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I-CONVERGENT DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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Abstract. In this article we introduce the sequence spaces $c_0^I(F,\triangle)$ and $l_\infty^I(F,\triangle)$ for the sequence of modulii $F=(f_k)$ and give some inclusion realations.The results here in proved are analogous to those by ASMA BEKTAS Cigdem (2003)[Soochow.J.Math.,(2003) 29(2) : 215-220].

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1. Introduction

Let ω,l_∞, c and c_0 be the set of all sequences of complex numbers, the linear spaces of bounded, convergent and null sequences $x=(x_k)$ with complex terms, respectively,normed by

$$||x||_\infty = \sup_k |x_k|, \text{ where } k \in \mathbb{N} = \{1, 2, \cdots\}.$$

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The idea of difference sequence spaces was introduced by Kizmaz [15]. In 1981, Kizmaz [15] defined the sequence spaces

$$l_\infty(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in l_\infty\},$$

$$c(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c\},$$

$$c_0(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c_0\},$$

where,

$$\triangle x = (x_k - x_{k+1}) \text{ and } \triangle^0 x = (x_k).$$

These are Banach spaces with the norm

$$||x||_\triangle = |x_1| + ||\triangle x||_\infty.$$

Difference sequence spaces have been studied by Vakeel.A.Khan and S. Tabassum [13], Vakeel A.Khan and K. Ebadullah [14] and many others.

The idea of modulus was structured in 1953 by Nakano.(See[19]).A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if

(i)f(t) $= 0$ if and only if $t = 0$,
(ii) f(t+u)$\leq$ f(t)+ f(u) for all t,u$\geq$0,
(iii) f is increasing, and
(iv) f is continuous from the right at zero.

Let X be a sequence space. Ruckle [20,21,22] defined the sequence space X(f) as

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}$$

for a modulus f. Kolk [16,17]gave an extension of $X(f)$by considering a sequence of moduli $F = (f_k)$,that is

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.$$

After then Gaur and Mursaleen[7] defined the following sequence spaces

$$l_\infty(F,\triangle)=\{x=(x_k):\triangle x\in l_\infty(F)\}$$

$$c_0(F,\triangle)=\{x=(x_k):\triangle x\in c_0(F)\}$$

for a sequence of moduli $F=(f_k)$ and gave the necessary and sufficient conditions for the inclusion relations between $X(\triangle)$ and $Y(F,\triangle)$,where X,Y $= l_\infty$ or c_0. Sequence of moduli have been studied by C.A.Bektas and R.Colak[1], Vakeel A.Khan [8,9,10], Vakeel A.Khan and Q.M.D.Lohani [11] and many others .

The notion of the statistical convergence was introduced by H.Fast[4].Later on it was studied by J.A.Fridy[5,6] from the sequence space point of view and linked it with the summability theory.

The notion of I-convergence is a generalization of the statistical convergence. The notion of I-convergence was studied at the initial stage by Kostyrko, Salat and Wilezynski [18]. Later on it was studied by Salat [23], Salat,Tripathy and Ziman [24],Demirci [3], Vakeel A.Khan and Khalid Ebadullah [12], Dems [2]and many others.

Let N be a non empty set. Then a family of sets I$\subseteq 2^N$(power set of N)is said to be an ideal if I is additive i.e A,B$\in$I $\Rightarrow$A$\cup$ B$\in$I and hereditary i.e A$\in$I, B$\subseteq$A$\Rightarrow$B$\in$I. A non-empty family of sets $\pounds(I)\subseteq 2^N$ is said to be filter on N if and only if $\Phi\notin\pounds$(I),for A,B$\in\pounds$(I) we have A$\cap$B$\in\pounds$(I) and for each A$\in\pounds$(I)and A$\subseteq$B implies B$\in\pounds$(I).

An Ideal I$\subseteq 2^N$ is called non-trivial if I$\neq 2^N$. A non-trivial ideal I$\subseteq 2^N$ is called admissible if $\{\{x\}:x\in N\}\subseteq$I. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal J$\neq$I containing I as a subset.For each ideal I, there is a filter $\pounds$(I) corresponding to I. i.e $\pounds$(I) $=\{K\subseteq N:K^c\in I\}$,where K^c = N-K.

