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FIXED POINT FOR TWO MAPPINGS IN N-CONE METRIC SPACES

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Abstract. In this paper, we obtain some new coincidence and common fixed point theorems for two mappings in the N-cone metric space, introduced by Malviya and Fisher [4]. The results presented in this paper improve and generalize the corresponding results in the literature.

Keywords: Fixed point; Coincidence point; N-Cone metric space.

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1. Introduction

In 2007, Huang and Zhang [1] have replaced the real numbers by ordering Banach space and defining cone metric space. They have proved some fixed point theorems of contractive mappings on cone metric spaces. The study of fixed point theorems in such spaces is followed by some other mathematicians;see [6-14]. In 2010, Aage and Salunke [2]introduced a generalized D^*-metric space which generalized cone metric space. And Beg *et al.* [3] introduced G-cone metric space.

Very recently, Malviya and Fisher [4] introduced the notion of N-cone metric space and proved fixed point theorems for asymptotically regular maps. This new notion generalized the

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notion of G-cone metric space and generalized D*-metric space. In 2015, Jerolina, Geeta and Neeraj [5] proved unique fixed point theorems for contractive maps in N-cone metric spaces.

In this paper, we prove some new fixed point theorems for two mappings in N-cone metric spaces. And our results also extend and improve recent related results.

2. Preliminaries

Consistent with Huang and Zhang [1], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed,non-empty and $P \neq \{0\}$;

(b) $\forall a,b \in R, a,b \geq 0, \forall x,y \in P$ imply that ax+$by \in P$;

(C) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y-x \in P$. A cone P is said to be normal if there exists a constant $K > 0$ such that for all $x,y \in E$, $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$. The least position number satisfying the above inequality is called the normal constant of P. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stand for $y-x \in intP$, $intP$ denotes the interior of P.

Definition 1.1. [4] Let X be a nonempty set. An N-cone metric on X is a function $N : X^3 \to E$ satisfies the following conditions: for all $x,y,z,a \in X$,

(1) $N(x,y,z) \geq 0$;

(2) $N(x,y,z) = 0$ if and only if $x = y = z$;

(3) $N(x,y,z) \leq N(x,x,a)+N(y,y,a)+N(z,z,a)$.

Then N is called an N-cone metric and (X,N) is called an N-cone metric space.

Remark 1.2. [4] It is easy to see that every generalized-D*-metric space is an N-cone metric space but in general, the converse is not true, see the following example.

Example Let $E = R^3, P = \{(x,y,z) \in E, x,y,z \geq 0\}, X = R$ and $N : X \times X \times X \longrightarrow E$ is defined by

$$N(x,y,z) = (\alpha(| y+z-2x | + | y-z |), \beta(| y+z-2x | + | y-z |), \gamma(| y+z-2x | + | y-z |)),$$

where α, β, γ are positive constants. Then (X,N) is an N-cone metric space but not a generalized-D*-metric space,because N is not symmetric.

Lemma 1.3. [4] *If (X,N) be an N-cone metric space, then for all $x,y \in X$, we have*

$$N(x,x,y) = N(y,y,x)$$

Proof. By the third condition of N-cone metric, we get

$$N(x,x,y) \leq N(x,x,x) + N(x,x,x) + N(y,y,x) = N(y,y,x)$$

and similarly

$$N(y,y,x) \leq N(y,y,y) + N(y,y,y) + N(x,x,y) = N(x,x,y).$$

Hence we obtain $N(x,x,y) = N(y,y,x)$.

Definition 1.4. [4] Let (X,N) be an N-cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$, there is an N such that for all $n > N$, $N(x_n,x_n,x) \ll c$, then $\{x_n\}$ is said to be convergent. If $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\{x_n\} \to x$ as $n \to \infty$.

Lemma 1.5. [4] *Let (X,N) be an N-cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ also converges to y, then $x = y$.*

Definition 1.6. [5] Let (X,N) be an N-cone metric space and $\{x_n\}$ be a sequence in X.If for any $c \in E$ with $0 \ll c$, there is an N such that for all $n,m > N$,$N(x_n,x_n,x_m) \ll c$, then $\{x_n\}$ is called a cauchy sequence in X.

Definition 1.7. [5] Let (X,N) be an N-cone metric space. If every cauchy sequence in X is convergent in X, then X is called a complete N-cone metric space.

Lemma 1.8. [5] *Let (X,N) be an N-cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequence in X and that $x_n \to x, y_n \to y$ as $n \to \infty$, then $N(x_n,x_n,y_n) \to N(x,x,y)$ as $n \to \infty$.*

Definition 1.9. [2] Let f and g be self maps on a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 1.10. [14] Let f and g be self maps on a set X. Then f and g are said to be weakiy compatible if they commute at every coincidence point.

