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THE DISCRETIZATION OF THE BLACK-SCHOLES OPTION PRICING MODEL WITH VOLATILE PORTFOLIO RISK MEASURE

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Abstract. We study in this paper, the discretization of the Black-Scholes option pricing model with volatile portfolio risk measure to obtain the variational formulation of the Black-Scholes option pricing model with volatile portfolio risk measure. This we shall do by using an implicit discretization in time and standard PI conforming finite elements in space with respect to a simplicial triangulation of the spatial domain.

Keywords: Sobolev space; nonlinear Black-Scholes equation; transaction cost; portfolio risk measure; Roth' methods.

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1. Introduction

For the pricing of option on equity shares, the Black-Scholes equation has become an indispensable tool for agents on the financial market. Under the assumption that the value of the underlying share evolves in time according to a stochastic differential equation and some further assumptions on the financial market, the equation can be derived by an application of lto's calculus. It represents a deterministic second order parabolic differential equation backward in time with the price of the option as the unknown and the interest rate and the volatility entering the equation as coefficient functions. Since analytical solutions in explicit form are only available in special cases, in general the equation must be solved by numerical methods based on appropriate discretization s in time and in space where the spatial variable is the value of the share.

In particular, a general approach of the numerical approximation, making use of finite difference, of the Cauchy problem for a multidimensional linear parabolic PDE of order $M \geq 2$, with

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bounded time and space-dependent coefficients, can be found in [12]. This approach is pursued under a strong setting, where the PDE problem has a classical solution.

The ¯finite difference method was also early applied to financial option pricing, the pioneering work being due to M. Brennan and E. S. Schwartz in 1978, and was, since then, widely researched in the context of the financial application, and extensively used by practitioners. For the references of the original publications and further major research, we refer to the review paper by Broadie and Detemple[4].

Most studies concerning the numerical approximation of PDE problems in Fi-nance consider the particular case where the PDE coefficients are constant (see, e.g., Barles et al. [1], Boyle and Tian [2], Fusai et al. [5], and Gilli et al. [6]). This occurs, namely, in option pricing under the Black-Scholes model (in one or several dimensions), when the the asset price vector's drift and volatility are taken constant. The simpler PDE, with constant coefficients, is obtained after a standard change of variables (see, e.g., Lamberton and Detemple [9] for the one-dimensional case, and Goncalves [7] for the multidimensional case).

This paper looked at the discretization of Black-Scholes equation with volatile portfolio risk measure using an implicit discretization in time and standard PI conforming finite elements in space with respect to a simplicial triangulation of the spatial domain.

2. The Model

Transaction costs as well as the volatile portfolio risk depend on the time –lag between two consecutive transactions. Minimizing their sum yields the optimal length of the hedge interval – time lag (for numerical example, see references in [11]). This leads to a fully nonlinear parabolic PDE. If transaction costs are taken into account perfect replication of the contingent claim is no longer possible. Modeling the short rate $r = r(t)$ by a solution to a one factor stochastic differential equation,

$$dS = \mu(s,t)dt + \sigma(s,t)dw, \tag{2.1}$$

where $\mu(S,t)dt$ represent a trend or drift of the process and $\sigma(S,t)$ represents volatility part of the process, the risk adjusted Black-Scholes equation can be viewed as an equation with a variable volatility coefficient

$$\partial_t V + \frac{\sigma^2(s,t)}{2} S^2 \left(1 - \mu(S\partial_S V)^{\frac{1}{3}}\right) \partial_S^2 V + rs\partial_S V - rV = 0, \tag{2.2}$$

where $\sigma^2(s,t)$ depends on a solution $V = V(s,t)$ and $\mu = 3\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{3}}$, since

$$\hat{\sigma}^2(s,t) = \sigma^2(1 - \mu(S\partial_S^2 V(S,t))^{\frac{1}{3}}.$$

Incorporating both transaction costs and risk arising from a volatile portfolio into equation (2.2) we have the change in the value of portfolio to become.

