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COMMON FIXED POINT THEOREMS IN COMPLEX VALUED b- METRIC SPACES

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Abstract. In this paper, common fixed point theorems for a pair of mappings satisfying certain rational contractions are established in the framework of complex valued b- metric spaces. Some examples are provided to support our main results.

Keywords: Cauchy sequence; common fixed point; complex valued b-metric spaces; weakly compatible mappings; contractive type mapping.

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1. Introduction

In 1922, Banach proved contraction principle [7] which provides a technique for solving existence problems in many branches of mathematical sciences and engineering. Subsequently Banach contraction principle was generalized, extended and improved by many authors in different ways. In 1998, Czerwik [9] introduced the concept of b- metric space. In 2011, Azam

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et al.[1] introduced the notion of complex valued metric spaces. After the establishment of complex valued metric spaces, many researchers have contributed with their work in this space. Rouzkard and Imdad [3] generalized Azam *et al.* [1] with an important note that complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. Indeed the definition of cone metric space banks on the underlying Banach space which is not a division Ring. However, in complex valued metric spaces, we can study improvements of a host of results of analysis involving divisions. Recently, Rao *et al.* [5] developed the notion of complex valued b- metric spaces and proved fixed point results. We give some common fixed point theorems for four mappings in complex valued b- metric spaces and obtain generalizations of Azam *et al.* [1], Rouzkard *et al.* [3], Mukheimer [2] and Nashine *et al.* [4].

2. Preliminaries

In what follows, we recall some definitions and notations that will be used in our note.

Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $\precsim$ on C as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(C1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;

(C4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(C2)$, $(C3)$ and $(C4)$ is satisfied and we write $z_1 \prec z_2$ if only $(C4)$ is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

The following definition is developed by Azam *et al.* [1].

Definition 2.1. [1] Let X be a nonempty set. A mapping $d : X \times X \to C$ satisfies the following conditions:

(CM1) $0 \precsim d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(CM2) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(CM3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

Definition 2.2. [5] Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to C$ satisfies the following conditions:

(CVBM1) $0 \precsim d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(CVBM2) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(CVBM3) $d(x,y) \precsim s[d(x,z) + d(z,y)]$ for all $x,y,z \in X$.

Then d is called a complex valued b-metric on X and (X,d) is called a complex valued b-metric space.

Example 2.1. Let $X = [0,1]$. Define the mapping $d : X \times X \to C$ by

$$d(x,y) = |x-y|^2 + i|x-y|^2, \ for \ all \ x,y \in X.$$

Then (X,d) is a complex valued b- metric space with $s = 2$.

Definition 2.3. [5] Let (X,d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in C$ such that $B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set $A \subseteq X$ whenever for every $0 \prec r \in C$ such that $B(x,r) \cap (X - A) \neq \phi$.

(iii) A subset $B \subseteq X$ is called open whenever each limit point of B is an interior point of B.

(iv) A subset $B \subseteq X$ is called closed whenever each limit point of B is belong to B.

(v) The family $F = \{B(x,r) : x \in X \ \ and \ \ 0 \prec r\}$ is a sub basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 2.4. [5] Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in C$ with $0 \prec c$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec c$ for all $n > n_0$, then $\{x_n\}$ is said to be converges to x and x is a limit point of $\{x_n\}$. We denote this by $x_n \to x$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = x$.

(2) If for every $c \in C$ with $0 \prec c$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$ where $m \in N$, then $\{x_n\}$ is said to be Cauchy sequence.

(3) If every Cauchy sequence is convergent in (X,d), then (X,d) is called a complete complex valued b- metric space.

Lemma 2.1. [5] *Let (X,d) be a complex valued b- metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.2. [5] *Let (X,d) be a complex valued b- metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in N$.*

Definition 2.5. Let A and S be self mappings on a set X, if $w = Ax = Sx$ for some x in X, then x is called coincidence point of A and S and w is called a point of coincidence of A and S.

Definition 2.6. A pair of self mappings $A, S : X \to X$ is called weakly compatible if A and S commute at their coincidence point.

3. Main results

In this section, we prove some common fixed point theorems for rational type contraction conditions. Our main result runs as follows.

Theorem 3.1. *Let A, B, S and T be four self-mappings of a complete complex valued b-metric spaces (X,d) satisfying*

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,

(ii) $d(Ax, By) \precsim \alpha d(Sx, Ty) + \beta \frac{d(Sx,Ax)d(By,Ty)}{1+d(Sx,Ty)} + \gamma \frac{d(Sx,By)d(Ax,Ty)}{1+d(Sx,Ty)}$,$x, y \in X$, where α, β and γ are non-negative reals such that $s(\alpha + \gamma) + \beta < 1$.

(iii) Pairs (A,S) and (B,T) are weakly compatible and $B(X)$ is a closed subspace of X.

Then A, B, S and T have a unique common fixed point.