Proposition 1.11. [2] *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

3. Main results

Lemma 3.1. *If (X,N) be an N-cone metric space, then for all $x,y \in X$, we have*

$$N(x,y,y) \leq 2N(y,y,x) \quad and \quad N(x,y,y) \leq N(x,x,y)\ .$$

Proof. By the third condition of N-cone metric, we get

$$N(x,y,y) \leq N(x,x,x) + N(y,y,x) + N(y,y,x) = 2N(y,y,x)$$

and similarly

$$N(x,y,y) \leq N(x,x,y) + N(y,y,y) + N(y,y,y) = N(x,x,y).$$

Hence we obtain $N(x,y,y) \leq 2N(y,y,x)$ and $N(x,y,y) \leq N(x,x,y)$.

Theorem 3.2. *Let (X,N) be an N-cone metric space, P be a normal cone with normal constant K and $f,g : X \to X$ be two mappings which satisfy the following conditions:*
(i) $f(X) \subset g(X)$;
(ii) $f(X)$ or $g(X)$ is complete;
(iii) $N(fx, fy, fz) \leq aN(gx, gy, gz) + bN(gx, fx, fx) + cN(gy, fy, fy) + dN(gz, fz, fz)$
for all $x,y,z \in X$, where $a,b,c,d \geq 0$, $a+4b+4c+2d < 1$, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible,then f and g have a unique common fixed point.

Proof. Taking $x_0 \in X$, we see that there exist $x_1 \in X$ such that $fx_0 = gx_1$. In this way, we construct sequences $\{gx_n\}$ with $fx_{n-1} = gx_n$. From Condition (iii) and Lemma 3.1, we have

$$\begin{aligned}
N(gx_{n+1}, gx_{n+1}, gx_n) &= N(fx_n, fx_n, fx_{n-1}) \\
&\leq aN(gx_n, gx_n, gx_{n-1}) + bN(gx_n, fx_n, fx_n) \\
&\quad + cN(gx_n, fx_n, fx_n) + dN(gx_{n-1}, fx_{n-1}, fx_{n-1}) \\
&= aN(gx_n, gx_n, gx_{n-1}) + bN(gx_n, gx_{n+1}, gx_{n+1}) \\
&\quad + cN(gx_n, gx_{n+1}, gx_{n+1}) + dN(gx_{n-1}, gx_n, gx_n) \\
&\leq aN(gx_n, gx_n, gx_{n-1}) + 2bN(gx_{n+1}, gx_{n+1}, gx_n) \\
&\quad + 2cN(gx_{n+1}, gx_{n+1}, gx_n) + 2dN(gx_n, gx_n, gx_{n-1}) \\
&= (a+2d)N(gx_n, gx_n, gx_{n-1}) + 2(b+c)N(gx_{n+1}, gx_{n+1}, gx_n).
\end{aligned}$$

This implies $N(gx_{n+1}, gx_{n+1}, gx_n) \leq qN(gx_n, gx_n, gx_{n-1})$, where $q = \frac{a+2d}{1-2(b+c)}$ and $0 < q < 1$. By repeated application of above inequality, we have

$$N(gx_{n+1}, gx_{n+1}, gx_n) \leq q^n N(gx_1, gx_1, gx_0).$$

For all $n, m \in N, n < m$, we see that

$$\begin{aligned}
N(gx_n, gx_n, gx_m) &\leq 2N(gx_n, gx_n, gx_{n+1}) + N(gx_m, gx_m, gx_{n+1}) \\
&= 2N(gx_n, gx_n, gx_{n+1}) + N(gx_{n+1}, gx_{n+1}, gx_m) \\
&\leq 2N(gx_n, gx_n, gx_{n+1}) + 2N(gx_{n+1}, gx_{n+1}, gx_{n+2}) + N(gx_m, gx_m, gx_{n+2}) \\
&\leq 2N(gx_n, gx_n, gx_{n+1}) + \ldots + 2N(gx_{m-2}, gx_{m-2}, gx_{m-1}) \\
&\quad + N(gx_m, gx_m, gx_{m-1}) \\
&\leq 2N(gx_n, gx_n, gx_{n+1}) + \ldots + 2N(gx_{m-2}, gx_{m-2}, gx_{m-1}) \\
&\quad + 2N(gx_m, gx_m, gx_{m-1}) \\
&\leq 2q^n N(gx_1, gx_1, gx_0) + \ldots + 2q^{m-1} N(gx_1, gx_1, gx_0) \\
&= 2q^n (1 + q + \ldots + q^{m-n-1}) N(gx_1, gx_1, gx_0) \\
&\leq \frac{2q^n}{1-q} N(gx_1, gx_1, gx_0).
\end{aligned}$$

Since $\frac{2q^n}{1-q}N(gx_1,gx_1,gx_0)\to 0$ as $n,m\to\infty$, we have $N(gx_n,gx_n,gx_m)\to 0$ as $n,m\to\infty$. Thus $\{gx_n\}$ is a Cauchy sequence.

