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RECURRENCE RELATIONS FOR MOMENTS OF LOWER GENERALIZED ORDER STATISTICS FROM EXPONENTIATED LOMAX DISTRIBUTION AND ITS CHARACTERIZATION

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Abstract: In this paper, recurrence relations for single and product moments of generalized order statistics from Exponentiated Lomax Distribution have been obtained. Specializations to order statistics and records have been made. Further, using a recurrence relation for single moments we obtain characterization of Exponentiated Lomax Distribution.

Keywords: Generalized order statistics - Order statistics – Records –Single and product moments – Recurrence relations - Exponentiated Lomax Distribution - Characterization.

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1. Introduction

A random variable X is said to have Exponentiated Lomax Distribution (*ELD*) if its probability density function (*pdf*) is given by (Abdul-Moniem and Abdel-Hameed [1]):

$$f(x)=\alpha\theta\lambda\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha-1}\left[1+\lambda x\right]^{-(\theta+1)};\ x>0,\ \alpha,\theta \text{ and } \lambda>0, \quad (1)$$

and the corresponding cumulative distribution function (CDF) is

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$$F(x) = \left[1 - (1 + \lambda x)^{-\theta}\right]^{\alpha}; \ x > 0, \ \alpha, \theta \text{ and } \lambda > 0. \tag{2}$$

Therefore, from (1) and (2), we have

$$F(x) = \frac{1}{\alpha\theta\lambda}\left[\theta\lambda x + \sum_{i=2}^{\theta+1}\binom{\theta+1}{i}(\lambda x)^{i}\right] f(x), \ \theta \text{ is positive integer}. \tag{3}$$

Not that: From (1), we can get the pdf for exponentiated Pareto, Pareto and Lomax distributions by taking $\lambda = 1$, $\lambda = \alpha = 1$ and $\alpha = 1$ respectively. More details on this distribution can be found in Abdul-Moniem and Abdel-Hameed [1].

The concept of generalized order statistics (*gos*) was introduced by Kamps [4] as a unified distribution theoretical set-up which contains a variety of models of ordered random variables with different interpretations. But when $F(\ )$ is an inverse distribution function, we need a concept of lower generalized order statistics (*lgos*), which was introduced by Pawlas and Szynal [12] as follows:

Let $n \in N$, $k \geq 1$, $m \in \Re$, , be the parameters such that

$\gamma_r = k + (n - r)(m + 1) > 0$, for all $0 \leq r \leq n$.

By the *lgos* from an absolutely continuous distribution function $F(x)$ with density function $f(x)$ we mean random variables $X'(1, n, m, k), \ldots, X'(n, n, m, k)$ having joint *pdf* of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[F(x_i)\right]^m f(x_i)\right)\left[F(x_n)\right]^{k-1} f(x_n), \tag{4}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

The *pdf* of r^{th} *lgos* is given by

$$f_{X'(r,n,m,k)}(x) = \frac{C_{r-1}}{\Gamma(r)}\left[F(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1}\left[F(x)\right], \ x \in \chi \tag{5}$$

where χ is the domain on which $f_{X'(r,n,m,k)}(x)$ is positive.

The joint *pdf* of r^{th} and s^{th} *lgos* is

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x, y) = \frac{C_{s-1}}{\Gamma(r)\Gamma(s-r)}\left[F(x)\right]^m f(x) g_m^{r-1}\left[F(x)\right]$$

$$\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}\left[F(y)\right]^{\gamma_s-1}f(y), \quad x>y, \tag{6}$$

where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i, \quad \gamma_i=k+(n-i)(m+1),$$

$$h_m(x)=\begin{cases}\dfrac{-1}{m+1}x^{m+1}, & m\neq -1\\ -\ln x, & m=-1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(1), \quad x\in[0,\ 1).$$

We shall also take $X'(0,n,m,k)=0$. If m = 0, k = 1, then $X'(r,n,m,k)$ reduces to the $(n-r+1)^{th}$ order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, ..., X_n$ and when m = -1, then $X'(r,n,m,k)$ reduces to the r^{th} k-lower record value (Pawlas and Szynal [12]).

