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ALEKSANDROV PROBLEM IN LINEAR N-NORMED SPACE

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Abstract. We study the Aleksandrov problem in linear n-normed space and give out a sufficient condition for n-isometry in linear n-normed space.

Keywords: Linear n-normed space; Aleksandrov problem; n-isometry.

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1. Introduction

Let (E,d_E) and (F,d_F) be metric spaces. A mapping $f:E\to F$ is called an isometry if $d_F(f(x),f(y))=d_E(x,y)$ for any $x,y\in E$. For fixed number $r>0$, f is said to preserve distance r if $d_E(x,y)=r$ implies $d_F(f(x),f(y))=r$ for any $x,y\in E$. Then r is called a preserved distance for the mapping f. Aleksandrov posed the question: Whether the existence of a single preserved distance for some mapping f implies f is an isometry (see [1]). Several papers have investigated the Aleksandrov problem (see [2-10]). In particular, Chu et al [2] begin to consider the Alksandrov problem in linear n-normed space. They introduce the concept of n-isometry and prove that Rassias and Šemrl's theorem holds under some conditions. In this paper, we generalize the concept of n-isometry and give out a sufficient condition for generalized n-isometry in the linear n-normed space.

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2. Main results

Definition 1. (Chu et al [2]) Let E be a real linear space with $dimE \geqslant n$ and $\|\cdot,\cdots,\cdot\| : E^n \to R$ a function. Then $(E,\|\cdot,\cdots,\cdot\|)$ is called a linear n-normed space if

(1) $\|x_1,\cdots,x_n\| = 0$ if and only $x_1,\cdots,x_n$ are linearly dependent;

(2) $\|x_1,\cdots,x_n\| = \|x_{j_1},\cdots,x_{j_n}\|$ for any permutation $(j_1,\cdots,j_n)$ of $(1,\cdots,n)$;

(3) $\|\alpha x_1,\cdots,x_n\| = |\alpha|\|x_1,\cdots,x_n\|$;

(4) $\|x+y,x_2,\cdots,x_n\| \leqslant \|x,x_2,\cdots,x_n\| + \|y,x_2,\cdots,x_n\|$.

for any $\alpha \in \mathbb{R}$ and any $x,y,x_2,\cdots,x_n \in E$. The function $\|\cdot,\cdots,\cdot\|$ is called the n-norm on E.

Definition 2. (Chu et al [2]) Let E,F be liner n-normed spaces and $f : E \to F$ a mapping. f is said to be an n-isometry if and only if

$$\|x_1 - x_0,\cdots,x_n - x_0\| = \|f(x_1) - f(x_0),\cdots,f(x_n) - f(x_0)\|$$

for all $x_0,x_1,\cdots,x_n \in E$.

We now generalize the concept of n-isometry as following.

Definition 3. Let E,F be liner n-normed spaces and $f : E \to F$ a mapping. f is said to be an n-isometry if and only if

$$\|x_1 - y_1,\cdots,x_n - y_n\| = \|f(x_1) - f(y_1),\cdots,f(x_n) - f(y_n)\|$$

for all $x_1,\cdots,x_n,y_1,\cdots,y_n \in E$.

Throughout this paper, n-isometry has its meaning in the sense of Definition 3 if not special specified.

Theorem 1. Let E,F be liner n-normed spaces, $\alpha > 0$ and $f : E \to F$ a surjection satisfying the following:

(1) $\|x_1 - y_1,\cdots,x_n - y_n\| \leqslant 1$, then $\|f(x_1) - f(y_1),\cdots,f(x_n) - f(y_n)\| \leqslant \|x_1 - y_1,\cdots,x_n - y_n\|$;

(2) $\|x_1 - y_1,\cdots,x_n - y_n\| \geqslant \alpha$, then $\|f(x_1) - f(y_1),\cdots,f(x_n) - f(y_n)\| \geqslant \alpha$.

Then f is an n-isometry.

Proof. Clearly $\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| = \|x_1-y_1,\cdots,x_n-y_n\|$ when $\|x_1-y_1,\cdots,x_n-y_n\| = 0$. Without loss of generality, we assume that $\|x_1-y_1,\cdots,x_n-y_n\| \neq 0$ throughout the proof.

