Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 4, 1078-1085
ISSN: 1927-5307

PERIODIC SOLUTION OF A NONLINEAR PROBLEM BY ELLIPTIC REGULARIZATION TECHNIQUES

MEFLAH MABROUK*

University of Kasdi Merbah Ouargla, Algeria

Faculty of Sciences, Department of Mathematics,

Abstract: In this paper we investigate for the existence and uniqueness of a function u = u(x,t), x∈ Ω, t∈]0,T[weak periodic solution of the nonlinear boundary value problem by the elliptic regularization techniques based on theory of monotone operators.

Keywords: Elliptic regularization, monotone operators, periodic solution, priori estimates.

2000 AMS Subject Classification: 35Q60

1. Introduction

G.Prodi [12]investigated the existence of the equation $u'' + u' - \Delta u + u + |u'|u' = f$ and Lions [6] study the system of equations. The author [7], [8] study similar problems governed by Lamé operator. In this work we study the fallowing problem.

$$(P)\begin{cases} u'' + u' - \Delta u + |u'|^{p-2}u' = f & in\ Q & (1'.1) \\ u = 0 & on\ \Sigma & (1'.2) \\ u(x,0) = u(x,T) & \forall\ x \in \Omega & (1'.3) \\ u'(x,0) = u'(x,T) & \forall\ x \in \Omega & (1'.4) \end{cases} \quad (1.1)$$

Where Ω is an open bounded domain of $IR^n_,$ with regular boundary Γ. We denote by Q the cylinder $IR^n_x \times IR_t$: Q = Ω ×]0,T[, with boundary Σ, h and f are functions. We look

*Corresponding author

E-mail address: meflah@ymail.com. (M. Mabrouk)

This research was supported by the Laboratory of applied mathematics, Kasdi Merbah university, Ouargla, Algeria.

Received March 5, 2012

for the existence and uniqueness of a function u = u(x,t), $x \in \Omega$, $t \in]0,T[$, solution of the problem.

2. Preliminaries:

We use the standard Lebesgue space $L^p(\Omega)$ and the Sobolev spaces with their usual products and norms.

Definition 2.1 Let V be a Banach space, V' be the topological dual of V. We define the operator B_δ: $V \to V'$ and we say that B_δ is:

1) Monotonous, if: $\langle B_\delta(u) - B_\delta(v), u - v\rangle \geq 0 \quad \forall\, u, v \in V$

2) Strictly monotonous , if: $\langle B_\delta(u) - B_\delta(v), u - v\rangle > 0 \quad \forall\, u, v \in V$

3) Coercive, if: $\lim_{\|v\|_V \to \infty}(\langle B_\delta(v), v\rangle / \|v\|_V) = \infty$

4) Hemi continuous, if: $t \to \langle B_\delta(u + tv), w\rangle$ is continuous in IR

Remark 2.1 The classical method of energy not efficacy, because when we multiply the both side of equation $(1'.1)$ by u′ and integrating on ($\Omega \times]0,T[$) and use the periodicity, we haven't the solution for $(1'.1)$.

3. Main results

The main results of the paper is given by the fallowing theorems

Theorem 3.1 Suppose that Ω is bounded open set in IR^n are given f, with $f \in L^q(Q)$

Then there exists a function $u = w_0 + w$ satisfying (1.1)

$$w_0 \in H_0^1(\Omega) + W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \tag{1.2}$$

$$w \in L^2\big(0,T; H_0^1(\Omega)\big) \tag{1.3}$$

$$w' \in L^p(Q) \tag{1.4}$$

Proof We use an approach of G. Prodi [12] we have

$$\begin{cases} u = w_0 + w \\ w_0 \text{ independent of } t \\ \int_0^T w\,dt = 0 \end{cases} \tag{1.5}$$

We introduce the Prodi idea (1. 5) in $(1'.1)$ we having

$$u'' + u' - \Delta u + |u'|^{p-2}u' - f = f + Lu_0 \tag{1.6}$$

We consider the derivative of (1.6) we obtain

$$\frac{d}{dt}(u'' + u' - \Delta u + |u'|^{p-2}u') = \frac{df}{dt} \tag{1.7}$$

And

$$\begin{cases} \int_0^T u dt = 0 \\ u(T) = u(0) \\ u'(x,0) = u'(x,T) \end{cases} \tag{1.8}$$