Recurrence relations for single and product moments of *lgos* from the inverse Weibull distribution are derived by Pawlas and Szynal [12]. Khan and Kumar [6, 7, 8] discussed *lgos* from the exponentiated Pareto, exponentiated Gamma and generalized exponential distributions respectively. Khan et al. [9] have established recurrence relations for moments of *lgos* from exponentiated Weibull distribution. Recurrence relations for single and product moments of *lgos* from the Frechet-type extreme value distribution are derived by Kumar [10]. Ahsanullah [2] and Mbah and Ahsanullah [11] characterized the uniform and power function distributions based on distributional properties of *lgos* respectively. Kamps [4] investigated the importance of recurrence relations of order statistics in characterization.

In this paper, we have established explicit expressions and some recurrence relations for single and product moments of *lgos* from *ELD*. Result for order statistics and r^{th} lower record values are deduced as special cases and a characterization of *ELD* has been obtained on using a recurrence relation for single moments.

2. Explicit expression for single moments of *lgos* for *ELD*

The single moments of *lgos* for *ELD* can be obtained from (1), (2) and (5) (when $m \neq -1$) as follows:

$$E\left[X'^{j}\left(r,n,m,k\right)\right]=\frac{\alpha\theta\lambda C_{r-1}}{\left(m+1\right)^{r-1}\Gamma\left(r\right)}\int_{0}^{\infty}x^{j}\left[1-\left(1+\lambda x\right)^{-\theta}\right]^{\alpha\gamma_{r}-1}$$

$$\left\{1-\left[1-\left(1+\lambda x\right)^{-\theta}\right]^{\alpha\left(m+1\right)}\right\}^{r-1}\left(1+\lambda x\right)^{-\left(\theta+1\right)}dx$$

Expanding $\left\{1-\left[1-\left(1+\lambda x\right)^{-\theta}\right]^{\alpha\left(m+1\right)}\right\}^{r-1}$ binomially, we get

$$E\left[X'^{j}\left(r,n,m,k\right)\right]=\frac{\alpha\theta\lambda C_{r-1}}{\left(m+1\right)^{r-1}\Gamma\left(r\right)}\sum_{i=0}^{r-1}\binom{r-1}{i}\left(-1\right)^{i}$$

$$\int_{0}^{\infty}x^{j}\left[1-\left(1+\lambda x\right)^{-\theta}\right]^{\alpha\left[\gamma_{r}+\left(m+1\right)i\right]-1}\left(1+\lambda x\right)^{-\left(\theta+1\right)}dx$$

Using the transformation $z=1-\left(1+\lambda x\right)^{-\theta}$, we get

$$E\left(X'^{j}\left(r,n,m,k\right)\right)=\frac{\alpha\lambda^{-j}C_{r-1}}{\left(m+1\right)^{r-1}\Gamma\left(r\right)}\sum_{i=0}^{r-1}\binom{r-1}{i}\left(-1\right)^{i}\int_{0}^{1}z^{\alpha\left[\gamma_{r}+\left(m+1\right)i\right]-1}\left[\left(1-z\right)^{\frac{-1}{\theta}}-1\right]^{j}dz$$

Expanding $\left[\left(1-z\right)^{\frac{-1}{\theta}}-1\right]^{j}$ binomially, we get

$$E\left(X'^{j}\left(r,n,m,k\right)\right)=\frac{\alpha\lambda^{-j}C_{r-1}}{\left(m+1\right)^{r-1}\Gamma\left(r\right)}\sum_{i=0}^{r-1}\sum_{a=0}^{j}\binom{r-1}{i}\binom{j}{a}\left(-1\right)^{i+j-a}\int_{0}^{1}z^{\alpha\left[\gamma_{r}+\left(m+1\right)i\right]-1}\left(1-z\right)^{\frac{-a}{\theta}}dz$$

$$=\frac{\alpha\lambda^{-j}C_{r-1}}{\Gamma\left(r\right)\left(m+1\right)^{r-1}}\sum_{i=0}^{r-1}\sum_{a=0}^{j}\binom{r-1}{i}\binom{j}{a}\left(-1\right)^{i+j-a}\sum_{b=0}^{\infty}\frac{\left(\frac{a}{\theta}\right)_{(b)}}{b!\left\{\alpha\left[\gamma_{r}+\left(m+1\right)i\right]+b\right\}},$$

$$\theta>j \text{ and } j=0,1,\ldots \quad (7)$$

where $\beta_{(i)}=\begin{cases}\beta\left(\beta+1\right)\ldots\left(\beta+i-1\right) & i>0\\ 1 & i=0\end{cases}$.