Proof. Consider a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \quad and \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2},$$

where $\{x_n\}$ is another sequence in X. First of all we show that $\{y_n\}$ is a Cauchy sequence of X, for this consider

$$\begin{aligned}
&d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1}) \\
&\precsim \alpha d(Sx_{2n}, Tx_{2n+1}) + \beta \frac{d(Sx_{2n}, Ax_{2n}) d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sx_{2n}, Tx_{2n+1})} + \gamma \frac{d(Sx_{2n}, Bx_{2n+1}) d(Ax_{2n}, Tx_{2n+1})}{1 + d(Sx_{2n}, Tx_{2n+1})} \\
&\precsim \alpha d(y_{2n-1}, y_{2n}) + \beta \frac{d(y_{2n-1}, y_{2n}) d(y_{2n+1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} + \gamma \frac{d(y_{2n-1}, y_{2n+1}) d(y_{2n}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} \\
&\precsim \alpha d(y_{2n-1}, y_{2n}) + \beta \frac{d(y_{2n-1}, y_{2n}) d(y_{2n+1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})}
\end{aligned}$$

or

$$|d(y_{2n}, y_{2n+1})| \leq \alpha |d(y_{2n-1}, y_{2n})| + \beta |d(y_{2n+1}, y_{2n})| \left| \frac{d(y_{2n-1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} \right|.$$

Since $|d(y_{2n-1}, y_{2n})| < |1 + d(y_{2n-1}, y_{2n})|$, we have

$$|d(y_{2n}, y_{2n+1})| \leq \alpha |d(y_{2n-1}, y_{2n})| + \beta |d(y_{2n+1}, y_{2n})|$$

and $|d(y_{2n}, y_{2n+1})| \leq \frac{\alpha}{1-\beta} |d(y_{2n-1}, y_{2n})|$. Similarly we obtain

$$|d(y_{2n+1}, y_{2n+2})| \leq \frac{\alpha}{1-\beta} |d(y_{2n}, y_{2n+1})|. \tag{0.1}$$

Since $s(\alpha + \gamma) + \beta < 1$ and $s \geq 1$, therefore with $\delta = \frac{\alpha}{1-\beta}$ and so for all $n \geq 0$ and consequently, we have

$$|d(y_{2n}, y_{2n+1})| \leq \delta |d(y_{2n-1}, y_{2n})| \leq \delta^2 |d(y_{2n-2}, y_{2n-1})| \leq \ldots \leq \delta^{2n} |d(y_0, y_1)|. \tag{0.2}$$

Finally, we can conclude that $\quad d(y_n, y_{n+1}) \precsim \delta^n d(y_0, y_1)$

for all $m > n, \quad m,n \in N$ and since $s\delta = \frac{s\alpha}{1-\beta} < 1$, we get

$$|d(y_n,y_m)| \leq s|d(y_n,y_{n+1})| + s|d(y_{n+1},y_m)|$$
$$\leq s|d(y_n,y_{n+1})| + s^2|d(y_{n+1},y_{n+2})| + s^2|d(y_{n+2},y_m)|$$
$$\leq s|d(y_n,y_{n+1})| + s^2|d(y_{n+1},y_{n+2})| + s^3|d(y_{n+2},y_{n+3})| + s^3|d(y_{n+3},y_m)|$$
$$\leq s|d(y_n,y_{n+1})| + s^2|d(y_{n+1},y_{n+2})| + s^3|d(y_{n+2},y_{n+3})| + \dots +$$
$$s^{m-n-2}|d(y_{m-3},y_{m-2})| + s^{m-n-1}|d(y_{m-2},y_{m-1})| + s^{m-n}|d(y_{m-1},y_m)|.$$

By using equation (0.2), we get

$$|d(y_n,y_m)| \leq s\delta^n|d(y_0,y_1)| + s^2\delta^{n+1}|d(y_0,y_1)| + s^3\delta^{n+2}|d(y_0,y_1)|$$
$$+ \dots + s^{m-n-2}\delta^{m-3}|d(y_0,y_1)| + s^{m-n-1}\delta^{m-2}|d(y_0,y_1)| + s^{m-n}\delta^{m-1}|d(y_0,y_1)|.$$

This implies that

$$|d(y_n,y_m)| \leq (s\delta^n + s^2\delta^{n+1} + s^3\delta^{n+2} + \dots + s^{m-n-2}\delta^{m-3} + s^{m-n-1}\delta^{m-2} +$$
$$s^{m-n}\delta^{m-1})|d(y_0,y_1)|.$$

This implies that

$$|d(y_n,y_m)| \leq (s^n\delta^n + s^{n+1}\delta^{n+1} + s^{n+2}\delta^{n+2} + \dots + s^{m-3}\delta^{m-3} + s^{m-2}\delta^{m-2} +$$
$$s^{m-1}\delta^{m-1})|d(y_0,y_1)|$$
$$= \sum_{t=n}^{m-1} s^t\delta^t|d(y_0,y_1)|$$
$$\leq \sum_{t=n}^{\infty}(s\delta)^t|d(y_0,y_1)|$$
$$= \frac{(s\delta)^n}{1-s\delta}|d(y_0,y_1)|.$$

Hence, $|d(y_n,y_m)| = \frac{(s\delta)^n}{1-s\delta}|d(y_0,y_1)| \to 0$ as $m,n \to \infty$, since $s\delta < 1$.

Thus, $\{y_n\}$ is a Cauchy sequence in X.

Since X is a complete, therefore there exist a point $z \in X$.

Such that $\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$.

Now since $B(X)$ is closed sub space of X and so $z \in B(X)$.

Since $B(X) \subseteq S(X)$, then there exists a point $u \in X$, such that $z = Su$.

Now we show that $Au = Su = z$, by condition (ii) of Theorem (3.1), we have

$$d(Au,z) \precsim s[d(Au,Bx_{2n+1}) + d(Bx_{2n+1},z)].$$

This implies that

$$\frac{1}{s}d(Au,z) \precsim \alpha d(Su,Tx_{2n+1}) + \frac{\beta d(Su,Au)d(Bx_{2n+1},Tx_{2n+1})}{1+d(Su,Tx_{2n+1})} + \frac{\gamma d(Su,Bx_{2n+1})d(Au,Tx_{2n+1})}{1+d(Su,Tx_{2n+1})}$$
$$+ d(Bx_{2n+1},z).$$

Letting $n \to \infty$, we have

$$\frac{1}{s}d(Au,z) \precsim \alpha d(z,z) + \beta(0) + \gamma(0)$$
$$\frac{1}{s}d(Au,z) = 0 \ \ or \ \ \Rightarrow |d(Au,z)| = 0 \ \ or \ \ Au = z.$$

Thus $Au = Su = z$.