We deduce to (1.7)

$$u'' + u' - \Delta u + |u'|^{p-2}u' - f = h_0 \ \ with\ h_0\ independent\ of\ t \tag{1.9}$$

For resolve (1.7) and (1.8) we denote. Δ = -A; $\beta(u') = |u'|^{p-2}u'$

We consider the functional space V:

$$V = \begin{cases} v: v\epsilon\, L^2\big(0,T,H_0^1\ (\Omega)\big);\quad v'\epsilon\, L^2\big(0,T,(H_0^1\ (\Omega)\big) \cap L^p(Q); \\ v''\epsilon\, L^2\big(0,T,L^2(\Omega)\big);\ \int_0^T v(t)dt = 0;\ \ v(T) = v(0);\ v'(T) = v'(0) \end{cases} \tag{1.10}$$

The Banach structure of V is defined by

$$\|v\|_V = \|v\|_{L^2\left(0,T,H_0^1\ (\Omega)\right)} + \|v'\|_{L^2\left(0,T,H_0^1\ (\Omega)\right)} + \|v\|_{L^p(Q)} + \|v\|_{L^2\left(0,T,L^2(\Omega)\right)}$$

We define the bilinear form:

$$b(u,v) = \int_0^T [(u'',v) + (u',v) + (Au,v) + (\beta(u'),v)]dt \tag{1.11}$$

The weak formulation of (1.7) and (1.8) is to find $u\epsilon V$ such that

$$b(u,v) = \int_0^T (f,v')dt \quad \forall\ v\epsilon V \tag{1.12}$$

Following some ideas of Lions for obtain the elliptic regularization, given δ > 0 and $u, v\ \epsilon$ we define

$$\pi_\delta(u,v) = \delta\int_0^T [(u'',v'') + (Au',v')]ds + \int_0^T (u'' + u' + Au + \beta(u'),v')ds. \tag{1.13}$$

The application $v \to \pi_\delta(u,v)$ is continuous on V so there exists an application

$\pi_\delta\ \epsilon\ V'$: $\pi_\delta(u,v) = (B_\delta(u),v)$ (1.14)

The linear operator $B_\delta\ :\ V\ \rightarrow V'$ satisfies the properties:

B_δ is bounded in V' for all bounded set in V and is a hemi continuous and is a strictly monotonous and is coercive. In view of these properties and as consequence of theorem of Lions [5], there exist unique a function $u_\delta \epsilon V$:

$$\pi_\delta(u_\delta, v) = \int_0^T (f, v')dt \quad \forall\, v \epsilon V \tag{1.15}$$

A priori estimates I

Explicitly the elliptic regularization (1.15) and setting $v = u_\delta$, we obtain:

$$\delta \int_0^T [\|u'_\delta\|^2 + |u''_\delta|^2]dt + \int_0^T [|u'_\delta|^2 + (\beta(u'_\delta), u'_\delta)]dt = \int_0^T (f, u_\delta)dt \tag{1.16}$$

Or

$$\int_0^T (\beta(u'), u')dt = \|u'\|^p_{L^p(Q)} \; And \int_0^T udt = 0 \Rightarrow \|u\|_{L^2\left(0,T,H_0^1\ (\Omega)\right)} \leq \mathrm{C}\ \|u'\|_{L^2\left(0,T,H_0^1\ (\Omega)\right)}$$

Then u'_δ is bounded in $L^p(Q)$ when $\delta \to 0$ (1.17)

$$\delta \int_0^T [|u''_\delta|^2 + |u'_\delta|^2 + \|u'_\delta\|^2]dt \leq \mathrm{C} \tag{1.18}$$

Or $\int_0^T udt = 0$

We have by (1.17)and (1.18)that u_δ is bounded in $L^p(Q)$ (1.19)

And $\delta \int_0^T \|u_\delta\|^2 dt \leq \mathrm{C}$ (1.20)

A priori estimates II

Exchange in (1.15) v with:

$$v(t) = \int_0^T u_\delta(s)d - \frac{1}{T}\int_0^T (T - s)u_\delta(s)ds \tag{1.21}$$