$$\partial_t V + \frac{\hat{\sigma}^2(s,t)}{2} S^2 \partial_S^2 V + rS\partial_S V - rV = (r_{TC} + r_{VP})S, \qquad (2.3)$$

where $r_{TC} = \frac{C|\Gamma|\hat{\sigma}S}{\sqrt{2\pi}} \frac{1}{\sqrt{\Delta t}}$ is the transaction costs measure, $r_{Vp} = \frac{1}{2} R\hat{\sigma}^4 S^2 \Gamma^2 \Delta t$ is the volatile portfolio risk measure and $\Gamma = \partial_S^2 V$.Minimizing the total risk with respect to the time lag Δt yields;

$$\min_{\Delta t}(r_{TC} + r_{VP}) = \frac{3}{2}\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{3}} \hat{\sigma}^2 |S\partial_S^2 V|^{\frac{4}{3}}.$$

For simplicity of solution and without loss of generality, we choose the minimized risk as

$$\left\{\min_{\Delta t}(r_{TC} + r_{VP})\right\}^{\frac{3}{2}} = AS^2 \partial_S^2 v, \qquad (2.4a)$$

with

$$A = (\frac{3}{2})^{\frac{3}{2}} \left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{2}} \hat{\sigma}^3. \qquad (2.4b)$$

They change in the value of the portfolio after minimizing the total risk with respect to time lag is given as

$$\partial_t V + \frac{\hat{\sigma}^2(s,t)}{2} S^2 \partial_S^2 V + rS\partial_S V - rV = AS^2 \partial_S^2 v, \qquad (2.5)$$

Since transaction cost and risk involved in the business is a drain to the portfolio. We have our equation now become

$$\partial_t V + \frac{\hat{\sigma}^2(s,t)}{2} S^2 \partial_S^2 V + rS\partial_S V - rV - AS^2 \partial_S^2 v = 0$$

which can also be written as

$$\frac{\partial V}{\partial t} + \frac{\hat{\sigma}^2(s,t)}{2} s^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV - AS^2 \frac{\partial^2 V}{\partial S^2} = 0 \qquad (2.6)$$

$$u(s,0) = u_0 \qquad (2.7)$$

For later discretization purposes, we truncate the domain in the variable S and consider (2.6),(2.7) on $\Omega \times (0,T), where\ \Omega \coloneqq (0,\bar{S})$

$$\frac{\partial V}{\partial t}+\frac{\hat{\sigma}^2(s,t)}{2}\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV-AS^2\frac{\partial^2 V}{\partial S^2}=0 \qquad (2.8)$$

$$u(\bar{S},t)=0 \qquad (2.9)$$

$$u(S,0)=u_0 \qquad (2.10)$$

3. Variational formulation for Black-Schole's equation with volatile portfolio risk measure

We use standard notation from Lebesque space and Sobolev space theory and denote by $\mathcal{D}(\Omega)$ the space of infinitely often differentiable function with compact support in $\Omega\subset\mathbb{R}_+$ and by $L^2(\Omega),\Omega\subseteq\mathbb{R}_+$,the Hilbert space of square integrable functions on Ω with inner product $(.,.)_{0,\Omega}$ and associated norm $\|.\|_{0,\Omega}$. We further refer to $H^1(\Omega)$ as the Hilbert space of square integrable functions with square integrable weak deriatives equipped with the norm $\|.\|_{1,\Omega}$.The Hilbert space $L^2\big((0,T)\big)$ and $H^1\big((0,T)\big)$ are define analogously.

In order to derive an appropriate variational formulation of (2.8)-(2.10) ,we introduce the weighted Sobolev space

$$V=\left\{v: v\in L^2(\Omega), S\frac{\partial v}{\partial S}\in L^2(\Omega)\right\}, \qquad (3.1)$$

endowed with the inner product

$$(v,w)_V:=\int_\Omega \left(v(S)w(S)+S^2\frac{\partial v}{\partial S}(S)\frac{\partial w}{\partial S}(S)\right)ds, \qquad (3.2)$$

Where $\frac{\partial v}{\partial S}$ stands for the weak derivative ,and we refer to $\|.\|_V$ as the associated norm .We define V_0 as the closure of $\mathcal{D}(\Omega)$ in V.Then ,it is easy to see that V_0 is closed subspace of V with $v(\bar{S})=0\ for\ v\in V_0$ .Moreover ,the following Poincare-Friedrichs inequality holds true:

Lemma 2.1.(Poincare-Friedrichs inequality) For all $v\in V_0$ there holds

$$\|v\|_{L^2(\Omega)}\leq 2|v|_V. \qquad (3.3)$$

Proof. Since $\mathcal{D}(\Omega)$ is dense in V_0, it suffices to prove (3.3) for $v\in\mathcal{D}(\Omega)$ .Obviously, we have

$$\|L\|^2_{L^2(\Omega)}=\int_\Omega v^2 ds=-2\int_\Omega Sv\frac{\partial v}{\partial S}(S)dS.$$

An application of Cauchy-Schwarz inequality to the right –hand side gives

$$\left|\int_{\Omega} Sv\frac{\partial v}{\partial S}(S)dS.\right| \le \left(\int_{\Omega}\left(S\frac{\partial v}{\partial S}(S)\right)^2 dS.\right)^{1/2}\left(\int_{\Omega} v(s)^2 dS\right)^{1/2}$$

from which we deduce the desired result.consequently,the semi-norm

$$|v|_V = \left(\int_{\Omega} S^2\left(\frac{\partial v}{\partial S}\right)^2 dS\right)^{1/2},$$

is in fact a norm on V_0 equivalent to $|.|_V$.We refer to V_0^* as the dual of V_0 with norm $|.|_{V_0^*}$ and to $\langle .,.\rangle_{V_0^*,V_0}$ as the dual pairing between V_0 and V_0^* .

we further denote by $L^2\big((0,T);L^2(\Omega)\big)$ the Hilbert space equipped with the norm

$$\|u\|^2_{L^2\left((0,T);L^2(\Omega)\right)} \coloneqq \int_0^T \|u(t)\|^2_{0,\Omega}\,dt$$

and define $L^2\big((0,T);V_0\big)$ and $\|.\|_{L^2((0,T);V_0)}$ analogously. Moreover, we introduce $H^1\big((0,T);V_0^*\big)$ as the Hilbert space with the norm

$$\|u\|^2_{H^1((0,T);V_0^*)} = \int_0^T \left(\|u(t)\|^2_{V_0^*} + \|u_t(t)\|^2_{V_0^*}\right)dt$$

where $\|u\|_{V_0^*} = \sup\limits_{v\in V_0}\frac{(u,v)}{|v|V}$.

Now multiplying (2.8) by $v \in V_0$ and integrating over Ω,we obtain

$$\begin{aligned} 0 = \int_{\Omega} \frac{\partial}{\partial t}u(S,t)v(S)dS + \int_{\Omega} \frac{\hat{\sigma}^2(S,t)S^2}{2}\frac{\partial^2}{\partial S^2}u(S,t)v(S)dS + r(t)\int_{\Omega} S\frac{\partial}{\partial S}u(S,t)dS \\ - r(t)\int_{\Omega} u(S,t)v(S)dS \\ - A\int_{\Omega} S^2\frac{\partial^2}{\partial S^2}u(S,t)v(S)dS. \qquad (3.4)\end{aligned}$$

Integrating by parts applying the fact that $v(\bar{S}) = 0$ results in

$$0 = \int_{\Omega}\frac{\partial}{\partial t}u(S,t)\,v(S)dS - \int_{\Omega}\frac{\hat{\sigma}^2(S,t)S^2}{2}\frac{\partial u}{\partial S}(S,t)\frac{\partial v}{\partial S}(S)dS + \int_{\Omega}\Big(\hat{\sigma}^2(S,t),S\hat{\sigma}(S,t) + \frac{\partial\hat{\sigma}}{\partial S}(S,t) - r(t)\Big)S\frac{\partial}{\partial S}u(S,t)v(S)dS - r(t)\int_{\Omega}u(S,t)v(S)dS - A\int_{\Omega}S^2\frac{\partial^2}{\partial S^2}u(S,t)v(S)dS. \quad (3.5)$$

In view of (3.5) ,we introduce the bilinear form $a_t(.,.)\colon V_0 \times V_0 \to \mathbb{R}$ according to

$$a_t(u,v) = \left(\frac{\hat{\sigma}^2}{2} S\frac{\partial u}{\partial S}, S\frac{\partial v}{\partial S}\right) + \left(\left(-r + \hat{\sigma}^2 + S\hat{\sigma}\frac{\partial \hat{\sigma}}{\partial S}\right) S\frac{\partial u}{\partial S}, v\right) - r(u,v) - A(S\frac{\partial u}{\partial S}, S\frac{\partial v}{\partial S}) \quad (3.6)$$

