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A CONSTRUCTIVE VERSION OF KY FAN'S COINCIDENCE THEOREM

YASUHITO TANAKA*

Faculty of Economics, Doshisha University, Kamigyo-ku, Kyoto, 602-8580, Japan

Abstract. In this paper we examine Ky Fan's coincidence theorem from the point of view of Bishop style constructive mathematics using constructive versions of Kakutani's fixed point theorem, Berge's maximum theorem and the separating hyperplane theorem.

Keywords: Ky Fan's coincidence theorem; Constructive mathematics.

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1. Introduction

We examine Ky Fan's coincidence theorem from the point of view of constructive mathematics à la Bishop ([2], [4], [5]) using constructive (approximate) versions of Kakutani's fixed point theorem, Berge's maximum theorem ([1]) and the separating hyperplane theorem ([5]).

In constructive mathematics a nonempty set is called an *inhabited* set. A set S is inhabited if there exists an element of S.

*Corresponding author

E-mail address: yasuhito@mail.doshisha.ac.jp (Y. Tanaka)

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Note that in order to show that S is inhabited, we cannot just prove that it is impossible for S to be empty: we must actually construct an element of S (see page 12 of [5]).

Also in constructive mathematics compactness of a set means *total boundedness with completeness*. A set S is finitely enumerable if there exist a natural number N and a mapping of the set $\{1, 2, \ldots, N\}$ onto S. An ε-approximation to S is a subset of S such that for each $x \in S$ there exists y in that ε-approximation with $\rho(x, y) < \varepsilon(\rho(x, y)$ is the distance between x and $y)$. S is totally bounded if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to S. Completeness of a set, of course, means that every Cauchy sequence in the set converges.

The constructive version of Ky Fan's coincidence theorem, which we will prove, is as follows;

Let $\delta > 0$, X be an inhabited, convex, compact subset of $\mathbf{R}^n$, and let $F,\ G$ be two totally bounded valued multi-functions (multi-valued functions or correspondences) with uniformly closed graph from X to $\mathbf{R}^n$ such that

for any $x \in X$ and any $p \in \mathbf{R}^n$ for which $p{\cdot}x \leq \inf\{p{\cdot}y|y \in X\}{+}\delta$, there exist $z \in F(x)$ and $w \in G(x)$ such that $p \cdot z > p \cdot w - \delta$.

Then for each $\eta > 0$ there exists $x^* \in X$ for which

$$\inf_{a \in F(x^*),\ b \in G(x^*)} \rho(a, b) < \eta,$$

and so for each $\varepsilon > \eta$, $U(F(x^*), \varepsilon) \cap U(G(x^*), \varepsilon)$ is inhabited, that is,

$$U(F(x^*), \varepsilon) \cap U(G(x^*), \varepsilon) \neq \emptyset.$$

Uniformly closed graph property is a stronger version of closed graph property. We define it in the next section.

In the next section we introduce a constructive version of the maximum theorem. In Section 3 we prove a constructive version of Kakutani's fixed point theorem, and in Section 4 we prove a constructive version of Ky Fan's coincidence theorem using these theorems and a constructive version of the separating hyperplane theorem according to [5].

2. Constructive version of the maximum theorem

In classical mathematics Berge's maximum theorem (see [1], [3], [6]) is expressed as follows:

> Let X, Y be metric spaces, f be a continuous function from $X \times Y$ to $\mathbf{R}$, and let F be a compact-valued continuous (upper and lower hemi-continuous) multi-function from X to the set of nonempty subsets of Y. Consider a maximization problem:
>
> (1) $$\text{maximize } f(x, y) \text{ subject to } y \in F(x).$$
>
> Then, this has a solution, and
>
> (1) the function φ defined by $\varphi(x) = \max_{y \in F(x)} f(x, y)$ from X to $\mathbf{R}$, is continuous in X, and
>
> (2) the multi-function $\Phi = \{y \in F(x) | f(x, y) = \varphi(x)\}$ from X to the set of nonempty subsets of Y is upper hemi-continuous (has a closed graph).