Case I: If $g(X)$ is complete, there exists $u\in g(X)$ such that $gx_n\to u$ as $n\to\infty$, so exist $p\in X$ such that $gp=u$.

Case II: If $f(X)$ is complete, then there exists $u\in f(X)$ such that $gx_n=fx_{n-1}\to u$, since $f(X)\subset g(X)$ we have $u\in g(X)$, so there exist $p\in X$ such that $gp=u$.

We claim that $fp=u$. Note that

$$\begin{aligned}
N(fp,fp,u) &\leq 2N(fp,fp,fx_n)+N(u,u,fx_n)\\
&\leq 2[aN(gp,gp,gx_n)+bN(gp,fp,fp)+cN(gp,fp,fp)\\
&\quad +dN(gx_n,fx_n,fx_n)]+N(u,u,gx_{n+1})\\
&= 2aN(u,u,gx_n)+2(b+c)N(u,fp,fp)\\
&\quad +2dN(gx_n,gx_{n+1},gx_{n+1})+N(u,u,gx_{n+1})
\end{aligned}$$

as $n\to\infty$. From Lemma 3.1, it shows that

$$\begin{aligned}
N(fp,fp,u) &\leq 2(b+c)N(u,fp,fp)\\
&\leq 4(b+c)N(fp,fp,u).
\end{aligned}$$

Hence $N(fp,fp,u)=0$ and $fp=u$. So $fp=gp=u$ and u is a point of coincidence of f and g.

Now we show that f and g have a unique point of coincidence. To this end, let us assume that there exists a point q in X such that $fq=gq$.

$$\begin{aligned}
N(fp,fp,fq) &\leq aN(gp,gp,gq)+bN(gp,fp,fp)\\
&\quad +cN(gp,fp,fp)+dN(gq,fq,fq)\\
&= aN(gp,gp,gq)\\
&= aN(fp,fp,fq).
\end{aligned}$$

Since $a<1$, so $N(fp,fp,fq)=0$. Tthus $fp=fq$. Therefore f and g have a unique point of coincidence. By Proposition 1.11, f and g have a unique common fixed point.

Corollary 3.3. *Let (X,N) be an N-cone metric space, P be a normal cone with normal constant K and $f,g : X \to X$ be two mappings which satisfy the following conditions:*

(i) $f(X) \subset g(X)$;

(ii) $f(X)$ or $g(X)$ is complete;

(iii) $N(fx, fy, fz) \leq kN(gx, gy, gz)$

for all $x,y,z \in X$, where $0 < k < 1$, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible,then f and g have a unique common fixed point.

Corollary 3.4. *Let (X,N) be an N-cone metric space, P be a normal cone with normal constant K and $f : X \to X$ be a mapping which satisfy the following conditions:*

(i) $f(X)$ is complete;

(ii) $N(fx, fy, fz) \leq aN(x,y,z) + bN(x, fx, fx) + cN(y, fy, fy) + dN(z, fz, fz)$

for all $x,y,z \in X$, where $a,b,c,d \geq 0$, $a+4b+4c+d < 1$, then f has a unique fixed point.

Theorem 3.5. *Let (X,N) be an N-cone metric space, P be a normal cone with normal constant K and $f,g : X \to X$ be two mappings which satisfy the following conditions:*

(i) $f(X) \subset g(X)$;

(ii) $f(X)$ or $g(X)$ is complete;

(iii) $N(fx, fy, fz) \leq a[N(gx, fy, fy) + N(gy, fx, fx)] + b[N(gy, fz, fz) + N(gz, fy, fy)]$
$+c[N(gx, fz, fz) + N(gz, fx, fx)]$ for all $x,y,z \in X$, where $a,b,c \geq 0$, $8a+4b+4c < 1$, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible,then f and g have a unique common fixed point.