$\Rightarrow u$ is a coincidence point of (A,S).

Since $A(X) \subseteq T(X)$ and now $z \in A(X)$, then there exists a point $v \in X$ such that $z = Tv$.

Now we show that $Bv = z$.

By condition (ii) of Theorem (3.1) and by $Au = Su = Tv = z$, we have

$$d(Au,Bv) = d(z,Bv) \precsim s[d(z,Ax_{2n}) + d(Ax_{2n},Bv)], \quad we \ \ have$$
$$\frac{1}{s}d(z,Bv) \precsim d(z,Ax_{2n}) + \alpha d(Sx_{2n},Tv) + \frac{\beta d(Sx_{2n},Ax_{2n})d(Bv,Tv)}{1+d(Sx_{2n},Tv)} + \frac{\gamma d(Sx_{2n},Bv)d(Ax_{2n},Tv)}{1+d(Sx_{2n},Tv)}$$
$$\Rightarrow \frac{1}{s}d(z,Bv) \precsim d(z,y_{2n}) + \alpha d(y_{2n-1},Tv) + \frac{\beta d(y_{2n-1},y_{2n})d(Bv,Tv)}{1+d(y_{2n-1},Tv)} + \frac{\gamma d(y_{2n-1},Bv)d(y_{2n},Tv)}{1+d(y_{2n-1},Tv)}.$$

Taking $n \to \infty$, we have

$\Rightarrow |d(z,Bv)| = 0 \Rightarrow Bv = z$. Hence $Bv = Tv = z$.

$\Rightarrow v$ is a coincidence point of (B,T).

Now we have $Au = Su = Tv = Bv = z$.

Since A and S are weakly compatible mapping then $ASu = SAu = Az = Sz$.

Now we show that z is a fixed point of A, $\ i.e.\ Az = z$.

If $Az \neq z$ then by condition (ii) of Theorem (3.1)

$$d(Az,z) \precsim s[d(Az,Bx_{2n+1}) + d(Bx_{2n+1}),z)]$$

$$\Rightarrow \frac{1}{s}d(Az,z) \precsim \alpha d(Sz,Tx_{2n+1}) + \frac{\beta d(Sz,Ax_{2n})d(Bx_{2n+1},Tx_{2n+1})}{1+d(Sz,Tx_{2n+1})} + \frac{\gamma d(Sz,Bx_{2n+1})d(Az,Tx_{2n+1})}{1+d(Sz,Tx_{2n+1})} + d(Bx_{2n+1},z)$$

$$\precsim \alpha d(Sz,y_{2n}) + \frac{\beta d(Sz,Az)d(y_{2n+1},y_{2n})}{1+d(Sz,y_{2n})} + \frac{\gamma d(Sz,y_{2n+1})d(Az,y_{2n})}{1+d(Sz,y_{2n})} + d(y_{2n+1},z).$$

Taking $n \to \infty$, we have

$$\frac{1}{s}d(Az,z) \precsim \alpha d(Sz,z) + \frac{\beta d(Sz,Az)d(z,z)}{1+d(Sz,z)} + \frac{\gamma d(Sz,z)d(Az,z)}{1+d(Sz,z)} + d(z,z)$$

$$\precsim \alpha d(Sz,z) + \frac{\gamma d(Sz,z)d(Az,z)}{1+d(Sz,z)}.$$

$$Or \quad \frac{1}{s}|d(Az,z)| \leq \alpha|d(Az,z)| + \gamma\left|\frac{d(Sz,z)}{1+d(Sz,z)}\right||d(Az,z)|$$

$$\Rightarrow \frac{1}{s}|d(Az,z)| \leq (\alpha+\gamma)|d(Az,z)|$$

$$\Rightarrow |d(Az,z)| \leq s(\alpha+\gamma)|d(Az,z)|, \quad since \quad s(\alpha+\gamma)+\beta < 1,$$

$$\Rightarrow |d(Az,z)| = 0 \Rightarrow Az = z. \quad Therefore \quad Az = Sz = z.$$

Since B and T are weakly compatible mapping then, $BTv = TBv \Rightarrow Bv = Tv = z$, we have $Bz = Tz$.

Now $d(Bz,z) \precsim s[d(Bz,Ax_{2n}) + d(Ax_{2n},z)]$

$$\Rightarrow \frac{1}{s}d(Bz,z) \precsim d(Ax_{2n},Bz) + d(Ax_{2n},z)$$

$$\precsim \alpha d(Sx_{2n},Tz) + \frac{\beta d(Sx_{2n},Ax_{2n})d(Bz,Tz)}{1+d(Sx_{2n},Tz)} + \frac{\gamma d(Sx_{2n},Bz)d(Ax_{2n},Tz)}{1+d(Sx_{2n},Tz)} + d(x_{2n},z)$$

$$\precsim \alpha d(y_{2n-1},Tz) + \frac{\beta d(y_{2n-1},y_{2n})d(Bz,Tz)}{1+d(y_{2n-1},Tz)} + \frac{\gamma d(y_{2n-1},Bz)d(y_{2n},Tz)}{1+d(y_{2n-1},Tz)} + d(y_{2n},z).$$

Taking $n \to \infty$, we have

$$\frac{1}{s}|d(Bz,z)| \leq \alpha|d(z,Tz)| + \beta\left|\frac{d(z,z)d(Bz,Tz)}{1+d(z,Tz)}\right| + \gamma\left|\frac{d(z,Bz)d(z,Tz)}{1+d(z,Tz)}\right| + |d(z,z)|$$

$$\leq \alpha|d(z,Tz)| + \gamma|d(z,Tz)|$$

$$\leq (\alpha+\gamma)|d(z,Tz)|$$

$\Rightarrow |d(Bz,z)| \leq s(\alpha+\gamma)|d(z,Tz)|$, since $s(\alpha+\gamma)+\beta<1$.