We verify that:

$$\begin{cases} \int_0^T vdt = 0 \quad \forall v \in V \\ v' = u_\delta \end{cases} \tag{1.22}$$

Taking into account (1.21) in (1.15) we get

$$\delta\int_0^T [(u''_\delta,u'_\delta)+(u'_\delta,u_\delta)+(Au'_\delta,u_\delta)]dt+\int_0^T [(u''_\delta,u_\delta)+(u'_\delta,u_\delta)+\|u_\delta\|^2]dt$$

$$+\int_0^T (\beta(u'_\delta),u'_\delta)dt=\int_0^T (f,u_\delta)dt \qquad (1.23)$$

By using periodicity of $u_\delta, u'_\delta \epsilon\, V$ we obtain:

$$\int_0^T (u''_\delta,u'_\delta)dt=\int_0^T (Au'_\delta,u_\delta)dt=0 \qquad (1.24)$$

And

$$\int_0^T \left(u''_\delta,u_\delta\right)dt=\left(u'_\delta(T),u_\delta(T)\right)-\left(u'_\delta(0),u_\delta(0)\right)-\int_0^T (u'_\delta,u'_\delta)dt$$

$$=-\int_0^T |u'_\delta|^2dt \qquad (1.25)$$

By (1.24) and (1.17) we have

$$\left|\int_0^T \left(u''_\delta,u_\delta\right)dt\right|\leq \mathrm{C} \quad when \ \ \delta\to 0 \qquad (1.26)$$

Also, from (1.17) and (1.19) we obtain:

$$\left|\int_0^T (\beta(u'_\delta),u_\delta)dt\right|\leq \|\beta(u'_\delta)\|_{L^p(Q)}\ \|u_\delta\|_{L^p(Q)}\ \leq \mathrm{C}' \qquad (1.27)$$

Combining (1.24), (1.26), (1.27) with (1.23) we deduce

$$\int_0^T \|u_\delta\|^2 dt\leq \mathrm{C} \qquad (1.28)$$

Passage to the limit

From (1.17) and (1.28) that there exists a subsequence from (u_δ), such that

$$u_\delta\to 0 \quad weak\ in \quad L^2\left(0,T,H_0^1\ (\Omega)\right) \qquad (1.29)$$

$$u'_\delta\to u' \quad weak\ in \quad L^p(Q) \qquad (1.30)$$

$$\beta(u'_\delta)\to\ \chi\ \ weak\ in \quad L^q(Q) \qquad (1.31)$$

Passage to the limit in (1.15) we obtain

$$\int_0^T [(-u',v'')+(Au,v')+(\chi,v')]dt=\int_0^T (f,v')dt \quad \forall v\in V \tag{1.32}$$

Use the convolution technical in (1.32) we have

$$\int_0^T (\chi,u'*\eta_\delta*\eta_\delta)dt=\int_0^T (f,u'*\eta_\delta*\eta_\delta)dt \quad \forall v\in V \tag{1.33}$$

When

$$\int_0^T (\chi,u'')dt=\int_0^T (f,u')dt \qquad \forall v\in V \tag{1.34}$$

Theorem 3.2 We consider two solutions u_1 and u_2 under the hypotheses of the theorem of existence to the problem (P). Then $u_1 = u_2$.

Proof We subtract the equations (1.9) corresponding to u_1 and u_2 and sitting $\phi = u_1 - u_2$ we have:

$$\phi''+\phi'+A\phi+\beta(u_1')-\beta(u_2') \tag{2.1}$$

Denoting by (η_δ) the regularizing sequence a similar argument by Brézis [2] we obtain:

$$\phi'*\eta_\delta*\eta_\delta=\phi*\eta'_\delta*\eta_\delta \tag{2.2}$$

Hence, by using (1.2) and (1.3), we have

$$\phi=\varphi+\phi_0 : \phi_0\in V \textit{ and } \varphi\in L^2\big(0,T,H_0^1\ (\Omega)\big) \tag{2.3}$$

From (2.2) we get

$$\phi'*\eta_\delta*\eta_\delta=\phi*\eta'_\delta*\eta_\delta=\varphi'*\eta_\delta*\eta_\delta \tag{2.4}$$