and when $m=-1$ that

$$E\left(X'^{j}\left(r,n,-1,k\right)\right)=\frac{\left(\alpha k\right)^{r}\lambda^{-j}}{\Gamma\left(r\right)}\sum_{a=0}^{j}\binom{j}{a}\left(-1\right)^{j-a}\sum_{b=0}^{\infty}\frac{\left(\frac{a}{\theta}\right)_{(b)}}{b!}\int_{0}^{\infty}t^{r-1}e^{-t(\alpha k+b)}dt$$

$$E\left(X'^{j}\left(r,n,-1,k\right)\right)=\left(\alpha k\right)^{r}\lambda^{-j}\sum_{a=0}^{j}\binom{j}{a}\left(-1\right)^{j-a}\sum_{b=0}^{\infty}\frac{\left(\frac{a}{\theta}\right)_{(b)}}{\left(\alpha k+b\right)^{r}b!}. \tag{8}$$

Note that: We can obtain the single moments of *lgos* for exponentiated Pareto distribution by taking $\lambda=1$ in (7) and (8), established by Khan and Kumar [6].

Special cases:

(1) The j^{th} moments of lower order statistics can be obtained by taking $m=0$, $k=1$ in (7) as follows

$$E\left(X'^{j}_{n-r+1:n}\right)=\alpha\lambda^{-j}C_{r:n}\sum_{i=0}^{r-1}\sum_{a=0}^{j}\binom{r-1}{i}\binom{j}{a}\left(-1\right)^{i+j-a}\sum_{b=0}^{\infty}\frac{\left(\frac{a}{\theta}\right)_{(b)}}{b!\left[\alpha\left(n-r+1+i\right)+b\right]}, \tag{9}$$

where $C_{r:n}=\dfrac{n!}{(r-1)!(n-r)!}$.

(2) The moments of lower record values can be obtained by taking $k=1$ in (8) as follows:

$$E\left(X'^{j}\left(r,n,-1,1\right)\right)=\alpha^{r}\lambda^{-j}\sum_{a=0}^{j}\binom{j}{a}\left(-1\right)^{j-a}\sum_{b=0}^{\infty}\frac{\left(\frac{a}{\theta}\right)_{(b)}}{\left(\alpha+b\right)^{r}b!}. \tag{10}$$

(3) We can obtain the moments of lower record values for the exponentiated Pareto distribution by taking $\lambda=1$ in (10), established by Shawky and Abu-Zinadah [13].

Recurrence relations for single moments of *lgos* from *ELD* can be obtained in the following theorems, when θ is positive integer.

The following an important relations proved by Khan et al. [9] which will be used to prove the following theorems.

For $2 \le r \le n$ and $k = 1, 2, \ldots$

$$E\left[X'^{j}(r,n,m,k)\right] - E\left[X'^{j}(r-1,n,m,k)\right]$$

$$= -\frac{j\ C_{r-1}}{\gamma_r \Gamma(r)} \int_{\alpha}^{\beta} x^{j-1} \left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right] dx \qquad (11)$$

$$E\left[X'^{j}(r-1,n,m,k)\right] - E\left[X'^{j}(r-1,n-1,m,k)\right]$$

$$= \frac{(m+1)\, jC_{r-1}}{\gamma_1 \Gamma(r-1)} \int_{\alpha}^{\beta} x^{j-1} \left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right] dx \qquad (12)$$

$$E\left[X'^{j}(r,n,m,k)\right] - E\left[X'^{j}(r-1,n-1,m,k)\right]$$

$$= -\frac{j\ C_{r-1}}{\gamma_1 \Gamma(r)} \int_{\alpha}^{\beta} x^{j-1} \left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right] dx \qquad (13)$$

Theorem 2.1 For *ELD* and for $2 \le r \le n$ and $k = 1, 2, \ldots$

$$E\left[X'^{j}(r-1,n,m,k)\right]$$

$$= \frac{j}{\theta\alpha\gamma_r} \sum_{i=2}^{\theta+1} \binom{\theta+1}{i} \lambda^{i-1} E\left[X'^{j+i-1}(r,n,m,k)\right] + \left(\frac{j}{\alpha\gamma_r} + 1\right) E\left[X'^{j}(r,n,m,k)\right] \qquad (14)$$