(i) We first prove that for $x_1,\cdots,x_n,y_1,\cdots,y_n \in E$, we have

$$\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| \leqslant \|x_1-y_1,\cdots,x_n-y_n\|. \tag{1.1}$$

Notice that there exist $n,m \in \mathbb{N}$ such that $\|x_1-y_1,\cdots,x_n-y_n\| \leqslant \frac{m}{n}$. Clearly (1.1) holds when $m=1$. For $m \geqslant 2$, put

$$z_i = y_1 + \frac{i}{m}(x_1-y_1)$$

for $i=0,1,2,\cdots,m$. Then we have $z_{i+1}-z_i = \frac{1}{m}(x_1-y_1)$ and

$$\begin{aligned}\|z_{i+1}-z_i,\cdots,x_n-y_n\| &= \|\frac{1}{m}(x_1-y_1),\cdots,x_n-y_n\| \\ &= \frac{1}{m}\|x_1-y_1,\cdots,x_n-y_n\| \\ &\leqslant \frac{1}{n} \\ &\leqslant 1\end{aligned}$$

for $i=0,1,2,\cdots,m-1$. Thus

$$\begin{aligned}\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| &\leqslant \sum_{i=0}^{m-1}\|f(z_{i+1})-f(z_i),\cdots,f(x_n)-f(y_n)\| \\ &\leqslant \sum_{i=0}^{m-1}\|z_{i+1}-z_i,\cdots,x_n-y_n\| \\ &= \sum_{i=0}^{m-1}\|\frac{1}{m}(x_1-y_1),\cdots,x_n-y_n\| \\ &= \|x_1-y_1,\cdots,x_n-y_n\|.\end{aligned}$$

(ii) Next we prove that if $\|x_1-y_1,\cdots,x_n-y_n\| \leqslant \alpha$, then

$$\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| = \|x_1-y_1,\cdots,x_n-y_n\|.$$

It follows (1.1) that

$$\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| \leqslant \|x_1-y_1,\cdots,x_n-y_n\|.$$

Suppose $\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| < \|x_1-y_1,\cdots,x_n-y_n\|$. Put $z_1 = y_1 + \frac{\alpha}{\|x_1-y_1,\cdots,x_n-y_n\|}(x_1-y_1)$. We have

$$\|z_1-y_1,x_2-y_2,\cdots,x_n-y_n\| = \alpha.$$

Then

$$\begin{aligned}
\alpha &\leqslant \|f(z_1)-f(y_1),f(x_2)-f(y_2),\cdots,f(x_n)-f(y_n)\| \\
&\leqslant \|f(z_1)-f(x_1),f(x_2)-f(y_2),\cdots,f(x_n)-f(y_n)\| \\
&\quad + \|f(x_1)-f(y_1),f(x_2)-f(y_2),\cdots,f(x_n)-f(y_n)\| \\
&< \|z_1-x_1,x_2-y_2,\cdots,x_n-y_n\| \\
&\quad + \|x_1-y_1,x_2-y_2,\cdots,x_n-y_n\| \\
&= \|(\frac{\alpha}{\|x_1-y_1,\cdots,x_n-y_n\|}-1)(x_1-y_1),\cdots,x_n-y_n\| \\
&\quad + \|x_1-y_1,\cdots,x_n-y_n\| \\
&= \alpha,
\end{aligned}$$

which is a contradiction. Hence $\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| = \|x_1-y_1,\cdots,x_n-y_n\|$.

(iii) We now prove that if $\|x_1-y_1,\cdots,x_n-y_n\| = \frac{n}{2}\alpha$ $(n\in\mathbb{N}, n\geqslant 2)$, then

$$\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| = \|x_1-y_1,\cdots,x_n-y_n\|.$$

Suppose $\|x_1-y_1,\cdots,x_n-y_n\| = \frac{n}{2}\alpha$ $(n\in\mathbb{N}, n\geqslant 2)$. Put

$$u = f(y_1) + \frac{\alpha}{2}\frac{f(x_1)-f(y_1)}{\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\|}.$$

Then $\|u-f(y_1),\cdots,f(x_n)-f(y_n)\| = \frac{\alpha}{2}$. Since f is surjective, there exists $v\in E$ such that $f(v)=u$. More we have

$$\|v-y_1,\cdots,x_n-y_n\| < \alpha.$$

Otherwise condition (2) implies $\|u-f(y_1),\cdots,f(x_n)-f(y_n)\|\geqslant\alpha$, which is a contradiction. Now it follows (ii) that

$$\|u-f(y_1),\cdots,f(x_n)-f(y_n)\|=\|v-y_1,\cdots,x_n-y_n\|=\frac{\alpha}{2}.$$

We assert that

$$\|u-f(x_1),\cdots,f(x_n)-f(y_n)\|\geqslant\frac{\alpha}{2}(n-1).$$

In fact if $\|u-f(x_1),\cdots,f(x_n)-f(y_n)\|<\frac{\alpha}{2}(n-1)$, that is

$$\|f(v)-f(x_1),\cdots,f(x_n)-f(y_n)\|<\frac{\alpha}{2}(n-1).$$

Since f is surjective, we can find $v_i\in E$ $(i=0,1,2,\cdots,n-1)$ such that $v_0=x_1$, $v_{n-1}=v$ and