$\Rightarrow |d(Bz,z)| = 0 \Rightarrow Bz = z.$

Therefore $Tz = Bz = z$.

Therefore z is a common fixed point of A,B,S and T.

Uniqueness: Let $w(\neq z)$ be another fixed point of A,B,S and T.

Then $Aw = Bw = Sw = Tw = w$.

$$\begin{aligned} d(z,w) &= d(Az,Bw) \\ &\precsim \alpha d(Sz,Tw)+\beta\frac{d(Sz,Az)d(Bw,Tw)}{1+d(Sz,Tw)}+\gamma\frac{d(Sz,Bw)d(Az,Tw)}{1+d(Sz,Tw)} \end{aligned}$$

$$|d(z,w)| \leq \alpha|d(z,w)|+\gamma\left|\frac{d(z,w)}{1+d(z,w)}\right|.|d(z,w)|$$

$$\Rightarrow |d(z,w)| \leq (\alpha+\gamma)|d(z,w)|, \ \ since \ \ s(\alpha+\gamma)+\beta<1, s\geq 1, \ \ therefore \ \ \alpha+\gamma<1.$$

$$\Rightarrow |d(z,w)| = 0 \Rightarrow z = w.$$

Hence z is a unique common fixed point of A,B,S and T.

Following example substantiates the genuineness of our result.

Example 3.1. Let (X,d) be a complex valued $b-$ metric space, where $X=[0,1]$ and $d: X\times X \to C$ with $d(x,y) = |x-y|^2+i|x-y|^2$.

Now to find s, we have

$$\begin{aligned} d(x,y) &= |x-y|^2+i|x-y|^2 \\ &\precsim |(x-z)+(z-y)|^2+i|(x-z)+(z-y)|^2 \\ &\precsim [|x-z|^2+|z-y|^2+2|x-z||z-y|]+i[|x-z|^2+|z-y|^2+2|x-z||z-y|] \\ &\precsim [|x-z|^2+|z-y|^2+|x-z|^2+|z-y|^2]+i[|x-z|^2+|z-y|^2+|x-z|^2+|z-y|^2] \\ &= 2\{[|x-z|^2+i|x-z|^2]+[|z-y|^2+i|z-y|^2]\} \end{aligned}$$

$$that \ is \ \ d(x,y) \precsim 2[d(x,z)+d(z,y)], \ \ where \ \ s=2.$$

Define A,B,S and $T: X \to X$ by $Ax = \frac{x}{24}, Bx = \frac{x^2}{32}, Sx = \frac{x}{3}$ and $Tx = \frac{x^2}{4}$.

Before discussing different cases, one needs to notice that

$$0 \precsim d(Ax,By), d(Sx,Ty), \frac{d(Sx,Ax)d(By,Ty)}{1+d(Sx,Ty)}, \frac{d(Sx,By)d(Ax,Ty)}{1+d(Sx,Ty)} \ \ in \ \ all \ \ aspects.$$

It is sufficient to show that $d(Ax,By) \precsim \alpha d(Sx,Ty), \forall x,y \in [0,1]$ and $s(\alpha+\gamma)+\beta<1$, $\alpha,\gamma,\beta \geq 0$.

$$d(Ax,By) = [|Ax-By|^2 + i|Ax-By|^2]$$

$$= \left[\left|\frac{x}{24}-\frac{y^2}{32}\right|^2 + i\left|\frac{x}{24}-\frac{y^2}{32}\right|^2\right]$$

$$= \frac{1}{64}\left[\left|\frac{x}{3}-\frac{y^2}{4}\right|^2 + i\left|\frac{x}{3}-\frac{y^2}{4}\right|^2\right] \tag{0.3}$$

$$d(Sx,Ty) = [|Sx-Ty|^2 + i|Sx-Ty|^2]$$

$$= \left[\left|\frac{x}{3}-\frac{y^2}{4}\right|^2 + i\left|\frac{x}{3}-\frac{y^2}{4}\right|^2\right]. \tag{0.4}$$

Following cases for $x,y \in [0,1]$ are discussed with $\alpha = \frac{1}{10}, \beta = \frac{1}{5}, \gamma = \frac{1}{20}$ and $s=2$.
Notice that $s(\alpha+\gamma)+\beta = 2\left(\frac{1}{10}+\frac{1}{20}\right)+\frac{1}{5} < 1$.

Case-I: For $x=0, y=0$.

Putting these values in equation (0.3) and (0.4) we find that $d(Ax,By) \precsim \alpha d(Sx,Ty)$ as $\frac{1}{64}\left[\left|\frac{x}{3}-\frac{y^2}{4}\right|^2 + i\left|\frac{x}{3}-\frac{y^2}{4}\right|^2\right] \precsim \alpha\left[\left|\frac{x}{3}-\frac{y^2}{4}\right|^2 + i\left|\frac{x}{3}-\frac{y^2}{4}\right|^2\right] \Rightarrow 0 \precsim 0$.

Case-II: For $x=1, y=0$.

From equation (0.3) and (0.4) we obtain $d(Ax,By) \precsim \alpha d(Sx,Ty)$ since

$$\frac{1}{576}[1+i] \precsim \frac{1}{90}[1+i]$$

Thus condition (ii) of Theorem (3.1) is satisfied.