Consequently ,the boundary value problem (2.8)-(2.10) has the following variational formulation :Find $u \in H^1\big((0,T);V_0^*\big) \cap L^2\big((0,T);V_0\big)$ $such$ that for all $v \in V_0$

$$\langle\frac{\partial u}{\partial t}, v\rangle V_0^*, V_0 + a_t(u,v) = 0, \quad (3.7a)$$

$$(u(.,0),v)_{0,\Omega} = (u_0,v)_{0,\Omega} \quad (3.7b)$$

We note that $H^1\big((0,T);V_0^*\big) \cap L^2\big((0,T);V_0\big)$ is continuously embedded in $C^0\big([0,T];L^2(\Omega)\big)(cf,e.g.,10)$,

Theorem 3.1. Suppose $u_0 \in L^2(\Omega)$. Then , the variational formulation (3.7a),(3.7b) has a unique solution .Moreover,for all $0 < t < T$ there holds

$$e^{-2\lambda t}\|u(t)\|_{0,\Omega}^2 + \frac{1}{2}\sigma_{min}^2 \int_0^t e^{-2\lambda s}|u(s)|_V^2 ds \le \|u_0\|_{0,\Omega}^2 \quad (3.8)$$

Proof. Existence can be shown by the Galerkin method ,ie.,by constructing a sequence $u_n \in C^1\big((0,T);V_n\big), n \in \mathcal{N}$,of solutions of(3.7a),(3.7b) in finite dimensional subspaces $V_n \subset V_0$ that are limit-dense in $V_0$and then passing to the limit.For details of the existence proof we refer to [10].Uniqueness readily follows from (3.8).For the proof of (3.8),we choose $v = u(t)e^{-2\lambda t}$ in (3.7a) and integrate over (0,t) which gives

$$\int_0^t \left(\frac{\partial u}{\partial t}, u(\tau)e^{-2\lambda\tau}\right) d\tau + \int_0^t a_\tau\big(u(\tau), u(\tau)e^{-2\lambda\tau}\big) d\tau = 0 \quad (3.9)$$

Integrating by parts,we obtain

$$\|u_0\|^2 = \|u(t)\|^2 e^{-2\lambda t} - \int_0^t \left(u, \frac{\partial u}{\partial t} e^{-2\lambda\tau} - 2\lambda u e^{-2\lambda\tau}\right) d\tau + \int_0^t e^{-2\lambda\tau} a_\tau\big(u(\tau),u(\tau)\big) d\tau.$$

Now setting

$$[[v]](t) \coloneqq \left(e^{-2\lambda t}\|v(t)\|^2 + \frac{1}{2}\sigma_{min}^2 \int_0^t e^{-2\lambda t}\, |v(\tau)|_V^2 d\tau\right)^{\frac{1}{2}},$$

An application of Garding’s inequality yields

$$\|u_0\|^2 \geq \|u(t)\|^2 e^{-2\lambda t} - \int_0^t \left(u, \frac{\partial u}{\partial t} e^{-2\lambda\tau} - 2\lambda u e^{-2\lambda\tau}\right) d\tau$$
$$+ \int_0^t e^{-2\lambda\tau} \left(\frac{1}{4}\sigma_{min}^2 |u|_V^2 - \lambda\|u\|^2\right) d\tau = \big[[v]\big]^2(t) - \frac{1}{4}\int_0^t e^{-2\lambda\tau}\,\sigma_{min}^2 |u|_V^2 d\tau$$
$$- \int_0^t \left(u, \frac{\partial u}{\partial t} e^{-2\lambda\tau} - 2\lambda u e^{-2\lambda\tau}\right) d$$
$$\geq \big[[v]\big]^2(t) - \int_0^t e^{-2\lambda\tau}\, a_\tau d\tau - \int_0^t \left(u(\tau)e^{-2\lambda\tau}, \frac{\partial u}{\partial t}\right) d\tau = \big[[v]\big]^2(t),$$

from which we deduce (3.8).

The stability estimate (3.8)motivates to consider the norm

$$\big[[v]\big](t) = \left(e^{-2\lambda t}\|v(t)\|^2 + \tfrac{1}{2}\sigma_{min}^2 \int_0^t e^{-2\lambda t}\,|v(\tau)|_V^2 d\tau\right)^{\frac{1}{2}}, \tag{3.10}$$

So that (3.8) reads

$$\big[[v]\big](t) \leq \|u_0\|. \tag{3.11}$$

Similar technique as in the proof of (3.8) allows to establish the following estimate.