$$f(v_i)=f(x_1)+\frac{i}{n-1}(f(v)-f(x_1))$$

for $i=0,1,2,\cdots,n-1$. Then

$$\begin{aligned}\|f(v_{i+1})-f(v_i),\cdots,f(x_n)-f(y_n)\|&=\frac{1}{n-1}\|f(v)-f(x_1),\cdots,f(x_n)-f(y_n)\|\\&<\frac{\alpha}{2}\end{aligned}$$

for $i=0,1,2,\cdots,n-2$. Hence

$$\|v_{i+1}-v_i,\cdots,x_n-y_n\|<\alpha,\ for\ i=0,1,2,\cdots,n-2.$$

Now it follows (ii) that

$$\|v_{i+1}-v_i,\cdots,x_n-y_n\|=\|f(v_{i+1})-f(v_i),\cdots,f(x_n)-f(y_n)\|<\frac{\alpha}{2}$$

for $i=0,1,2,\cdots,n-2$. Thus

$$\begin{aligned}\|x_1-y_1,\cdots,x_n-y_n\|&\leqslant\|v-x_1,\cdots,x_n-y_n\|+\|v-y_1,\cdots,x_n-y_n\|\\&\leqslant\sum_{i=0}^{n-2}\|v_{i+1}-v_i,\cdots,x_n-y_n\|+\|v-y_1,\cdots,x_n-y_n\|\\&<\frac{\alpha}{2}(n-1)+\frac{\alpha}{2}\\&=\frac{n}{2}\alpha,\end{aligned}$$

which is a contradiction. Hence $\|u-f(x_1),\cdots,f(x_n)-f(y_n)\| \geqslant \frac{\alpha}{2}(n-1)$. On the other hand, we have

$$
\begin{aligned}
&\|u-f(x_1),\cdots,f(x_n)-f(y_n)\| \\
&=\|(1-\frac{\alpha}{2\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\|})(f(y_1)-f(x_1)),\cdots,f(x_n)-f(y_n)\| \\
&=\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\|-\frac{\alpha}{2}.
\end{aligned}
$$

So

$$
\begin{aligned}
\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| &= \|u-f(x_1),\cdots,f(x_n)-f(y_n)\|+\frac{\alpha}{2} \\
&\geqslant \frac{n}{2}\alpha.
\end{aligned}
$$

Now $\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| = \|x_1-y_1,\cdots,x_n-y_n\|$ follows from (1.1).

(iv) We finally prove f is n-isometry. For $x_1,\cdots,x_n,y_1,\cdots,y_n \in E$, there exists $n \in \mathbb{N}$ such that $\|x_1-y_1,\cdots,x_n-y_n\| < \frac{n}{2}\alpha$. It follows (1.1) that

$$\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| \leqslant \|x_1-y_1,\cdots,x_n-y_n\|.$$

Suppose $\|f(x_1)-f(y_1),\cdots,f(x_n)-f(y_n)\| < \|x_1-y_1,\cdots,x_n-y_n\|$. Put $z_1 = y_1+\frac{\frac{n}{2}\alpha}{\|x_1-y_1,\cdots,x_n-y_n\|}(x_1-y_1)$. We have

$$\|z_1-y_1,x_2-y_2,\cdots,x_n-y_n\| = \frac{n}{2}\alpha.$$

Thus

$$\begin{aligned}
\frac{n}{2}\alpha &= \|f(z_1)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n)\| \\
&\leqslant \|f(z_1)-f(x_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n)\| \\
&\quad + \|f(x_1)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n)\| \\
&< \|z_1-x_1, x_2-y_2, \cdots, x_n-y_n\| \\
&\quad + \|x_1-y_1, x_2-y_2, \cdots, x_n-y_n\| \\
&= \|(\frac{\frac{n}{2}\alpha}{\|x_1-y_1, \cdots, x_n-y_n\|}-1)(x_1-y_1), \cdots, x_n-y_n\| \\
&\quad + \|x_1-y_1, \cdots, x_n-y_n\| \\
&= \frac{n}{2}\alpha,
\end{aligned}$$

which is a contradiction. Hence $\|f(x_1)-f(y_1), \cdots, f(x_n)-f(y_n)\| = \|x_1-y_1, \cdots, x_n-y_n\|$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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