is a tensor induced of the symplectic group Sp(2n, q), then H preserves a decomposition of V as $V_1 \otimes V_2 \otimes \dots \otimes V_r$, where V_i are isomorphic, each V_i has dimension 2m, dim V =2n =$(2m)^r$, and the set of V_i is permuted by H, so H stabilize the tensor product decomposition $F^{2m} \otimes F^{2m} \otimes \dots \otimes F^{2m}$, where $F = F_q$. Thus, $H/Z \leq PSp(2m, q){:}S_r$. Consequently, there are no C_7 groups in PSp(6, q) since 6 is not a proper power.

Lemma 3.2.8: There are two maximal C_8 groups in G which are $PSGO^+(6, q)$ and $PSGO^-(6, q)$.

Proof:

The groups in this class are stabilizers of forms, this means H is the normalizers of one classical groups PSL(2n, q), $PO^{\in}(2n, q)$ or PSU(2n, q) as a subgroup of PSp(2n, q). But from [5] and [10], if q is even, then the normalizers of $PO^{+}(2n, q)$ and $PO^{-}(2n, q)$ are maximal subgroups of PSp(2n, q) except when n = 2 and $\in = -$. Consequently, In C_8, there are two irreducible maximal subgroups in PSp(6, q) that are $PSGO^{+}(6, q)$ and $PSGO^{-}(6, q)$. Which prove the points (8) and (9) of theorem 3.1.

In the following, we will find the maximal subgroups of class C_9 of Result 3.1.9:

4. The maximal subgroups of C_9:

In Corollary 4.1, we will find the primitive non abelian simple subgroups of G. In Theorem 4.2, we will find the maximal primitive subgroups H of G which have the property that the minimal normal subgroup M of H is not abelian group and simple. We will prove this Theorem 4.2 by finding the normalizers of the groups of Corollary 4.1 and determine which of them are maximal.

Corollary 4.1: If M is a non abelian simple group of a primitive subgroup H of G, then M is isomorphic to one of the following groups:

- PSp(6, 2);
- $PSO^{-}(6, q)$, where $q = 2^k$;
- $PSO^{+}(6, q)$, where $q = 2^k$;
- $P\Omega^{-}(6, q) \cong PSU(4, q)$, where $q = 2^k$;
- $P\Omega^{-}(6, 2) \cong PSU(4, 2)$;
- $P\Omega^{+}(6, 2) \cong PSL(4, 2)$,
- PSU(3, 3);

Proof:

Let H be a primitive subgroup of G with a minimal normal subgroup M of H which is not abelian and simple. So, we will discuss the possibilities of M of H according to:

(I) M contains transvections, {section 4.1}.

(II) M is a finite primitive subgroup of rank three, {section 4.2}.

4.1 Primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian is generated by transvections:

Definition 4.1.1: An element $T \in GL(n, q)$ is called *a transvection* if T satisfies $\text{rank}(T - I_n) = 1$ and $(T - I_n)^2 = 0$. The collineation of projective space induced by a transvection is called *elation*. *The axis* of the transvection is the hyperplane $\text{Ker}(T - I_n)$; this subspace is fixed elementwise by T, Dually, *the centre* of T is the image of $(T - I_n)$.

To find the primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian and is generated by transvections, we will use the following result of Kantor {see [9]}:

Result 4.1.2:

Let H be a proper irreducible subgroup of $Sp(2n, q^i)$ generated by transvections. Then H is one of:

1. $Sp(2n, q)$;

2. $O^{\pm}(n, q^i)$ for q even;

3. S_{2n} or S_{2n+1};

4. $SL(2, 5) < Sp(2, 9^i)$;

5. Dihedral subgroups of $Sp(2, 2^i)$.

In the following Corollary, we will find the primitive subgroups of PSp(6, q) which generated by transvections:

Corollary 4.1.3: If M is a non abelian simple group and contains some transvections, then M is isomorphic to one of the groups:

(i) PSp(6, q′), where $q' = 2^{k'}$ and k′ is a prime number divides k;

(ii) $PSO^-(6, q)$;

(iii) $PSO^+(6, q)$.

Proof:

We will prove this Corollary by discussing the different possibilities of Result (4.1.2), thus M is isomorphic to one of the following groups:

1. From Lemma 3.2.5, PSp(6, q′)$\subset$ G, where $q' = 2^{k'}$ and k′ is a prime number divides k;

2. From Lemma 3.2.8, $PSO^-(6, q)$ and $PSO^+(6, q)$ are maximal subgroups of PSp(6, q).

3. $S_6 \not\subset G$, since, the irreducible 2-modular characters for S_6 by GAP are:
[[1, 1], [4, 2], [16, 1]]

(gap> CharacterDegrees(CharacterTable("S6")mod 2);)

And none of these characters of degree 6.