Lemma 3.2 .For any $u \in H^1\big((0,T);V_0^*\big) \cap L^2\big((0,T);V_0\big) \subset C^0\big([0,T];L^2(\Omega)\big)$ there holds

$$\left\|e^{-\lambda\tau}, \frac{\partial u}{\partial t}\right\|_{L^2\left((0,T);V_0^*\right)} \leq \sqrt{2}\,\frac{\mu}{\sigma_{min}}\|u_0\| \tag{2.12}$$

Proof .In view of lemma 3.1 and (3.9) we get

$$\left|\int_0^t \left(\frac{\partial u}{\partial t}, u(\tau)e^{-2\lambda\tau}\right) d\tau\right| \leq \int_0^t a_\tau\big(u(\tau), u(\tau)e^{-2\lambda\tau}\big) d\tau \leq \sqrt{2}\,\frac{\mu}{\sigma_{min}}\big[[u]\big](T)|v|,$$

whence

$$\left\|e^{-\lambda\tau}, \frac{\partial u}{\partial t}\right\|_{L^2\left((0,T);V_0^*\right)} \leq \sqrt{2}\,\frac{\mu}{\sigma_{min}}\|u_0\|.$$

4. Result.

For the discretization of the variational (3.7a),(3.7b) of the Black-Scholes option pricing equation with volatile portfolio risk measure we use Rothe's method,ie we first consider a semi discretization in time by the implicit Euler scheme which amounts to the solution of a sub problem for each time step. The sub problems are than approximated by continuous, piecewise linear finite elements with respect to simplicial triangulations of the spatial domain Ω.

4.1 Semi discretization in Time

We consider a partition of the interval [0,T] into subinterval $[t_{n-1}, t_n], 1 \leq n \leq N$,such that

$$0 = t_0 < t_1 < \cdots < t_n = T$$

Set $\vartriangle t_n := t_n - t_{n-1}$,$\vartriangle t := max\{\vartriangle t_n\}$ and

$$\rho_{\Delta t} := \max_{2\leq n\leq N} \frac{\Delta t_n}{\Delta t_{n-1}} \quad (4.1)$$

For continuous f on [0,T],we introduce the notation $f^n = f(t_n)$.The semi discrete problem arising from implicit Euler scheme is as follows :Find $(u^n)_{2\leq n\leq N} \in V_0$ such that

$$(u^n - u^{n-1}, v)_{0,\Omega} + \Delta t_n a_{t_n}(u^n, v) = 0, v \in V_{0}, 1 \leq n \leq N, \quad (4.2a)$$

$$u^0 = u_0 \quad (4.2b)$$

The existence and uniqueness of the solution $u^n \in V_0$ of (4.2a),(4,2b) can be show for sufficiently small time step Δt_n.

Theorem 4.1 The time step restriction

$$\Delta t_n < \frac{1}{2\lambda} \quad (4.3)$$

And the semi discrete problem (4.2a),(4.2b) admits a unique solution.

Proof. We note that (4.2a) can be equivalently written as

$$c_n(u^n, v) = (u^{n-1}, v)_{0,\Omega} \quad ,v \in V_0,$$

where the bilinear form $c_n(.,.): V_0 \times V_0 \to \mathbb{R}$ is given by

$$c_n(v, w) = \Delta t_n a_{t_n}(v, w) + (v, w)_{0,\Omega} \qquad v, w \in V_0.$$

The bilinear form $c_n(.,.): V_0 \times V_0 \to \mathbb{R}$ is bounded. Moreover,taking additional (4.3) into account, there exists a constant $\alpha > 0$ such that

$$c_n(v, v) \geq \alpha \|v\|_V^2 \quad ,v \in V_0.$$

Hence ,the assertion follows from the Lax-Milgram Lemma(cf,eg.,[3]).