Case-III: For $x=\frac{1}{2}, y=\frac{1}{4}$. $d(Ax,By) \precsim \alpha d(Sx,Ty)$ is true as
$\frac{1}{64}\left[\left|\frac{29}{192}\right|^2 + i\left|\frac{29}{192}\right|^2\right] \precsim \alpha\left[\left|\frac{29}{192}\right|^2 + i\left|\frac{29}{192}\right|^2\right]$
$\frac{1}{2359296}[1+i] \precsim \frac{1}{36864}[1+i]$.

Case-IV: For $x=1, y=1$. We get, $d(Ax,By) \precsim \alpha d(Sx,Ty)$ as
$\frac{1}{64}\left[\left|\frac{1}{3}-\frac{1}{4}\right|^2 + i\left|\frac{1}{3}-\frac{1}{4}\right|^2\right] \precsim \alpha\left[\left|\frac{1}{3}-\frac{1}{4}\right|^2 + i\left|\frac{1}{3}-\frac{1}{4}\right|^2\right]$
$\frac{1}{64}\left[\left|\frac{1}{12}\right|^2 + i\left|\frac{1}{12}\right|^2\right] \precsim \alpha\left[\left|\frac{1}{12}\right|^2 + i\left|\frac{1}{12}\right|^2\right]$
$\frac{1}{9216}[1+i] \precsim \frac{1}{1440}[1+i]$.

Thus, all conditions of Theorem (3.1) are satisfied. Notice that the point $0 \in X$ remains fixed under mappings A,B,S and T and is indeed unique.

If we put $S=T=I$ (Identity mapping) in Theorem (3.1) we get the following Corollary.

Corollary 3.1. *Let A and B be two self-mappings of complete complex valued b-metric spaces (X,d) satisfying:*

$$d(Ax,By) \precsim \alpha d(x,y) + \beta \frac{d(x,Ax)d(y,By)}{1+d(x,y)} + \gamma \frac{d(x,By)d(y,Ax)}{1+d(x,y)}.$$

$\forall\ x,y \in X$, *where* α,β *and* γ *are non-negative reals such that* $s(\alpha+\gamma)+\beta < 1$.

Then A and B have a unique common fixed point.

Remark 3.1. If we put $\gamma = 0$ in Corollary (3.1) then we get Theorem (2.1) of Aiman A. Mukheimer [2].

Remark 3.2. If we put $s = 1$ in Corollary (3.1) then we get Theorem (2.1) of Rouzkard and Imdad [3].

Remark 3.3. If we put $s = 1$ and $\gamma = 0$ in Corollary (3.1) then we get Theorem (4) of Azam A. *et al.* [1].

If we set $S = T$ in Theorem (3.1), then we get another corollary.

Corollary 3.2. Let A,B and S be three self-mappings of a complete complex valued b-metric spaces (X,d) satisfying

(i) $A(X) \subseteq S(X)$ and $B(X) \subseteq S(X)$,

(ii) $d(Ax,By) \precsim \alpha d(Sx,Sy) + \beta \frac{d(Sx,Ax)d(By,Sy)}{1+d(Sx,Sy)} + \gamma \frac{d(Sx,By)d(Ax,Sy)}{1+d(Sx,Sy)}$

$\forall\ x,y \in X$, where α,β and γ are non-negative reals such that $s(\alpha+\gamma)+\beta < 1$.

(iii) If pairs (A,S) and (B,S) are weakly compatible and $B(X)$ is a closed subspace of X.

Then A,B and S have a unique common fixed point.

Theorem 3.2. *Let A,B,S and T be four self-mappings of a complete complex valued b-metric spaces (X,d) satisfying*

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,

(ii)

$$d(Ax,By) \precsim \alpha d(Sx,Ty) + \beta \frac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,By)+d(Ty,Ax)+d(Sx,Ty)} \tag{0.5}$$

$\forall\ x,y \in X$, *where* α,β *are non-negative reals such that* $s(\alpha+s\beta) < 1$.

(iii) Pairs (A,S) and (B,T) are weakly compatible and $B(X)$ is a closed subspace of X.

Then A,B,S and T have a unique common fixed point.

Proof. Consider a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \ \ and \ \ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, \tag{0.6}$$

where $\{x_n\}$ is another sequence in X.

To prove $\{y_n\}$ is a Cauchy sequence of X,

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Ax_{2n}, Bx_{2n+1}) \\
&\precsim \alpha d(Sx_{2n}, Tx_{2n+1}) + \beta \frac{d(Sx_{2n}, Ax_{2n}) d(Tx_{2n+1}, Bx_{2n+1})}{1 + d(Sx_{2n}, Bx_{2n+1}) + d(Tx_{2n+1}, Ax_{2n}) + d(Sx_{2n}, Tx_{2n+1})} \\
&\precsim \alpha d(y_{2n-1}, y_{2n}) + \beta \frac{d(y_{2n-1}, y_{2n}) d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n})} \\
&\precsim \alpha d(y_{2n-1}, y_{2n}) + \beta \frac{d(y_{2n-1}, y_{2n}) d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n+1}) + d(y_{2n-1}, y_{2n})}
\end{aligned}$$

$$\begin{aligned}
or \ \ |d(y_{2n}, y_{2n+1})| &\leq \alpha |d(y_{2n-1}, y_{2n})| + \beta \left| \frac{d(y_{2n-1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n}) + d(y_{2n-1}, y_{2n})} \right| |d(y_{2n}, y_{2n+1})| \\
&\leq \alpha |d(y_{2n-1}, y_{2n})| + s\beta |d(y_{2n-1}, y_{2n})| \left| \frac{d(y_{2n}, y_{2n-1}) + d(y_{2n-1}, y_{2n+1})}{1 + d(y_{2n}, y_{2n-1}) + d(y_{2n-1}, y_{2n+1})} \right|.
\end{aligned}$$

$$Since, \quad \left| \frac{d(y_{2n}, y_{2n-1}) + d(y_{2n-1}, y_{2n+1})}{1 + d(y_{2n}, y_{2n-1}) + d(y_{2n-1}, y_{2n+1})} \right| < 1.$$