Proof From (3) and (11), we have

$$E\left[X'^{j}(r,n,m,k)\right] - E\left[X'^{j}(r-1,n,m,k)\right]$$

$$= -\frac{j\ C_{r-1}}{\alpha\gamma_r \Gamma(r)} \int_0^{\infty} x^{j-1} \left[\bar{F}(x)\right]^{\gamma_r - 1} \left[x + \frac{1}{\theta\lambda} \sum_{i=2}^{\theta+1} \binom{\theta+1}{i} (\lambda x)^i\right] f(x) g_m^{r-1}\left[F(x)\right] dx$$

$$= -\frac{j\ C_{r-1}}{\alpha\gamma_r \Gamma(r)} \left\{ \int_0^{\infty} x^{j} \left[\bar{F}(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1}\left[F(x)\right] dx \right.$$
$$\left. + \frac{1}{\theta} \sum_{i=2}^{\theta+1} \binom{\theta+1}{i} \lambda^{i-1} \int_0^{\infty} x^{j+i-1} \left[\bar{F}(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1}\left[F(x)\right] dx \right\}$$

$$=-\frac{j}{\alpha\gamma_r}\left\{E\left[X'^{j}(r,n,m,k)\right]+\frac{1}{\theta}\sum_{i=2}^{\theta+1}\binom{\theta+1}{i}\lambda^{i-1}E\left[X'^{j+i-1}(r,n,m,k)\right]\right\}$$

This is implies that

$$E\left[X'^{j}(r-1,n,m,k)\right]$$

$$=\frac{j}{\theta\alpha\gamma_r}\sum_{i=2}^{\theta+1}\binom{\theta+1}{i}\lambda^{i-1}E\left[X'^{j+i-1}(r,n,m,k)\right]+\left(\frac{j}{\alpha\gamma_r}+1\right)E\left[X'^{j}(r,n,m,k)\right]$$

The prove is complete.

Remark 2.1 For $m=0,\ k=1$, the recurrence relations of *lgos* reduces to the recurrence relations of lower order statistics as

$$E\left[X'^{j}_{n-r+2:n}\right]$$

$$=\frac{j}{\theta\alpha(n-r+1)}\sum_{i=2}^{\theta+1}\binom{\theta+1}{i}\lambda^{i-1}E\left[X'^{j+i-1}_{n-r+1:n}\right]+\left(\frac{j}{\alpha(n-r+1)}+1\right)E\left[X'^{j}_{n-r+1:n}\right] \quad (15)$$

Remark 2.2 For $m=-1,\ k=1$, the recurrence relations of *lgos* reduces to the recurrence relations of lower record values as

$$E\left[X'^{j}(r-1,n,-1,1)\right]$$

$$=\frac{j}{\alpha\theta}\sum_{i=2}^{\theta+1}\binom{\theta+1}{i}\lambda^{i-1}E\left[X'^{j+i-1}(r,n,-1,1)\right]+\left(\frac{j}{\alpha}+1\right)E\left[X'^{j}(r,n,-1,1)\right] \quad (16)$$

Remark 2.3 Sitting $\lambda=1$ in Remark 2.2, we get the recurrence relations for single moments of lower record values from exponentiated Pareto, established by Shawky and Abu-Zinadah [13].

Theorem 2.2 For *ELD* and for $2\le r\le n$ and $k=1,2,\ldots$

$$E\left[X'^{j}(r-1,n,m,k)\right]-E\left[X'^{j}(r-1,n-1,m,k)\right]$$

$$=\frac{j(m+1)(r-1)}{\alpha\gamma_r\gamma_1}\left\{E\left[X'^{j}(r,n,m,k)\right]+\frac{1}{\theta}\sum_{i=2}^{\theta+1}\binom{\theta+1}{i}\lambda^{i-1}E\left[X'^{j+i-1}(r,n,m,k)\right]\right\}$$

(17)

Proof Results can be obtained from (3) and (12).