Proof. Take $x_0 \in X$, there exist $x_1 \in X$ such that $fx_0 = gx_1$, in this way we construct sequences $\{gx_n\}$ with $fx_{n-1} = gx_n$, then from the condition(iii), we have

$$\begin{aligned}
N(gx_{n+1}, gx_{n+1}, gx_n) &= N(fx_n, fx_n, fx_{n-1}) \\
&\leq a[N(gx_n, fx_n, fx_n) + N(gx_n, fx_n, fx_n)] \\
&\quad +b[N(gx_n, fx_{n-1}, fx_{n-1}) + N(gx_{n-1}, fx_n, fx_n)] \\
&\quad +c[N(gx_n, fx_{n-1}, fx_{n-1}) + N(gx_{n-1}, fx_n, fx_n)]
\end{aligned}$$

$$\begin{aligned}
&= a[N(gx_n,gx_{n+1},gx_{n+1})+N(gx_n,gx_{n+1},gx_{n+1})] \\
&\quad +b[N(gx_n,gx_n,gx_n)+N(gx_{n-1},gx_{n+1},gx_{n+1})] \\
&\quad +c[N(gx_n,gx_n,gx_n)+N(gx_{n-1},gx_{n+1},gx_{n+1})] \\
&= 2aN(gx_n,gx_{n+1},gx_{n+1})+(b+c)N(gx_{n-1},gx_{n+1},gx_{n+1}) \\
&\leq 4aN(gx_{n+1},gx_{n+1},gx_n)+(b+c)N(gx_{n-1},gx_{n-1},gx_n) \\
&\quad +2(b+c)N(gx_{n+1},gx_{n+1},gx_n).
\end{aligned}$$

So $(1-4a-2b-2c)N(gx_{n+1},gx_{n+1},gx_n)\leq(b+c)N(gx_n,gx_n,gx_{n-1})$, which implies

$$N(gx_{n+1},gx_{n+1},gx_n)\leq qN(gx_n,gx_n,gx_{n-1}),$$

where $q=\frac{(b+c)}{1-(4a+2b+2c)}$ and $0<q<1$. By repeated application of above inequality, we have

$$N(gx_{n+1},gx_{n+1},gx_n) \leq q^nN(gx_1,gx_1,gx_0).$$

For all $n,m\in N, n<m$, we see that

$$\begin{aligned}
N(gx_n,gx_n,gx_m) &\leq 2N(gx_n,gx_n,gx_{n+1})+N(gx_m,gx_m,gx_{n+1}) \\
&= 2N(gx_n,gx_n,gx_{n+1})+N(gx_{n+1},gx_{n+1},gx_m) \\
&\leq 2N(gx_n,gx_n,gx_{n+1})+2N(gx_{n+1},gx_{n+1},gx_{n+2})+N(gx_m,gx_m,gx_{n+2}) \\
&\leq 2N(gx_n,gx_n,gx_{n+1})+\ldots+2N(gx_{m-2},gx_{m-2},gx_{m-1}) \\
&\quad +N(gx_m,gx_m,gx_{m-1}) \\
&\leq 2N(gx_n,gx_n,gx_{n+1})+\ldots+2N(gx_{m-2},gx_{m-2},gx_{m-1}) \\
&\quad +2N(gx_m,gx_m,gx_{m-1}) \\
&\leq 2q^nN(gx_1,gx_1,gx_0)+\ldots+2q^{m-1}N(gx_1,gx_1,gx_0) \\
&= 2q^n(1+q+\ldots+q^{m-n-1})N(gx_1,gx_1,gx_0) \\
&\leq \frac{2q^n}{1-q}N(gx_1,gx_1,gx_0).
\end{aligned}$$

Since $\frac{2q^n}{1-q}N(gx_1,gx_1,gx_0)\to 0$ as $n,m\to\infty$, we see implies that $N(gx_n,gx_n,gx_m)\to 0$ as $n,m\to\infty$. Thus $\{gx_n\}$ is a Cauchy sequence.

Case I: If $g(X)$ is complete, there exists $u \in g(X)$ such that $gx_n \to u$ as $n \to \infty$, so exist $p \in X$ such that $gp = u$.

Case II:If $f(X)$ is complete, then there exists $u \in f(X)$ such that $gx_n = fx_{n-1} \to u$, since $f(X) \subset g(X)$ we have $u \in g(X)$, so there exist $p \in X$ such that $gp = u$.