In constructive mathematics, however, we cannot prove that the maximization problem (1) has a solution in a compact set $F(x)$ even if f is uniformly continuous with respect to y in $F(x)$. Instead we can prove that f has the supremum in $F(x)$ (see Corollary 2.2.7 in [5]).

We present some definitions. Let X and Y be metric spaces.

Definition 2.1. *A function f from X to Y is uniformly continuous if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in X$, if $\rho(x, x') < \delta$, then $\rho(f(x), f(x')) < \varepsilon$. The number δ depends on only ε.*

Definition 2.2. *The graph of a multi-function F from X to the set of inhabited subsets of Y is*

$$G(F) = \cup_{x \in X} \{x\} \times F(x).$$

If $G(F)$ is a closed set, we say that F has a closed graph. It implies the following fact:

If $(x_n)_{n\geq 1}$ *and* $(y_n)_{n\geq 1}$ *are sequences such that for each* n $y_n \in F(x_n)$*, and if* $x_n \longrightarrow x$*, then there exists* $y \in F(x)$ *such that* $y_n \longrightarrow y$.

This means

If, for each $\varepsilon > 0$*, there exists* n_0 *such that* $\rho(x_n, x) < \varepsilon$ *when* $n \geq n_0$*, then for each* $\varepsilon > 0$*, there exists* n_0' *such that* $\rho(y_n, F(x)) < \varepsilon$*, that is,* $\rho(y_n, y) < \varepsilon$ *for some* $y \in F(x)$*, when* $n \geq n_0'$.

The numbers n_0 and n_0' depend on x and ε. Further we require a uniform version of this property, and call such a multi-function *a multi-function with uniformly closed graph*, or say that a *multi-function has a uniformly closed graph*. It means that n_0 and n_0' depend on only ε not on x. In this case we say that if $x_n \longrightarrow x$, then $y_n \longrightarrow y$ uniformly in Definition 2.2.

We define continuity of multi-functions in this paper as follows:

Definition 2.3. *A multi-function* F *from* X *to the set of inhabited subsets of* Y *is continuous if*

(1) *it has a uniformly closed graph, and*

(2) *For every sequence* $(x_n)_{n\geq 1}$ *such that* $x_n \longrightarrow x$ *and* $y \in F(x)$*, there exist a sequence* $(y_n)_{n\geq 1}$ *such that* $y_n \in F(x_n)$ *and* $y_n \longrightarrow y$.

This means

If, for each $\varepsilon > 0$*, there exists* n_0 *such that* $\rho(x_n, x) < \varepsilon$ *when* $n \geq n_0$*, then for each* $\varepsilon > 0$*, there exists* n_0' *such that* $\rho(y_n, y) < \varepsilon$ *when* $n \geq n_0'$.

The numbers n_0 *and* n_0' *depend on* x *and* ε*. Further we require a uniform version of this property. It means that* n_0 *and* n_0' *depend on only* ε *not on* x.

This condition corresponds to lower-hemicontinuity in classical mathematics.

As stated above, if $F(x)$ is compact and f is uniformly continuous, the supremum

$$\sup_{y\in F(x)} f(x, y) \tag{2}$$

of f in $F(x)$ exists. We define a function $\varphi : X \to \mathbf{R}$ by

$$\varphi(x) = \sup_{y\in F(x)} f(x, y).$$

We also define a multi-function Φ from X to the set of inhabited subsets of Y by

$$\Phi(x) = \{y \in F(x) | \varphi(x) \geq f(x,y) \geq \varphi(x) - \varepsilon\}$$

with $\varepsilon > 0$.

We show the following theorem.