Theorem 2.3 For *ELD* and for and $k = 1, 2, \ldots$

$$E\left[X'^{j}\left(r-1, n-1, m, k\right)\right]$$

$$= \frac{j}{\theta\alpha\gamma_1}\sum_{i=2}^{\theta+1}\binom{\theta+1}{i}\lambda^{i-1}E\left[X'^{j+i-1}\left(r,n,m,k\right)\right]+\left(\frac{j}{\alpha\gamma_1}+1\right)E\left[X'^{j}\left(r,n,m,k\right)\right] \quad (18)$$

Proof Results can be obtained from (3) and (13).

3. Explicit expression for product moments of *lgos* for *ELD*

Using (6) and binomially expansion, the explicit expression for the product moments of *lgos* $X'(r,n,m,k)$ and $X'(s,n,m,k)$, can be obtained when $m \neq 1$ as

$$E\left[X'^{i}\left(r,n,m,k\right)X'^{j}\left(s,n,m,k\right)\right] = \frac{\alpha\theta\lambda\, C_{s-1}}{\Gamma(r)\Gamma(s-r)(m+1)^{s-2}}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\binom{r-1}{a}\binom{s-r-1}{b}$$

$$(-1)^{a+b}\int_0^{\infty} x^{i}\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha\left[(s-r+a-b)(m+1)\right]-1}(1+\lambda x)^{-(\theta+1)} I(x)dx$$

(19)

where

$$I(x) = \alpha\theta\lambda\int_0^{x} y^{j}\left[1-(1+\lambda y)^{-\theta}\right]^{\alpha\left[\gamma_s+(m+1)b\right]-1}(1+\lambda y)^{-(\theta+1)}dy$$

Setting $z = 1-(1+\lambda y)^{-\theta}$, we get

$$I(x) = \alpha\lambda^{-j}\sum_{c=0}^{j}\binom{j}{c}(-1)^{j-c}\sum_{l=0}^{\infty}\frac{\left(\frac{c}{\theta}\right)_{(l)}\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha\left[\gamma_s+(m+1)b\right]+l}}{l!\left\{\alpha\left[\gamma_s+(m+1)b\right]+l\right\}}, \quad \theta > j$$

Substituting the above result of $I(x)$ in (19), we get

$$E\left[X'^{i}(r,n,m,k)X'^{j}(s,n,m,k)\right]=\frac{\alpha^2\theta\lambda^{1-j}\ C_{s-1}}{\Gamma(r)\Gamma(s-r)(m+1)^{s-2}}$$

$$\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{c=0}^{j}\binom{r-1}{a}\binom{s-r-1}{b}\binom{j}{c}(-1)^{a+b+j-c}\sum_{l=0}^{\infty}\frac{\left(\frac{c}{\theta}\right)_{(l)}}{l!\left\{\alpha\left[\gamma_s+(m+1)b\right]+l\right\}}$$

$$\int_0^{\infty}x^{i}\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha\left[\gamma_s+(s-r+a)(m+1)\right]+l-1}(1+\lambda x)^{-(\theta+1)}dx$$

Again, we setting $w=1-(1+\lambda x)^{-\theta}$, we get

$$E\left[X'^{i}(r,n,m,k)X'^{j}(s,n,m,k)\right]=\frac{\alpha^2\lambda^{(i-j)}\ C_{s-1}}{\Gamma(r)\Gamma(s-r)(m+1)^{s-2}}$$

$$\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{c=0}^{j}\sum_{d=0}^{i}\binom{r-1}{a}\binom{s-r-1}{b}\binom{j}{c}\binom{i}{d}(-1)^{a+b+j+i-c-d}$$

$$\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{\left(\frac{c}{\theta}\right)_{(l)}\left(\frac{d}{\theta}\right)_{(p)}}{l!\,p!\left\{\alpha\left[\gamma_s+(m+1)b\right]+l\right\}\left\{\alpha\left[\gamma_r+(s-r+a)(m+1)\right]+l+p\right\}},$$

$$\theta>\max(i,j),\quad i,j=0,1,2,\ldots \qquad (20)$$

and when $m=-1$ that

$$E\left[X'^{i}(r,n,-1,k)X'^{j}(s,n,-1,k)\right]=\frac{k^{s}}{\Gamma(r)\Gamma(s-r)}\int_0^{\infty}x^{i}\left[-\ln F(x)\right]^{r-1}\frac{f(x)}{F(x)}I(x)dx$$