4. $S_7 \subset G$, since the irreducible 2-modular characters for S_7 by GAP are:
[[1, 1], [6, 1], [8, 1], [14, 1], [20, 1]]

(gap> CharacterDegrees(CharacterTable("S7")mod 2);)

Thus there is one irreducible character of degree 6 but the symmetric group S_7 is not a simple group.

5. SL(2, 5) $\not\subset$ G, since the irreducible 2-modular characters for SL(2, 5) by GAP are:
[[1, 1], [2, 2], [4, 1]]

(gap> CharacterDegrees(CharacterTable("L2(5)") mod 2);)

And none of these characters of degree 6.

6. If M is a Dihedral subgroups of $Sp(2, 2^i)$, then $M \not\subset G$, since M is not a simple group.

4.2 Primitive subgroups H of G which have the property that a minimal normal subgroup M of H which is not abelian is a finite primitive subgroup of rank three:

A group G has rank 3 in its permutation representation on the cosets of a subgroup K if there are exactly 3 (K, K)-double cosets. Indeed, the rank of a transitive permutation group is the number of orbits of the stabilizer of a point, thus if we consider PSp(2m, q), $m \geq 2$ and q is of a prime power, as group of permutations of the absolute points of the corresponding projective space, then PSp(2m, q) is a transitive group of rank 3. Indeed, the pointwise stabilizer of PSp(2m, q) has 3 orbits of lengths 1, $q(q^{2m-2} - l)/(q - 1)$ and q^{2m-1} {see [8] and [22]}.

In this section, we will consider the minimal normal subgroup M of H is not abelian and a finite primitive subgroup of rank three, so will use the classification of Kantor and Liebler {Result 4.2.2} for the primitive groups of rank three {see [8]}. The following Corollary is the main result of this section:

Corollary 4.2.1: If M is a non abelian simple group which is a finite primitive subgroup of rank three group of H, then M is isomorphic to one of the following groups:

1. $P\Omega^-(6, q) \cong PSU(4, q)$, where $q = 2^k$;
2. $P\Omega^+(6, 2) \cong PSL(4, 2)$;
3. $P\Omega^-(6, 2) \cong PSU(4, 2)$,
4. PSU(3, 3);
5. PSp(6, 2);

Proof:

Let M is not an abelian finite primitive subgroup of rank three of H, and will use the classification of Kantor and Liebler {Result 4.2.2} for the primitive groups of rank three {see [8]}. So, we will prove Corollary 4.2.1 by series of Lemmas 4.2.3 through Lemmas 4.2.18 and Result 4.2.2.

Result 4.2.2:

If Y acts as a primitive rank 3 permutation group on the set X of cosets of a subgroup K of Sp(2n-2, q), $\Omega^{\pm}(2n, q)$, $\Omega(2n-1, q)$ or SU(n, q). Then for $n \geq 3$, Y has a simple normal subgroup M^*, and $M^* \subseteq Y \subseteq Aut(M^*)$, where M^* as follows:

(i) M = Sp(4, q), SU(4, q), SU(5, q), $\Omega^-(6, q)$, $\Omega^+(8, q)$ or $\Omega^+(10, q)$.
(ii) M = SU(n, 2), $\Omega^{\pm}(2n, 2)$, $\Omega^{\pm}(2n, 3)$ or $\Omega(2n-1, 3)$.
(iii) M = $\Omega(2n-1, 4)$ or $\Omega(2n-1, 8)$;
(iv) M = SU(3, 3);
(v) SU(3, 5);
(vi) SU(4, 3);
(vii) Sp(6, 2);
(viii) $\Omega(7, 3)$;
(ix) SU(6, 2);

In the following, we will discuss the different possibilities of Result 4.2.2;

Lemma 4.2.3: If M = PSp(4, q), then $M \not\subset G$.

Proof:

PSp(2n, q), $n \geq 2$, has no projective representation in G of degree less than $\frac{1}{2}(q^n-1)$, if q is odd, and $\frac{1}{2}(q^{n-1})(q^{n-1}-1)(q-1)$ if q is even, {see [13] and [15]}, thus PSp(4, q), has no projective representation in G for all $n \geq 2$, thus $M \not\subset G$.

Lemma 4.2.4: PSU(4, q) $\cong P\Omega^-(6, q) \subset G$.

Proof:

From Lemma 3.2.8, $P\Omega^{-}(6, q)$ is irreducible subgroup of G, consequentially $PSU(4, q) \cong P\Omega^{-}(6, q) \subset G$. Which prove the point (1) of Corollary 4.2.1.

Lemma 4.2.5: $PSU(5, q) \not\subset G$.