Then we have

$$|d(y_{2n}, y_{2n+1})| \leq (\alpha + s\beta) |d(y_{2n-1}, y_{2n})|. \tag{0.7}$$

Similarly

$$|d(y_{2n+1}, y_{2n+2})| \leq (\alpha + s\beta) |d(y_{2n}, y_{2n+1})|. \tag{0.8}$$

Applying the similar argument as in Theorem (3.1) we arrive at

$$|d(y_n, y_m)| = \frac{(s\delta)^n}{1 - s\delta} |d(y_0, y_1)|.$$

Hence, $|d(y_n, y_m)| = \frac{(s\delta)^n}{1-s\delta} |d(y_0, y_1)| \to 0$ as $m, n \to \infty$, since $s\delta < 1$. Thus, $\{y_n\}$ is a Cauchy sequence in X.

Since X is a complete, therefore there exist a point $z \in X$.

Such that $\lim\limits_{n\to\infty} Ax_{2n} = \lim\limits_{n\to\infty} Tx_{2n+1} = \lim\limits_{n\to\infty} Bx_{2n+1} = \lim\limits_{n\to\infty} Sx_{2n+2} = z$.

Now since $B(X)$ is closed sub space of X and so $z \in B(X)$.

Since $B(X) \subseteq S(X)$, then there exist a point $u \in X$, such that $z = Su$.

Now we show that $Au = Su = z$, by equation (0.5), we have

$$d(Au,z) \precsim s[d(Au,Bx_{2n+1}) + d(Bx_{2n+1},z)]$$

$$\frac{1}{s}d(Au,z) \precsim \alpha d(Su,Tx_{2n+1}) + \frac{\beta d(Su,Au)d(Tx_{2n+1},Bx_{2n+1})}{1+d(Su,Bx_{2n+1})+d(Tx_{2n+1},Au)+d(Su,Tx_{2n+1})} + d(Bx_{2n+1},z).$$

Letting $n \to \infty$, we have

$$\frac{1}{s}d(Au,z) \precsim \alpha d(Su,z) + \frac{\beta d(Su,Au)d(z,z)}{1+d(Su,z)+d(z,Au)+d(Su,z)} + d(z,z)$$

$$\Rightarrow \frac{1}{s}d(Au,z) = 0 \ \ or \ \ \Rightarrow |d(Au,z)| = 0 \ \ or \ \ Au = z.$$

Thus $Au = Su = z$.

$\Rightarrow u$ is a coincidence point of (A,S).

Since $A(X) \subseteq T(X)$ and now $z \in A(X)$, then there exist a point $v \in X$ such that $z = Tv$.

Now we show that $Bv = z$. By equation (0.5) and by $Au = Su = Tv = z$, we have

$$d(z,Bv) \precsim s[d(z,Ax_{2n}) + d(Ax_{2n}),Bv)],$$

$$\frac{1}{s}d(z,Bv) \precsim d(z,Ax_{2n}) + \alpha d(Sx_{2n},Tv) + \frac{\beta d(Sx_{2n},Ax_{2n})d(Tv,Bv)}{1+d(Sx_{2n},Bv)+d(Tv,Ax_{2n})+d(Sx_{2n},Tv)}$$

Letting $n \to \infty$, we have

$\frac{1}{s}d(z,Bv) \precsim d(z,z) + \alpha d(z,Tv) + \frac{\beta d(z,z)d(Bv,Tv)}{1+d(z,Bv)+d(Tv,z)+d(z,Tv)}$

$\Rightarrow |d(z,Bv)| = 0 \Rightarrow Bv = z.$

Hence $Bv = Tv = z$.

$\Rightarrow v$ is a coincidence point of (B,T).

Now we have $Au = Su = Tv = Bv = z$.

Since A and S are weakly compatible mapping then $ASu = SAu = Az = Sz$.

Now we show that z is a fixed point of A.

On contrary if $Az \neq z$ then by equation (0.5)

$$d(Az,z) \precsim s[d(Az,Bx_{2n+1}) + d(Bx_{2n+1},z)]$$

$$\frac{1}{s}d(Az,z) \precsim \alpha d(Sz,Tx_{2n+1}) + \frac{\beta d(Sz,Az)d(Tx_{2n+1},Bx_{2n+1})}{1+d(Sz,Bx_{2n+1})+d(Tx_{2n+1},Sz)+d(Sz,Tx_{2n+1})} + d(Bx_{2n+1},z)$$

$$\frac{1}{s}d(Az,z) \precsim \alpha d(Sz,y_{2n}) + \frac{\beta d(Sz,Az)d(y_{2n+1},y_{2n})}{1+d(Sz,y_{2n+1})+d(y_{2n},Sz)+d(Sz,y_{2n})} + d(y_{2n+1},z)$$

Taking $n \to \infty$,

$$\frac{1}{s}d(Az,z) \precsim \alpha d(Sz,z) + \frac{\beta d(Sz,Az)d(z,z)}{1+d(Sz,z)+d(z,Az)+d(Sz,z)} + d(z,z)$$

$$\frac{1}{s}d(Az,z) \precsim \alpha d(Sz,z)$$

$$|d(Az,z)| \leq \alpha s|d(Az,z)|, \quad since \quad s(\alpha+s\beta)<1$$

$$\Rightarrow |d(Az,z)| = 0 \Rightarrow Az = z.$$

$$Therefore \quad Az = Sz = z.$$

Since B and T are weakly compatible mapping then, $BTv = TBv \Rightarrow Bv = Tv = z$, we have $Bz = Tz$.