(21)

where

$$I(x)=\int_0^{x}y^{j}\left[\ln F(x)-\ln F(y)\right]^{s-r-1}\left[F(y)\right]^{k-1}f(y)dy$$

Setting $z=\ln F(x)-\ln F(y)$, we get

$$I(x)=\lambda^{-j}\Gamma(s-r)\sum_{a=0}^{j}\binom{j}{a}(-1)^{j-a}\sum_{l=0}^{\infty}\frac{\left(\frac{a}{\theta}\right)_{(l)}\left[F(x)\right]^{\frac{l}{\alpha}+k}}{l!(\alpha k+l)^{s-r}},\quad \theta>j$$

Substituting the above result of $I(x)$ in (21) and simplifying the result, we get

$$E\left[X'^{i}(r,n,-1,k)X'^{j}(s,n,-1,k)\right]=(\alpha k)^{s}\lambda^{-j}\sum_{a=0}^{j}\sum_{b=0}^{i}\binom{j}{a}\binom{i}{b}(-1)^{j+i-a-b}$$

$$\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{\left(\frac{a}{\theta}\right)_{(l)}\left(\frac{b}{\theta}\right)_{(p)}}{l!p!(\alpha k+l)^{s-r}(\alpha k+l+p)^{r}},$$

$$\theta>\max(i,j),\quad i,j=0,1,2,\ldots \qquad (22)$$

Special cases:

(1) The product moments of lower order statistics can be obtained by taking $m=0,\ k=1$ in (20) as follows

$$E\left[X'^{i}_{n-r+1:n}X'^{j}_{n-s+1:n}\right]=\alpha^{2}\lambda^{(i-j)}\,C_{r,s:n}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{c=0}^{j}\sum_{d=0}^{i}\binom{r-1}{a}\binom{s-r-1}{b}\binom{j}{c}\binom{i}{d}$$

$$(-1)^{a+b+j+i-c-d}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{\left(\frac{c}{\theta}\right)_{(l)}\left(\frac{d}{\theta}\right)_{(p)}}{l!p!\left[\alpha(n-s+1+b)+l\right]\left[\alpha(n-r+1+b)+l+p\right]},$$

(23)

where $C_{r,s:n}=\frac{n!}{(r-1)!(s-r+1)!(n-s)!}$.

(2) The product moments of lower record values can be obtained by taking $k=1$ in (22) as follows:

$$E\left[X'^{i}(r,n,-1,1)X'^{j}(s,n,-1,1)\right]=\alpha^{s}\lambda^{-j}\sum_{a=0}^{j}\sum_{b=0}^{i}\binom{j}{a}\binom{i}{b}(-1)^{j+i-a-b}$$

$$\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{\left(\frac{a}{\theta}\right)_{(l)}\left(\frac{b}{\theta}\right)_{(p)}}{l!p!(\alpha+l)^{s-r}(\alpha+l+p)^{r}}, \qquad (24)$$

(3) We can obtain the product moments of lower record values for the exponentiated Pareto distribution by taking $\lambda=1$ in (24), established by Shawky and Abu-Zinadah [13].

Theorem 3.1 For *ELD* and for θ is positive integer, $1\le r<s\le n-1$ and $k=1,2,\ldots$

$$E\left[X'^{i}(r,n,m,k)X'^{j}(s-1,n,m,k)\right]=\frac{j}{\theta\alpha\gamma_{s}}\sum_{u=2}^{\theta+1}\binom{\theta+1}{u}\lambda^{u-1}$$

$$E\left[X'^{i}(r,n,m,k)X'^{j+u-1}(s,n,m,k)\right]+\left(\frac{j}{\alpha\gamma_r}+1\right)E\left[X'^{i}(r,n,m,k)X'^{j}(s,n,m,k)\right]$$

(25)

Proof From the following relation (Khan *et al*. [9])

$$E\left[X'^{i}(r,n,m,k)X'^{j}(s,n,m,k)\right]-E\left[X'^{i}(r,n,m,k)X'^{j}(s-1,n,m,k)\right]$$

$$=-\frac{j\ C_{s-1}}{\gamma_s\Gamma(r)\Gamma(s-r)}\int_{\alpha}^{\beta}\int_{\alpha}^{x}x^{i}y^{j}\left[F(x)\right]^{m}f(x)g_m^{r-1}\left[F(x)\right]$$

$$\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}\left[F(y)\right]^{\gamma_s}dydx$$

and using (3), (25) will be achieved.