Theorem 2.1. *Let X, Y be metric spaces, f be a uniformly continuous function from $X \times Y$ to $\mathbf{R}$, and let F be a compact-valued continuous multi-function from X to the set of inhabited subsets of Y. Let φ and Φ be defined above. Then*

(1) *φ is uniformly continuous in X, and*

(2) *Φ has a uniformly closed graph.*

Proof. Consider sequences $(x_n)_{n\geq 1}$ in X and $(y_n)_{n\geq 1}$ in Y such that $y_n \in \Phi(x_n)$, $x_n \longrightarrow x$ and $y_n \longrightarrow y$ uniformly. To say that $y_n \in \Phi(x_n)$ means that $y_n \in F(x_n)$ and $f(x_n, y_n) \geq \varphi(x_n) - \varepsilon$. Since F is a continuous multi-function, we have $y \in F(x)$, and for every $y' \in F(x)$ there exist sequences $(x_n)_{n\geq 1}$ and $(y'_n)_{n\geq 1}$ such that $y'_n \in \Phi(x_n)$, $x_n \longrightarrow x$ and $y'_n \longrightarrow y'$ uniformly. Assume $f(x, y') > f(x, y) + \varepsilon$. For all sufficiently large n we have (note $f(x_n, y'_n) \leq \varphi(x_n)$)

$$f(x_n, y_n) \geq \varphi(x_n) - \varepsilon \geq f(x_n, y'_n) - \varepsilon > f(x_n, y_n) + \varepsilon - \varepsilon = f(x_n, y_n),$$

which is absurd. Hence $\varphi(x) - \varepsilon \leq f(x, y') \leq f(x, y) + 2\varepsilon$, and $\varphi(x) - 3\varepsilon \leq f(x, y)$. Thus $y \in \Phi(x)$, and Φ has a uniformly closed graph.

Consider $x, x' \in X$ and $y \in \Phi(x)$, $y' \in \Phi(x')$. We have $\rho(\varphi(x), f(x, y)) \leq \varepsilon$ and $\rho(\varphi(x'), f(x', y')) \leq \varepsilon$. Therefore, if $\rho(f(x, y), f(x', y')) < \varepsilon$, we have $\rho(\varphi(x), \varphi(x')) < 3\varepsilon$. It means that φ is uniformly continuous because f is uniformly continuous.

This completes the proof.

3. Constructive version of Kakutani's fixed point theorem

It is well known that Brouwer's fixed point theorem cannot be constructively proved. On the other hand, Sperner's lemma which is used to prove Brouwer's theorem, however, can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's fixed point theorem using Sperner's lemma (See [9] and [10]). Let x be a point in a compact metric space X, and consider a uniformly continuous function f from X into itself. According to [9] and [10] f has an ε-approximate fixed point; that is, for each $\varepsilon > 0$, there exists $x \in X$ such that $\rho(x, f(x)) < \varepsilon$[1].

Consider an n-dimensional simplex Δ as a compact metric space. Let F be a compact and convex-valued multi-function F from Δ to the set of its inhabited subsets. Denote the distance between $F(x)$ and $x \in \Delta$ by $\rho(F(x), x)$, that is,

$$\rho(F(x), x) = \inf_{y \in F(x)} \rho(y, x).$$

This exists since $F(x)$ is a compact subset of a compact metric space, and so it is located (see [5]). An inhabited subset S of a metric space X is called located if for each $x \in X$ the distance

$$\rho(x, S) = \inf_{s \in S} \rho(x, s)$$

exists.

We define an ε-approximate fixed point of a multi-function F as follows:

Definition 3.1. *For each $\varepsilon > 0$ x is an ε-approximate fixed point of a multi-function F if $\rho(x, F(x)) < \varepsilon$.*

The approximate version of Kakutani's fixed point theorem is as follows:

Theorem 3.1. *If F is a compact and convex-valued multi-function with uniformly closed graph from an n-dimensional simplex Δ to the set of its inhabited subsets, then for each ε, F has an ε-approximate fixed point.*

Proof. Let Δ be an n-dimensional simplex, and consider m-th subdivision of Δ. Subdivision in the case of 2-dimensional simplex is illustrated in Figure 1.