For the sequence $(u^m)_{2\leq m\leq n}, n \leq N$,we introduce a discrete norm $[[u^m]]_n$ as the discrete analogue of $[[u]](t)$ (cf.(3.10)) according to

$$[[u^m]]_n := (\left(\prod_{i=1}^{n}(1-2\lambda\Delta t_i)\right)\|u^n\|_{0,\Omega}^2 + \tag{4.4}$$

$$\frac{1}{2}\sigma_{min}^2 \sum_{m=1}^{n} \Delta t_m \left(\prod_{i=1}^{m-1}(1-2\lambda\Delta t_i)\right)|u^m|_V^2 \;)^{1/2} \tag{4.5}$$

As a counterpart of (3.11) we obtain:

Lemma 4.2 Under the assumption of theorem 3.1 there holds

$$[[u^m]]_n \leq \|u^0\|_{0,\Omega}. \tag{4.6}$$

Proof .By Young's inequality we have

$$(1-2\lambda\Delta t_n)\|u^n\|_{0,\Omega}^2 + \frac{1}{2}\Delta t_n \sigma_{min}^2 |u^n|_V^2 \leq \|u^{n-1}\|_{0,\Omega}^2 \tag{4.7}$$

Multtiplication of (4.7) by $\prod_{i=1}^{n-1}(1-2\lambda\Delta t_i)$ and summation over n gives the assertion .

Given the sequence $(u^n)_{2\leq n\leq N}$ of solutions of (4.2a),(4.2b),we introduce the function $u_{\Delta t}$ $0n$ $[0,T]$ by

$$u_{\Delta t}(t)\mathrm{I}_{[t_{n-1},t_n]} := u_{n-1} + (\Delta t_n)^{-1}(t-t_{n-1})(u_n-u_{n-1}), \quad 1 \leq n \leq N \tag{4.8}$$

which obviously is affine on each interval $[t_{n-1}, t_n]$,, $\quad 1 \leq n \leq N$.

lemma 3.3.If there exists a positive constant $\alpha \leq \frac{1}{2}$ such that for

$$\Delta t \leq \frac{\alpha}{\lambda} \tag{4.9}$$

And for any family $(v^n)_{\,0\leq n\leq N} \in V_0^{N+1}$ *there holds*

$$\frac{1}{8}[[v^m]]_n^2 \leq [[v_{\Delta t}]]^2(t_n) \leq max(2{,}1+\rho\Delta t)[[v^m]]_n^2 + \frac{1}{2}\sigma_{min}^2|v^0|_V^2, \tag{4.10}$$

(see[8] for proof)

4.2. Fully Discretization

Given a null sequence $\mathcal{H}$ of positive real numbers ,for the discretization of the semidiscrete problem (4.2a),(4.3b) in space,we use continuous ,piecewise linear finite elements with respect to a family of simplicial triangulation $T_{nh}, 1 \leq n \leq N, of\ \Omega. For\ T \in T_{nh}, we\ denote\ by\ S_{min}(T), S_{max}(T)$ the endpoints of T and refer to $h_T := S_{max}(T) - S_{min}(T)$ as the length of T and to $h_n := max\{h_T \,/\, T \in T_{nh}\}$ as the maximal size of the intervals in T_{nh}. Moreover ,for $D \subseteq \Omega$we refer to $\mathcal{N}(D)$ as the set of nodes of $T_{nh}\ in\ D$ and associate with each $T \in T_{nh}$ the patch w_T according to

$$w_T := \bigcup \{T' \in T_{nh} \mathrm{I} \mathcal{N}_{nh}(T') \cap \mathcal{N}_{nh}(T) \neq \emptyset\}. \qquad (4.11)$$

We assume that the family of triangulations is locally quasi-uniform in the sense that there exists a constant $\rho > 0$ such that for two adjacent elements T,T'$\in T_{nh}$ there holds

$$h_T \leq \rho h_{T'} \quad , \qquad h \in \mathcal{H}. \qquad (4.12)$$

For each $h \in \mathcal{H}$,we define the finite element spaces by

$$V_{nh} := \{v_h^n \in C^0(\Omega) | v_h^n|_T \in P^1(T), T \in T_{nh}\}, \qquad (4.13)$$

$$V_{nh}^0 := V_{nh} \cap V_0 \qquad (4.14)$$

where $P'(T)$ stands for the linear space of polynomials of degree 1 on T.