Remark 3.1 Under the assumption given in Theorem 3.1 with k = 1, m = 0 , we get the recurrence relation for product moment of lower order statistics and at k = 1, m = −1, we deduce the recurrence relations for product moments of lower record values from *ELD*.

Remark 3.2 At k = 1, m = −1 and $\lambda=1$, we deduce the recurrence relations for product moments of lower record values from exponentiated Pareto distribution, proved by Shawky and Abu-Zinadah [13].

4. Characterization

Theorem 4.1 Let *X* be a non-negative random variable having an absolutely continuous distribution function *F* (*x*) with *F* (0) = 0 and 0 < *F* (*x*) < 1 for all *x* > 0, then

$$E\left[X'^{j}(r-1,n,m,k)\right]$$

$$=\frac{j}{\theta\alpha\gamma_r}\sum_{i=2}^{\theta+1}\binom{\theta+1}{i}\lambda^{i-1}E\left[X'^{j+i-1}(r,n,m,k)\right]+\left(\frac{j}{\alpha\gamma_r}+1\right)E\left[X'^{j}(r,n,m,k)\right]$$

(26)

if and only if $F(x)=\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha}$

Proof The necessary part follows immediately from equation (14). On the other hand if the recurrence relation in equation (26) is satisfied, then on using equation (5), we have

$$\frac{C_{r-2}}{\Gamma(r-1)}\int_0^\infty x^j\left[F(x)\right]^{\gamma_r+m} f(x)g_m^{r-2}\left[F(x)\right]dx$$

$$=\frac{j}{\theta\alpha\gamma_r}\sum_{i=2}^{\theta+1}\binom{\theta+1}{i}\frac{\lambda^{i-1}C_{r-1}}{\Gamma(r)}\int_0^\infty x^{j+i-1}\left[F(x)\right]^{\gamma_r-1} f(x)g_m^{r-1}\left[F(x)\right]dx$$
$$+\left(\frac{j}{\alpha\gamma_r}+1\right)\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^j\left[F(x)\right]^{\gamma_r-1} f(x)g_m^{r-1}\left[F(x)\right]dx$$

Integrating the lift hand side of the above equation, by parts, we get

$$\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^j\left[F(x)\right]^{\gamma_r-1} f(x)g_m^{r-1}\left[F(x)\right]dx+\frac{j}{\gamma_r}\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^{j-1}\left[F(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx$$

$$=\frac{j}{\theta\alpha\gamma_r}\sum_{i=2}^{\theta+1}\binom{\theta+1}{i}\frac{\lambda^{i-1}C_{r-1}}{\Gamma(r)}\int_0^\infty x^{j+i-1}\left[F(x)\right]^{\gamma_r-1} f(x)g_m^{r-1}\left[F(x)\right]dx$$
$$+\left(\frac{j}{\alpha\gamma_r}+1\right)\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^j\left[F(x)\right]^{\gamma_r-1} f(x)g_m^{r-1}\left[F(x)\right]dx$$

Which is implies that

$$\frac{j\ C_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty x^{j-1}\left[F(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right]$$
$$\left\{F(x)-\frac{1}{\theta\alpha\lambda}\sum_{i=2}^{\theta+1}\binom{\theta+1}{i}(\lambda x)^i f(x)-\frac{x}{\alpha}f(x)\right\}dx=0 \qquad (27)$$

Now applying a generalization of the Muntz-Szasz theorem (Hwang and Lin [3]) to equation (27), we get

$$\frac{f(x)}{F(x)}=\left[\frac{1}{\theta\alpha\lambda}\sum_{i=2}^{\theta+1}\binom{\theta+1}{i}(\lambda x)^i+\frac{x}{\alpha}\right]^{-1}$$

which prove that

$$F(x)=\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha};\quad x\ge 0,\ \theta,\ \lambda \text{ and } \alpha>0$$

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