Consider sufficiently fine partition of Δ, and define a uniformly continuous function $f^m : \Delta \longrightarrow \Delta$ as follows. If x is a vertex of a simplex constructed by m-th subdivision of Δ, $f^m(x) = y$ for some $y \in F(x)$. For other $x \in \Delta$ we define f^m by a convex combination

[1] We have also presented proofs of an approximate version of Brouwer's fixed point theorem in [7] and [8].

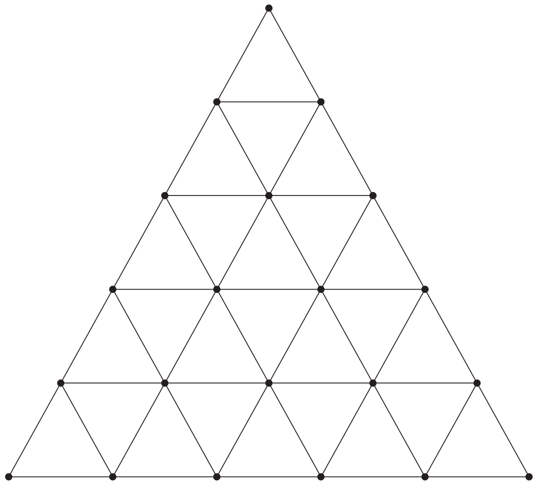

FIGURE 1. Subdivision of 2-dimensional simplex

of the values of F at vertices of the simplex x_0^m, x_1^m, $\dots$, x_n^m. Let $\sum_{i=0}^n \lambda_i = 1$, $\lambda_i \geq 0$,

$$f^m(x) = \sum_{i=0}^n \lambda_i f^m(x_i^m) \text{ with } x = \sum_{i=0}^n \lambda_i x_i^m.$$

Since f^m is clearly uniformly continuous, it has an ε-approximate fixed point. Let x^* be an ε-approximate fixed point of f^m, then for each $\frac{\varepsilon}{2} > 0$ there exists $x^* \in \Delta$ which satisfies

$$\rho(x^*, f^m(x^*)) < \frac{\varepsilon}{2}.$$

If the partition of Δ is sufficiently fine, the distance between vertices of a simplex, $\rho(x_i^m, x_j^m)$, $i \neq j$, is sufficiently small. Since F has a uniformly closed graph, for each $y_i \in F(x_i^m)$ and some $y_j \in F(x_j^m)$ we have $\rho(y_i, y_j) < \frac{\varepsilon}{2}$, and for each $y_j \in F(x_j^m)$ and some $y_i \in F(x_i^m)$ we have $\rho(y_i, y_j) < \frac{\varepsilon}{2}$. Since x^* is expressed as $x^* = \sum_{i=0}^n \lambda_i x_i^m$, if $\rho(x_i^m, x_j^m)$ is sufficiently small for each i and j, $\rho(x^*, x_i^m)$ is also sufficiently small for each i. Therefore, for each $y_i \in F(x_i^m)$ and some $y_i^* \in F(x^*)$ we have $\rho(y_i, y_i^*) < \frac{\varepsilon}{2}$. y_i^*'s for different x_i^m's may be different. But, since $F(x^*)$ is convex,

$$y^* = \sum_{i=0}^n \lambda_i y_i^* \in F(x^*).$$

Since, for each i $\rho(y_i, y_i^*) < \frac{\varepsilon}{2}$ and $f^m(x^*) = \sum_{i=0}^n \lambda_i f^m(x_i^m) = \sum_{i=0}^n \lambda_i y_i$,we have

$$\rho(f^m(x^*), y^*) < \frac{\varepsilon}{2}.$$

From $\rho(x^*, f^m(x^*)) < \frac{\varepsilon}{2}$, we obtain

$$\rho(x^*, y^*) < \varepsilon.$$

This means

$$\rho(x^*, F(x^*)) < \varepsilon.$$

Thus, x^* is an ε-approximate fixed point of F.

This completes the proof.