Assuming that $u_0 \in V_{Ih}$, the fully discrete problem reads as follows; find $(u_h^n)_{1 \leq n \leq N}, u_h^n \in V_{nh}^0, 1 \leq n \leq N, such\ that$

$$(u_h^n - u_h^{n-1}, v_n)_{0,\Omega} + \Delta t_n a_{t_n}(u_h^n, v_h) = 0 \,, v_h \in V_{nh}^0, \qquad (4.15)$$

$$u_h^0 = u_o \,, \qquad (4.16)$$

Theorem 4.2 Assume (4.3) hold true. Then ,the fully discrete problem admits a unique solution. Moreover, for the sequence $(u_h^m)_{1 \leq m \leq N,} 1 \leq n \leq N$,we have the stability estimate

$$\left[[u_h^m]\right]_n \leq \|u^0\|_{0,\Omega} \qquad (4.17)$$

Proof : Existence and uniqueness follow from the Lax-Milgram lemma, since $V_{nh}^0 \subset V_{0,} 1 \leq n \leq N$.The estimate is an immediate consequence of lemma 4.2.

As in 4.1 (cf.(4.8)) we define $u_{h,\Delta t}$ as the piecewise affine function

$$u_{h,\Delta t}(t)\big|_{[t_{n-1},t_n]} := P_h^n u_h^{n-1} + (\Delta t_n)^{-1}(t - t_{n-1})(u_h^n - P_h^n u_h^{n-1}), 1 \leq n \leq N, \qquad (4.18)$$

where $P_h^n u_h^{n-1}$ is the $L^2 - projection\ of\ u_h^{n-1}\ onto\ V_{nh}^0$.

Conclusion.

We obtained the variational formulation of the Black-Scholes option pricing model with volatile portfolio risk measure. We use the Rothe's method For the discretization of the variational (3.7a),(3.7b) of the Black-Scholes option pricing equation with volatile portfolio risk measure, ie we first consider a semi discretization in time by the implicit Euler scheme which amounts to the solution of a sub problem for each time step. The sub problems are than approximated by continuous, piecewise linear finite elements with respect to simplicial triangulations of the spatial domain Ω

REFERENCE

[1] Barles, G., Ch. Daher, and M. Romano. Convergence of Numerical Schemes for Parabolic Equations Arising in Finance Theory. Math. Models Methods Appl. Sci., 5(1) (1995), 125-143.

[2] Boyle P. P., and Y. Tian. An Explicit Finite Di®erence Approach to the Pricing of Barrier Options. Appl. Math. Finance 5(1998), 17-43.

[3] Braess, D(2007). Finite elements.3rd Edition.Cambridge University press,Cambridge.

[4] Broadie, M., and J. B. Detemple. *Option Pricing: Valuation Models and Applications.* Manag. Sci. 50(9) (2004), 1145-1177.

[5] Fusai, G., S. Sanfelici, and A. Tagliani. *Practical Problems in the Numerical Solution of PDE's in Finance.* Rend. Studi Econ. Quant. 2001, 105-132.

[6] Gilli, M., E. Kellezi, and G. Pauletto. Solving Finite Difference Schemes Arising inTrivariate Option Pricing. J. Econ. Dynam. Control 26(9-10) (2002), 1499-1515.

[7] Goncalves F. F. and Grossinho M. R. Numerical approximation of multidimensional parabolic partial differential equations arising in financial mathematics Cemapre,Iseg, Technical University of Lisbon, Rua do Quelhas 6(2008), 1200-781.

[8] Huifang Li. Adaptive finite element approximation of the Black-scholes Equation on Residual-type a Posteriori Error Estimator. A Dissertation presented to the faculty of the Department of Mathematics University of Houston, (2009).

[9] Lamberton, D., and B. Lapeyre. *Introduction to Stochastic Calculus Applied to Fi-nance.* Chapman and Hall. U.K. (1996).

[10] Renard, R and Roger ,R.C. An introduction to Partial Differential Equations.Springer,Berlin-Heidelbag-New York. (1993).

[11] Osu B.O. and Olunkwa C. A solution by stochastic iteration method for nonlinear Black-Scholes equation with transaction cost and volatile portfolio risk in Hilbert space. *International Journal of Mathematical Analysis and Applications.* 1(3) (2014),43-48

[12] Thomee, V. *Finite Difference Methods for Linear Parabolic Equations.* Handbook of Numerical Analysis I, North-Holland (1990), 3-196.