We show that z is a fixed point of B. On contrary if $Bz \neq z$, then by equation (0.5).

$$d(Bz,z) \precsim s[d(Bz,Ax_{2n}) + d(Ax_{2n},z)]$$

$$\Rightarrow \frac{1}{s}d(Bz,z) \precsim d(Ax_{2n},Bz) + d(Ax_{2n},z)$$

$$\frac{1}{s}d(Bz,z) \precsim \alpha d(Sx_{2n},Tz) + \frac{\beta d(Sx_{2n},Ax_{2n})d(Tz,Bz)}{1+d(Sx_{2n},Bz)+d(Tz,Ax_{2n})+d(Sx_{2n},Tz)} + d(Ax_{2n},z)$$

$$\frac{1}{s}d(Bz,z) \precsim \alpha d(y_{2n-1},Tz) + \frac{\beta d(y_{2n-1},y_{2n})d(Tz,Bz)}{1+d(y_{2n-1},Bz)+d(Tz,y_{2n})+d(y_{2n-1},Tz)} + d(y_{2n},z).$$

$Taking \ n \to \infty$,

$$\frac{1}{s}|d(Bz,z)| \leq \alpha|d(z,Tz)| + \beta\left|\frac{d(z,z)d(Bz,Tz)}{1+d(z,Bz)+d(Tz,z)+d(z,Tz)}\right| + |d(z,z)|$$

$$\frac{1}{s}|d(Bz,z)| \leq \alpha|d(Bz,z)|$$

$$|d(Bz,z)| \leq \alpha s|d(Bz,z)|. \ Since \ s(\alpha+s\beta)<1.$$

$$\Rightarrow |d(Bz,z)| = 0 \Rightarrow Bz = z.$$

Therefore $Tz = Bz = z$.

Therefore z is a common fixed point of A, B, S and T.

Uniqueness: Let w be another fixed point of A, B, S and T.

Then $Aw = Bw = Sw = Tw = w$.

$$\begin{aligned} d(z,w) &= d(Az,Bw) \\ &\precsim \alpha d(Sz,Tw) + \beta \frac{d(Sz,Az)d(Bw,Tw)}{1+d(Sz,Bw)+d(Tw,Az)+d(Sz,Tw)} \end{aligned}$$

$$|d(z,w)| \leq \alpha |d(z,w)|, Since \;\; s(\alpha + s\beta) < 1.$$

$$\Rightarrow |d(z,w)| = 0 \Rightarrow z = w.$$

Hence z is a unique common fixed point of A, B, S and T.

Example 3.2. Let (X,d) be a complex valued $b-$ metric space, where $X = [0,1]$ and $d: X \times X \to C$ with $d(x,y) = |x-y|^2 + i|x-y|^2$.

Now to find s, we have

$$\begin{aligned} d(x,y) &= |x-y|^2 + i|x-y|^2 \\ &\precsim |(x-z)+(z-y)|^2 + i|(x-z)+(z-y)|^2 \\ &\precsim [|x-z|^2 + |z-y|^2 + 2|x-z||z-y|] + i[|x-z|^2 + |z-y|^2 + 2|x-z||z-y|] \\ &\precsim [|x-z|^2 + |z-y|^2 + |x-z|^2 + |z-y|^2] + i[|x-z|^2 + |z-y|^2 + |x-z|^2 + |z-y|^2] \\ &= 2\{[|x-z|^2 + i|x-z|^2] + [|z-y|^2 + i|z-y|^2]\} \end{aligned}$$

$$d(x,y) \precsim 2[d(x,z) + d(z,y)]. \;\; Here \;\; s = 2.$$

Define A, B, S and $T : X \to X$ by $Ax = \frac{x}{6}, Bx = \frac{x^2}{12}, Sx = x$ and $Tx = \frac{x^2}{2}$.

Before discussing different cases, one needs to notice that

$$0 \precsim d(Ax,By), d(Sx,Ty), \frac{d(Sx,Ax)d(By,Ty)}{1+d(Sx,By)+d(Ty,Ax)+d(Sx,Ty)} \;\; in \;\; all \;\; aspects.$$

It is sufficient to show that $d(Ax,By) \precsim \alpha d(Sx,Ty), \forall x,y \in [0,1]$ and $s(\alpha + s\beta) < 1$, $\alpha, \beta \geq 0$.

$$\begin{aligned} d(Ax,By) &= [|Ax-By|^2 + i|Ax-By|^2] \\ &= \left[\left|\frac{x}{6} - \frac{y^2}{12}\right|^2 + i\left|\frac{x}{6} - \frac{y^2}{12}\right|^2\right] \\ &= \frac{1}{36}\left[\left|x - \frac{y^2}{12}\right|^2 + i\left|x - \frac{y^2}{12}\right|^2\right] \\ d(Sx,Ty) &= [|Sx-Ty|^2 + i|Sx-Ty|^2] \\ &= \left[\left|x - \frac{y^2}{2}\right|^2 + i\left|x - \frac{y^2}{2}\right|^2\right]. \end{aligned}$$

For $\alpha = \frac{1}{4}, \beta = \frac{1}{10}$, and $s = 2$, following cases for $x, y \in [0,1]$ are discussed

Case-I: For $x = 0, y = 0$.