Proof:

PSU(n, q), $n \geq 3$, has no projective representation in G of degree less than $q(q^{n-1}-1)/(q+1)$, if n is odd, and $(q^{n}-1)/(q+1)$, if n is even, {see [13] and [15]}, thus PSU(5, q), has no projective representation in G for all $q \geq 2$, thus $PSU(5, q) \not\subset G$.

Lemma 4.2.6: $P\Omega^{+}(8, q) \not\subset G$, $P\Omega^{+}(10, q) \not\subset G$.

Proof:

$P\Omega^{+}(2n, q)$, $n \geq 4$, $q \neq 2, 3, 5$, has no projective representation in G of degree less than $(q^{n-1}-1)(q^{n-2}+1)$, and $P\Omega^{+}(2n, q)$, $n \geq 4$, $q = 2$, 3 or 5, has no projective representation in G of degree less than $q^{n-2}(q^{n-1}-1)$, {see [13] and [15]}, but these bounds are greater than 6 for all $n \geq 4$, thus $P\Omega^{+}(8, q) \not\subset G$ and $P\Omega^{+}(10, q) \not\subset G$.

Lemma 4.2.7: if M = PSU(n, 2), then $PSU(4, 2) \subset G$

Proof:

In our case, 2n-2 = 6, thus n = 4 and, the irreducible 2-modular characters for PSU(4, 2) by GAP are:

[[1, 1], [4, 2], [6, 1], [14, 1], [20, 2], [64, 1]]

{ gap> CharacterDegrees(CharacterTable("U4(2)")mod 2); }

Thus, there is one irreducible character of degree 6, so $PSU(4, 2) \subset G$. Which prove the point (3) of Corollary 4.2.1.

Lemma 4.2.8: If $M = P\Omega^{\pm}(2n, 2)$, then $M \not\subset G$.

Proof:

In our case n = 4, thus we need to consider $P\Omega^{\pm}(8, 2)$:

- $P\Omega^{+}(2n, q)$, $n \geq 4$, $q = 2$ has no projective representation in G of degree less than $q^{n-2}(q^{n-1}-1)$, {see [13] and [15]}, but this bound is greater than 6 for all $n \geq 4$ and $q = 2$, thus $P\Omega^{+}(8, 2) \not\subset G$.

- $P\Omega^{-}(2n, q)$, $n \geq 4$, has no projective representation in G of degree less than $(q^{n-1} + 1)(q^{n-2} - 1)$, {see [13] and [15]}, but this bound is greater than 6 for all $n \geq 4$ and $q = 2$, thus $P\Omega^{-}(8, 2) \not\subset G$.

Lemma 4.2.9: If $M = P\Omega^{\pm}(2n, 3)$, then $M \not\subset G$.

Proof:

In our case n = 4, thus we need to consider $P\Omega^{\pm}(8, 3)$:

- $P\Omega^{+}(2n, q)$, $n \geq 4$, $q = 2$ has no projective representation in G of degree less than $q^{n-2}(q^{n-1}-1)$, {see [13] and [15]}, but this bound is greater than 6 for all $n \geq 4$ and $q = 3$, thus $P\Omega^{+}(8, 3) \not\subset G$.

- $P\Omega^{-}(2n, q)$, $n \geq 4$, has no projective representation in G of degree less than $(q^{n-1} + 1)(q^{n-2} - 1)$, {see [13] and [15]}, but this bound is greater than 6 for all $n \geq 4$ and $q = 3$, thus $P\Omega^{-}(8, 3) \not\subset G$.

Lemma 4.2.10: If $M = P\Omega(2n-1, 3)$, then $M \not\subset G$.

Proof:

In our case n = 4, thus, we have PΩ(7, 3) $\not\subset$ G, since PΩ(2n+1, q), n ≥ 3, q = 3, has no projective representation in G of degree less than $q^{n-1}(q^{n-1}- 1)$, {see [13] and [15]}, which is greater than 6 for all n ≥ 3 and q = 3.

Lemma 4.2.11: If M = PΩ(2n-1, 4), then M $\not\subset$ G.

Proof:

In our case n = 4, thus we have PΩ(7, 4) $\not\subset$ G. since, PΩ(2n+1, q) ≅ PSp(2n, q) for q even, then PΩ(7, 4) ≅ PSp(7, 4), and PSp(2n, q), n ≥ 2, has no projective representation in G of degree less than $½(q^{n-1})(q^{n-1}-1)(q-1)$ if q is even {see [13] and [15]}, which is greater than 6 for all n ≥ 2 and q = 4.

Lemma 4.2.12: If M = PΩ(2n-1, 8), then M $\not\subset$ G.