Definition 1.1. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{k \in N : |x_k - L| \geq \epsilon\} \in$ I. In this case we write I-lim $x_k = L$.

Definition 1.2. A sequence $(x_k) \in \omega$ is said to be I-null if L = 0 .In this case we write I-lim $x_k = 0$.

Definition 1.3. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number m = m(ϵ) such that $\{k \in N : |x_k - x_m| \geq \epsilon\} \in$ I.

Definition 1.4. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists M >0 such that $\{k \in N : |x_k| > M\}$

We need the following lemmas.

Lemma 1.5. The condition $\sup_k f_k(t) < \infty, t > 0$ holds if and only if there is a point $t_0 > 0$ such that $\sup_k f_k(t_0) < \infty$.(See[1,7]).

Lemma 1.6. The condition $\inf_k f_k(t) > 0$ holds if and only if there exists a point $t_0 > 0$ such that $\inf_k f_k(t_0) > 0$.(See[1,7]).

Lemma 1.7. Let K∈ $\pounds$(I) and M⊆N. If M∉I, then M∩K ∉I.(See[24]).

Lemma 1.8. If I $\subset 2^N$ and M⊆N. If M ∉I, then M∩K ∉I.(See[18]).

2. Main results

In this article we introduce the following classes of sequence spaces.

$$c_0^I(F,\triangle) = \{(x_k) \in \omega : I - \lim f_k(|\triangle x_k|) = 0\} \in I$$

$$l_\infty^I(F,\triangle) = \{(x_k) \in \omega : \sup_k f_k(|\triangle x_k|) < \infty\} \in I.$$

Theorem 2.1. For a sequence $F = (f_k)$ of moduli,the following statements are equivalent:

(a) $l_\infty^I(\triangle) \subseteq l_\infty^I(F,\triangle)$

(b) $c_0^I(\triangle) \subset l_\infty^I(F,\triangle)$

(c) $\sup_k f_k(t) < \infty, (t > 0)$

Proof. (a)implies (b) is obvious.

(b)implies (c). Let $c_0^I(\triangle) \subset l_\infty^I(F,\triangle)$. Suppose that (c) is not true. Then by Lemma 1.5

$$\sup_k f_k(t) = \infty \text{ for all } t > 0,$$

and therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(\frac{1}{i}) > i, \text{ for, } i = 1, 2, 3....... \tag{1}$$

Define $x = (x_k)$ as follows

$$x_k = \begin{cases} \frac{1}{i}, \text{ if } k = k_i, i = 1, 2, 3.......; \\ 0, \text{ otherwise.} \end{cases}$$

Then $x \in c_0^I(\triangle)$ but by (1), $x \notin l_\infty^I(F,\triangle)$ which contradicts (b).Hence (c)must hold.

(c)implies (a). Let (c) be satisfied and $x \in l_\infty^I(\triangle)$. If we suppose that $x \notin l_\infty^I(F,\triangle)$ then

$$\sup_k f_k(|\triangle x_k|) = \infty \text{ for } \triangle x \in l_\infty^I$$

If we take t $= |\triangle x|$ then $\sup_k f_k(t) = \infty$ which contradicts (c).
Hence $l_\infty^I(\triangle) \subseteq l_\infty^I(F,\triangle)$.

Theorem 2.2. If $F = (f_k)$ is a sequence of moduli, then the following statements are equivalent:

(a) $c_0^I(F,\triangle) \subseteq c_0^I(\triangle)$,

(b) $c_0^I(F,\triangle) \subset l_\infty^I(\triangle)$,

(c) $\inf_k f_k(t) > 0, (t > 0)$.

Proof.(a) implies (b) is obvious.