Show that

$$\int_0^T (\phi'',\phi'*\eta_\delta*\eta_\delta)dt=0 .$$

When

$$\int_0^T \frac{d}{dt}(\phi',\phi'*\eta_\delta*\eta_\delta)d=\int_0^T (\phi'',\phi'*\eta_\delta*\eta_\delta)dt+\int_0^T (\phi',\phi''*\eta_\delta*\eta_\delta)dt$$

$$=2\int_0^T (\phi'',\phi'*\eta_\delta*\eta_\delta)dt=0 \tag{2.5}$$

Therefore

$$\int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta)dt \quad = \frac{1}{2}\int_0^T \frac{d}{dt}(\phi', \phi' * \eta_\delta * \eta_\delta)dt = 0 \tag{2.6}$$

$\phi'$and η_δ periodic then we have

$$\int_0^T (\phi', \phi' * \eta_\delta * \eta_\delta)dt = \int_0^T (A\phi, \phi' * \eta_\delta * \eta_\delta)dt = 0 \tag{2.7}$$

From (2.1); (2.6) and (2.7) we obtain:

$$\int_0^T (\beta(u'_1) - \beta(u'_2), \phi' * \eta_\delta * \eta_\delta) = \ 0 \tag{2.8}$$

Passage to the limit in (2.8) we have

$$\int_0^T (\beta(u'_1) - \beta(u'_2), u'_1 - u'_2)dt = \ 0 \tag{2.9}$$

Where

$$u'_1 - u'_2 = 0 \Rightarrow u'_1 = u'_2 \tag{2.10}$$

This implies that

$\phi = u_1$-$u_1 = \theta$, θ independent of t (2.11)

From (2.7) *and* (2.11) *we obtain*

$$\int_0^T (A\theta, \theta)dt = 0 \quad \forall\theta \in V \tag{2.12}$$

We deduce from (1.2)

$$\theta \in H_0^1\ (\Omega) + W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \tag{2.13}$$

Using theorem of Green we have $(A\theta,\theta) = \|\theta\|^2$. (2.14)

By (2.12) and (2.13) and (2.14) we obtain the uniqueness.

Acknowledgement: The author would like his thanks to the Prof. B. Merouani and LAMA Ferhat Abbas and LMA Kasdi Merbah University.

REFERENCES

[1] Adams, Sobolev Spaces, Academic Press, (1976).

[2] H. Brezis, Analyse fonctionnelle, théorie et applications. Masson (1983).

[3] H. Brezis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, Ann, Ins.Fourier; 18, (1968), 115-175.

[4] G.Duvaut, J.L. Lions, Les inéquations en mécanique et en physique. Dunod. Paris. (1972).

[5] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod. (1969).

[6] Luc Tartar, Topics in non lineair analysis. Université de Paris-Sud, Publications Mathematiques d'Orsay, novembre (1978).

[7] M. Meflah, Study of Nonlinear Elasticity Problem by Elliptic Regularization with Lamé system, Int. J. of Mathematical Archive-2(5), May 2011, Page 693-697, ISSN 2229-5046.

[8] M. Meflah, A Nonlinear Elasticity Problem by Elliptic Regularization Technics, Int. J. Contemp. Math. Sciences, Vol 6, 2011, no. 25, 1221-1229, ISSN 1312-7586

[9] M. Meflah, B Merouani, A Nonlinear Elasticity Problem Governed by Lamé System, Applied Mathematical Sciences, Vol.4, 2010, no. 36,1785-1796.

[10] B. Merouani, M. Meflah, A. Boulaouad, The Generalized and perturbed Lamé system, Applied Mathematical SciencesVol.2, 2008,no. 49,2425-2430

[11] F. Messelmi, B. Merouani, M. Meflah, Nonlinear thermoelasticity problem, Analele Universitatii Oradea, Fasc. Matematica,Tom XV (2008), 207-217.

[12] G. Prodi, Soluzioni periodiche dell'equazione delle onde com termine dissipativo nonlineare. Rend. Sem.Mat. Pandova, 35 (1965).

[13] V. Patron, P. Perline, Méthode de la théorie mathématique de l'élasticité, Editions Mir, Moscou, 1981.

[14] Taylor M.E, Partial differential non linear equations III Springer. (1996).

[15] K. Yosida, Functional Analysis, Springer-Verlag Berlin Heidelberg New York 1968.