We find that $d(Ax,By) \precsim \alpha d(Sx,Ty)$ as
$\frac{1}{36}\left[\left|x - \frac{y^2}{12}\right|^2 + i\left|x - \frac{y^2}{12}\right|^2\right] \precsim \alpha\left[\left|x - \frac{y^2}{2}\right|^2 + i\left|x - \frac{y^2}{2}\right|^2\right] \Rightarrow 0 \precsim 0.$
Which satisfied the condition (0.5) of Theorem (3.2)

Case-II: For $x = 1, y = 0$.

$d(Ax,By) \precsim \alpha d(Sx,Ty)$ is true as $\frac{1}{36}[|1|^2 + i] \precsim \alpha[|1|^2 + i] \Rightarrow \frac{1}{36}[1+i] \precsim \frac{1}{4}[1+i]$.

Similarly, for the **Case-III** when $x = \frac{1}{2}, y = \frac{1}{4}$ and **Case-IV** for $x = 1, y = 1$.

We get, $d(Ax,By) \precsim \alpha d(Sx,Ty)$.

Which satisfied the condition (0.5) of Theorem (3.2).

Thus, all conditions of Theorem (3.2) are satisfied. Notice that the point $0 \in X$ remains fixed under mappings A, B, S and T and is indeed unique.

If we set $S = T$ in Theorem 3.2 then we get the following Corollary.

Corollary 3.3. *Let A, B and S be three self-mappings of a complete complex valued b-metric spaces (X,d) satisfying*

(i) $A(X) \subseteq S(X)$ and $B(X) \subseteq S(X)$,

(ii) $d(Ax,By) \precsim \alpha d(Sx,Sy) + \beta \frac{d(Sx,Ax)d(Sx,By)}{1+d(Sx,By)+d(Sy,Ax)+d(Sx,Sy)}$

$\forall\, x, y \in X$. where α, β are non-negative reals such that $s(\alpha + s\beta) < 1$.

(iii) If pairs (A,S) and (B,S) are weakly compatible and $B(X)$ is a closed subspace of X.

Then A, B and S have a unique common fixed point.

If we set $S = T = I$ (Identity mapping) in Theorem (3.2) then we get the following Corollary.

Corollary 3.4. *Let A and B be two self-mappings of a complete complex valued b-metric spaces (X,d) satisfying:*

$d(Ax,By) \precsim \alpha d(x,y) + \beta \frac{d(x,Ax)d(y,By)}{1+d(x,By)+d(y,Ax)+d(x,y)}$

$\forall\, x, y \in X$. where α, β are non-negative reals such that $s(\alpha + s\beta) < 1$.

Then A and B have a unique common fixed point.

Next, the role of denominator is discussed, resulting following theorem.

Theorem 3.3. *Let A,B,S and T be four self-mappings of a complete complex valued b-metric spaces (X,d) satisfying*

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,

(ii) $d(Ax,By) \precsim \alpha d(Sx,Ty) + \beta \frac{d(Sx,Ax)d(By,Ty)}{d(Sx,By)+d(Ty,Ax)+d(Sx,Ty)}$

$\forall\, x,y \in X$ such that $x \neq y, d(Sx,By)+d(Ty,Ax)+d(Sx,Ty) \neq 0$, where α,β are non-negative reals with $s(\alpha+s\beta)<1$ or $d(Ax,By)=0$ if $d(Sx,By)+d(Ty,Ax)+d(Sx,Ty)=0$.

(iii) Pairs (A,S) and (B,T) are weakly compatible and $B(X)$ is a closed subspace of X.

Then A,B,S and T have a unique common fixed point.

Proof. Proof follows immediately as consequence of previous results.

If we set $S=T$ in Theorem (3.3) then we get the following Corollary.

Corollary 3.5. *Let A,B and S be three self-mappings of a complete complex valued b-metric spaces (X,d) satisfying*

(i) $A(X) \subseteq S(X)$ and $B(X) \subseteq S(X)$,

(ii) $d(Ax,By) \precsim \alpha d(Sx,Sy) + \beta \frac{d(Sx,Ax)d(Sy,By)}{d(Sx,By)+d(Sy,Ax)+d(Sx,Sy)}$

$\forall\, x,y \in X$ such that $x \neq y, d(Sx,By)+d(Sy,Ax)+d(Sx,Sy) \neq 0$, where α,β are non-negative reals with $s(\alpha+s\beta)<1$ or $d(Ax,By)=0$ if $d(Sx,By)+d(Sy,Ax)+d(Sx,Sy)=0$.

(iii) Pairs (A,S) and (B,S) are weakly compatible and $B(X)$ is a closed subspace of X.

Then A,B and S have a unique common fixed point.

If we put $S=T=I$ (Identity mapping) in Theorem (3.3), then we get the following Corollary.

Corollary 3.6. *Let A and B be two self-mappings of a complete complex valued b-metric spaces (X,d) satisfying:*

$$d(Ax,By) \precsim \alpha d(x,y) + \beta \frac{d(x,Ax)d(y,By)}{d(x,By)+d(y,Ax)+d(x,y)}$$

$\forall\, x,y \in X$ such that $x \neq y, d(x,By)+d(y,Ax)+d(x,y) \neq 0$, where α,β are non-negative reals with $s(\alpha+s\beta)<1$ or $d(Ax,By)=0$ if $d(x,By)+d(y,Ax)+d(x,y)=0$.

Then A and B have a unique common fixed point.

Remark 3.4. If we put $S=1$ in Corollary 3.6 then we get, Theorem 3.1 of Nashine, Imdad, Hasan [4].

Conflict of Interests

The authors declare that there is no conflict of interests.

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