(b)implies (c). Let $c_0^I(F,\triangle) \subset l_\infty^I(\triangle)$.Suppose that (c) does not hold.
Then, by lemma 1.6 ,

$$\inf_k f_k(t) = 0,\ (t > 0), \tag{2}$$

and therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(i^2) < \frac{1}{i} \text{ for } i = 1, 2, ...$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} i^2, \text{if } k = k_i,\ i = 1, 2, 3.......; \\ 0,\ \text{otherwise.} \end{cases}$$

By (2) $x \in c_0^I(F,\triangle)$ but $x \notin l_\infty^I(\triangle)$ which contradicts (b).Hence (c) must hold.

(c)implies (a). Let (c) holds and $x \in c_0^I(F,\triangle)$ that is

$$I - \lim_k f_k(|\triangle x_k|) = 0$$

. Suppose that $x \notin c_0^I(\triangle)$.Then for some number $\epsilon_0 > 0$ and positive integer k_0 we have $|\triangle x_k| \leq \epsilon_0$ for $k > k_o$.Therefore $f_k(\epsilon_0) \geq f_k(|\triangle x_k|)$ for $k \geq k_o$ and hence $\lim_k f_k(\epsilon_0) > 0$ which contradicts our supposition that $x \notin c_0^I(\triangle)$.
Thus $c_0^I(F,\triangle) \subseteq c_0^I(\triangle)$.

Theorem 2.3. The inclusion $l_\infty^I(F,\triangle) \subseteq c_0^I(\triangle)$ holds if and only if

$$\lim_k f_k(t) = \infty, \text{ for, } t > 0. \tag{3}$$

Proof. Let $l_\infty^I(F,\triangle) \subseteq c_0^I(\triangle)$ such that $\lim_k f_k(t) = \infty$ for t >0 does not hold.Then there is a number $t_0 > 0$ and a sequence (k_i) of positive integers such that

$$f_{k_i}(t_0) \leq M < \infty. \tag{4}$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_0, \text{ if } k = k_i, \ i = 1,2,3.......; \\ 0, \text{ otherwise.} \end{cases}$$

Thus $x \in l_\infty^I(F,\triangle)$,by (4).But $x \notin c_0^I(\triangle)$, so that (3) must hold If $l_\infty^I(F,\triangle) \subseteq c_0^I(\triangle)$.
Conversely , let (3) hold. If $x \in l_\infty^I(F,\triangle)$, then $f_k(|\triangle x_k|) \leq M < \infty$
for k = 1,2,3.......Suppose that $x \notin c_0^I(\triangle)$.Then for some number $\epsilon_0 > 0$ and positive integer k_0 we have $|\triangle x_k| < \epsilon_0$ for $k \geq k_0$.Therefore $f_k(\epsilon_0) > f_k(|\triangle x_k|) \leq M$ for k$\geq k_0$ which contradicts (3).Hence $x \in c_0^I(\triangle)$.

Theorem 2.4. The inclusion $l_\infty^I(\triangle) \subseteq c_0^I(F,\triangle)$ holds, if and only if

$$\lim_k f_k(t) = 0 \text{ for } t > 0. \tag{5}$$

Proof. Suppose that $l_\infty^I(\triangle) \subseteq c_0^I(F,\triangle)$ but (5) does not hold.

Then

$$\lim_k f_k(t_0) = l \neq 0, \text{ for some } t_0 > 0. \tag{6}$$

Define the sequence $x = (x_k)$ by

$$x_k = t_0 \sum_{v=0}^{k-1} (-1) \begin{bmatrix} k-v \\ k-v \end{bmatrix}$$

for k = 1,2,3..........Then $x \notin c_0^I(F,\triangle)$, by (6).Hence (5)must hold.

Conversly , let $x \in l_\infty^I(\triangle)$ and suppose that (5) holds.

Then $|\triangle x_k| \leq M < \infty$ for k = 1,2,3.....

Therefore $f_k(|\triangle x_k|) \leq f_k(M)$ for k = 1,2,3..... and

$\lim_k f_k(|\triangle x_k|) \leq \lim_k f_k(M) = 0$, by (5).

Hence $x \in c_0^I(F,\triangle)$.

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