If a set X is homeomorphic to Δ, then a multi-function from X to the set of inhabited subsets of X with uniformly closed graph has ε-approximate fixed points.

4. Constructive version of Ky Fan's coincidence theorem

In this section we will prove a constructive (an approximate) version of Ky Fan's coincidence theorem. We follow the procedures of the proof in [6].

First we introduce the following constructive separating hyperplane theorem according to Theorem 5.2.9 in [5] and Theorem 4.3 in Chapter 7 of [2].

> Let F and G be bounded convex subsets of a separable normed linear space X, whose algebraic difference $\{y-x: \ x \in F, \ y \in G\}$ is located, and whose mutual distance
>
> $$d = \inf\{\rho(y, x): \ x \in F, \ y \in G\}$$
>
> is positive. Then for each $\varepsilon > 0$ there exists a normable linear functional u on X of norm 1 such that
>
> $$u(y) > u(x) + d - \varepsilon.$$

If F and G are totally bounded, the algebraic difference between F and G is located because the function $(x, y) \to \rho(y, x)$ is uniformly continuous on the totally bounded set $F \times G$ and so maps that set onto a totally bounded set. Let X be a subset of $\mathbf{R}^n$. Then, we get the following theorem.

Theorem 4.1. *Let F and G be totally bounded convex subsets of $\mathbf{R}^n$, whose mutual distance*

$$d = \inf\{\rho(y,x) :\ x \in F,\ y \in G\}$$

is positive. Then, for each $\varepsilon > 0$ there exists a vector $p \in \mathbf{R}^n$ with $||p|| = 1$ such that

$$p \cdot y > p \cdot x + d - \varepsilon\ (x \in F, y \in G).$$

Assume $d > \varepsilon$, and let $\delta = d - \varepsilon$. Then, $p \cdot y > p \cdot x + \delta$.

Next we prove the following lemma which is based on Lemma 3.3.1 in [6].

Lemma 4.1. *Let X be an inhabited, convex, compact subset of $\mathbf{R}^n$, and let f be a uniformly continuous function from X to $\mathbf{R}^n$. Then, for each $\varepsilon > 0$ there exists $x^* \in X$ such that $f(x^*) \cdot x^* \leq f(x^*) \cdot x + \varepsilon$ for all $x \in X$.*

Proof. For each $x \in X$ we define a multi-functions Φ from X to the set of subsets of X by

$$\Phi(x) = \left\{ y \in X | \forall y' \in X\ \left(f(x) \cdot y \leq f(x) \cdot y' + \frac{\varepsilon}{2} \right) \right\}.$$

$\Phi(x)$ is inhabited and convex. By the constructive version of the maximum theorem the multi-function Φ has a uniformly closed graph, and by the constructive version of Kakutani's fixed point theorem, for each $\delta > 0$ there exists $x^* \in X$ such that $\rho(x^*, \Phi(x^*)) < \delta$. For sufficiently small δ, since $f(x) \cdot x$ is uniformly continuous, we have $f(x^*) \cdot x^* \leq f(x^*) \cdot x + \varepsilon$ for all $x \in X$.

This completes the proof.

Now we prove the following theorem which is based on Theorem 3.3.2 and Theorem 3.3.3 in [6].

Theorem 4.2. *Let $\delta > 0$, X be an inhabited, convex, compact subset of $\mathbf{R}^n$, and let F, G be two totally bounded valued multi-functions with uniformly closed graph from X to the set of subsets of $\mathbf{R}^n$ such that:*

> *for any $x \in X$ and any $p \in \mathbf{R}^n$ for which $p \cdot x \leq \inf\{p \cdot y | y \in X\} + \delta$, there exist $u \in F(x)$ and $v \in G(x)$ such that $p \cdot u \geq p \cdot v - \delta$.*

Then for each $\eta > 0$ there exists $x^* \in X$ for which

$$\inf_{a \in F(x^*),\ b \in G(x^*)} \rho(a, b) < \eta,$$

and so for each $\varepsilon > \eta$, the set $U(F(x^*), \varepsilon) \cap U(G(x^*), \varepsilon)$ is inhabited.