Proof:

In our case n=4, thus we have PΩ(7, 8) $\not\subset$ G. since, PΩ(2n+1, q) ≅ PSp(2n, q) for q even, then PΩ(7, 8) ≅ PSp(7, 8), and PSp(2n, q), n ≥ 2, has no projective representation in G of degree less than $½(q^{n-1})(q^{n-1}-1)(q-1)$ if q is even {see [13] and [15]}, which is greater than 6 for all n ≥ 2 and q = 8.

Lemma 4.2.13: PSU(3, 3) ⊂ G.

Proof:

The irreducible 2-modular characters for PSU(3, 3) by GAP are:
[[1, 1], [6, 1], [14, 1], [32, 2]],

{gap> CharacterDegrees(CharacterTable("U3(3)")mod 2);}

Then there is one irreducible character of degree 6, thus PSU(3, 3) ⊂ G. Which prove the point (4) of Corollary 4.2.1.

Lemma 4.2.14: PSU(3, 5) $\not\subset$ G.

Proof:

Since the irreducible 2-modular characters for PSU(3, 5) by GAP are:
[[1, 1], [20, 1], [28, 3], [104, 1], [144, 2]]
(gap> CharacterDegrees(CharacterTable("U3(5)")mod 2);)
And none of these characters of degree 6.

Lemma 4.2.15: PSU(4, 3) $\not\subset$ G.

Proof:

Since the irreducible 2-modular characters for PSU(3, 5) by GAP are:
[[1, 1], [20, 1], [34, 2], [70, 4], [120, 1], [640, 2],[896, 1]]
(gap> CharacterDegrees(CharacterTable("U4(3)")mod 2);)
And none of these characters of degree 6.

Lemma 4.2.16: PSp(6, 2) $\subset$ G.

Proof:

From Corollary 4.1.3 PSp(6, 2) $\subset$ G which prove the point (5) of Corollary 4.2.1.

Lemma 4.2.17: PΩ(7, 3) $\not\subset$ G.

Proof:

See the proof of Lemma 4.2.110.

Lemma 4.2.18: PSU(6, 2) $\not\subset$ G.

Proof:

Since the irreducible 2-modular characters for PSU(6, 2) by GAP are:
[[1, 1], [20, 1], [34, 1], [70, 2], [154, 1], [400, 1], [896, 2], [1960, 1], [3114, 1], [32768, 1]]
(gap> CharacterDegrees(CharacterTable("U6(2)")mod 2);)
And none of these characters of degree 6.

Now, we will determine the maximal primitive group of C_9:

Theorem 4.2: If H is a maximal primitive subgroup of G which has the property that a minimal normal subgroup M of H is not abelian group, then H is isomorphic to one of the following subgroups of G:

(i) $P\Omega^-(6, q) \cong PSU(4, q)$, where $q = 2^k$;

(ii) $PSGO^+(6, q)$, where $q = 2^k$;

(iii) PGSU(3, 3);

Proof:

We will prove this theorem by finding the normalizers N of the groups of Corollary 4.1 and determine which of them are maximal:

From [4], the normalizer of Sp(2n, k) in SL(2n, k) is SGSp(2n, k) = GSp(2n, k) $\cap$ SL(2n, k). From [11], the normalizer of SU(n, k) in SL(n, k) is SGU(n, k) = GU(n, k) $\cap$ SL(n, k). From [10], the normalizer of SO(n, k) in SL(n, k) is SGO(n, k) = GO(n, k) $\cap$ SL(n, k). Thus,

- If Y = PSp(6, 2), then N = PSGSp(6, 2) but in PSp(6, q), PSGSp(6, 2) = PSp(6, 2), in this case Y is a subgroup of PSp(6, q′), where $q' = 2^{k'}$ and k′ is a prime number divides k, thus Y is not a maximal subgroup of G.

- If $Y = PSO^-(6, q)$, then $N = PSGO^-(6, q)$, which prove the point (9) of theorem 3.1.

- If $Y = PSO^+(6, q)$, then $N = PSGO^+(6, q)$, which prove the point (8) of theorem 3.1.

- If Y = $P\Omega^-(6, q) \cong PSU(4, q)$, where $q = 2^k$, then N = $PSGO^-(6, q) \cong PSGU(4, q)$, which prove the point (8) of theorem 3.1.

- If Y = $P\Omega^+(6, 2) \cong PSL(4, 2)$, then N = $PSGO^+(6, 2)$, but $PSGO^+(6, 2) \subset PSGO^+(6, q)$ where $q = 2^k$, thus Y is not a maximal subgroup of G.

- If Y = PSU(3, 3), then N = PSGU(3, 3). Which prove the point (10) of theorem 3.1.

This completes the proof of theorem 3.1.

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