We claim that $fp = u$. Note that

$$\begin{aligned}
N(fp, fp, u) &\leq 2N(fp, fp, fx_n) + N(u, u, fx_n) \\
&\leq 2a[N(gp, fp, fp) + N(gp, fp, fp)] \\
&\quad +2b[N(gp, fx_n, fx_n) + N(gx_n, fp, fp)] \\
&\quad +2c[N(gp, fx_n, fx_n) + N(gx_n, fp, fp)] + N(u, u, gx_{n+1}) \\
&= 4aN(gp, fp, fp) + 2(b+c)N(gp, gx_{n+1}, gx_{n+1}) \\
&\quad +2(b+c)N(gx_n, fp, fp) + N(u, u, gx_{n+1})
\end{aligned}$$

as $n \to \infty$. From Lemma 3.1, it shows that

$$\begin{aligned}
N(fp, fp, u) &\leq (4a+2b+2c)N(u, fp, fp) \\
&\leq (8a+4b+4c)N(fp, fp, u).
\end{aligned}$$

Hence $N(fp, fp, u) = 0$ and $fp = u$. So $fp = gp = u$ and u is a point of coincidence of f and g.

Now we show that f and g have a unique point of coincidence. To this end, assume that there exists a point q in X such that $fq = gq$.

$$\begin{aligned}
N(fp, fp, fq) &\leq a[N(gp, fp, fp) + N(gp, fp, fp)] \\
&\quad +b[N(gp, fq, fq) + N(gq, fp, fp)] \\
&\quad +c[N(gp, fq, fq) + N(gq, fp, fp)] \\
&= (b+c)[N(fp, fq, fq) + N(fq, fp, fp)] \\
&\leq (b+c)[N(fp, fp, fq) + 2N(fp, fp, fq)] \\
&= 3(b+c)N(fp, fq, fq).
\end{aligned}$$

In view of $3(b+c)<1$, one has $N(fp,fp,fq)=0$. Thus $fp=fq$. Therefore f and g have a unique point of coincidence. By Proposition 1.11, f and g have a unique common fixed point.

Corollary 3.6. *Let (X,N) be an N-cone metric space, P be a normal cone with normal constant K and $f:X\to X$ be a mapping which satisfy the following conditions:*
(i) $f(X)$ is complete;
(ii)

$$\begin{aligned} N(fx,fy,fz) \quad &\le \quad a[N(x,fy,fy)+N(y,fx,fx)]+b[N(y,fz,fz) \\ &\quad +N(z,fy,fy)]+c[N(x,fz,fz)+N(z,fx,fx)] \end{aligned}$$

for all $x,y,z\in X$, where $a,b,c\ge 0$, $8a+4b+4c<1$, then f has a unique fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] L.G. Huang, X. Zhang,Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468-1476.

[2] C.T. Aage, J.N. Salunke, Some fixed points theorems in generalized D*-metric spaces, Appl. Sci. 12 (2010), 1-13.

[3] B. Ismat, A. Mujahia, N. Talat, Generalized cone metric spaces, J. Nonlinear. Sci. Appl. 3 (2010), 21-31.

[4] N. Malviya, B. Fisher, N-cone metric space and fixed pointS of asymptotically regular maps, Filomat, in press.

[5] F. Jerolina, M. Geeta, M. Neeraj, Some fixed points theorems for contractive maps in N-cone metric space, Doi:10.1007/s40096-015-0145-x.

[6] Y.J. Piao, C.H. Li, Common fixed points for a countable family of non-self mappings in cone metric spaces with the convex property, Chin. Quart. J. Math. 29 (2014), 221-230.

[7] H.A. Hamed, K. Erdal, O.R. Donal, S. Priya, Fixed points of generalized contractive mappings of integral type, Fixed Point Theory Appl. 2014 (2014), 213.

[8] N.A. Assad, W.A. Kirk, Fixed point theorems for set valued mappings of contractive type, Pacific J. Math. 43 (1972), 553-562.

[9] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. 29 (2002), 531-536.

[10] B.S. Choudhury, N. Metiya, The point of coincidence and common fixed point for a pair of mappings in cone metric spaces, Comput. Math. Appl. 60 (2010), 1686-1695.

[11] X.Y. Pai, S.Y. Liu, H.H. Jiao,Some new coincidence and common fixed point theorems in cone metric spaces, Afr. Math. 24 (2013), 135-144.

[12] J.G. Metha, M.L. Joshi, On complete cone metric space and fixed point theorem, J. Sci. Res. 3 (2011), 303-309

[13] X.J. Huang, C.X. Zhu, X. Wen, L. Ljubica, Some common fixed point theorems for a family of non-self mappings in cone metric spaces, Fixed Point Theory Appl. 2013 (2013), 144.

[14] X. Fan, Y. Liu, M. Song, Y.Z Ma,Some common fixed point theorems in cone metric spaces, Adv. Fixed Point Theory. 4 (2014), 365-377.