Proof. Let $\delta > 0$. For each $p \in \mathbf{R}^n$ define

$$P(p) = \{x \in X | \forall u \in F(x)\ \forall v \in G(x)\ \ (p \cdot u + \delta < p \cdot v)\},$$

and denote by $\bar{P}(p)$ the closure of $P(p)$ in X. Then, $\bar{P}(p)$ is a totally bounded subset of X. Suppose that the assertion of this theorem is false. Then, for any $x \in X$

$$d = \inf_{a \in F(x),\ b \in G(x)} \rho(a, b) > 0,$$

and by Theorem 4.1 there exists $p_x \in \mathbf{R}^n$ and $t_x \in \mathbf{R}$ such that

$$F(x) \subset \{u \in \mathbf{R}^n | p_x \cdot u + \delta < t_x\}, \text{and } G(x) \subset \{v \in \mathbf{R}^n | p_x \cdot v > t_x\};$$

hence $x \in \bar{P}(p_x)$. By the uniformly closed graph property of F and G, for $x' \in X$ such that $\rho(x, x')$ is positive and sufficiently small,

$$F(x') \subset \{u \in \mathbf{R}^n | p_x \cdot u + \delta < t_x\}, \text{and } G(x') \subset \{v \in \mathbf{R}^n | p_x \cdot v > t_x\}; \tag{3}$$

Since X is totally bounded, for each $\gamma > 0$ there exists a finitely enumerable γ-approximation to X, $\{x_1, x_2, \ldots, x_m\}$, such that for any $y \in X$ $\rho(y, x_i) < \gamma$ for some $x_i \in \{x_1, x_2, \ldots, x_m\}$. Let $0 < \gamma < \rho(x, x')$ for some x' which satisfies (3) for each x, and $\rho(x, x') < \xi$ for $\xi > 0$. Then X is covered by $\bar{P}(p_{x_1})$, $\bar{P}(p_{x_2})$, $\ldots$, $\bar{P}(p_{x_m})$, where $\bar{P}(p_{x_i})$ is a closure of

$$P(p_{x_i}) = \{x \in X | \forall u \in F(x_i)\ \forall v \in G(x_i)\ \ (p_{x_i} \cdot u + \delta < p_{x_i} \cdot v)\}$$

for each i. Thus $X \subset \bigcup_{i=1}^m \bar{P}(p_{x_i})$. Then, according to Theorem 6.15 in Chapter 4 of [2] there exist nonnegative continuous functions $g_1, g_2, \ldots, g_m$ on X such that $\sum_{i=1}^m g_i \leq 1$ and $\sum_{i=1}^m g_i(x) = 1$ for all x in X, and each g_i is positive only in $\bar{P}(p_{x_i})$. Define a continuous function from X to $\mathbf{R}^n$ by $f(y) = \sum_{i=1}^m g_i(y) p_{x_i}$. Then for all $y \in X$, $u \in F(x)$, $v \in G(x)$, it follows that $f(y) \cdot u + \delta < f(y) \cdot v$. By Lemma 4.1, however, there exists $y^* \in X$ such that $f(y^*) \cdot y^* \leq f(y^*) \cdot y + \delta$ for all $y \in X$, so in view of the assumption of this theorem, there exist $u^* \in F(y^*)$ and $v^* \in G(y^*)$ such that $f(y^*) \cdot u^* \geq f(y^*) \cdot v^* - \delta$, a contradiction. Therefore, there exists $x^* \in X$ for which

$$\inf_{a \in F(x^*),\ b \in G(x^*)} \rho(a, b) < \eta,$$

and so for each $\varepsilon > \eta$ $U(F(x^*), \varepsilon) \cap U(G(x^*), \varepsilon)$ is inhabited.

This completes the proof.

The coincidence theorem is widely used in mathematical economics and game theory, for example, a proof of the existence of a core in an NTU (non-transferable utility) game.

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