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TOPOLOGICAL ENTROPY OF GENERALIZED DISCRETE DYNAMICAL SYSTEM

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Abstract: In this paper, the authors introduce Bowen's topological entropy for generalized discrete dynamical system. This is achieved through the consideration of the generalized spanning sets and generalized separated sets. Besides, we obtain that our definition keeps some properties which are held by the classic definition of topological entropy introduced for metric sets. Finally, we further discuss the relations of the generalized spanning set and generalized separated set between systems and their product system. These results further extend the scope of the research on discrete dynamical system and generalize the existing results to a very general case.

Keywords: discrete dynamical system; topological entropy; chaos; spanning set; separated set.

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1. Introduction

In recent years, there has been an increasing interest in the study of discrete dynamical system, which can describe organism, information processing, numerical simulation and many other problems in society. Considerable literatures have been

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appeared to study their properties such as chaos [1], stability [2], bifurcation [3], pseudo-random-distribution [4] and so on. Note that these results on dynamical properties of systems are restricted to systems with fixed parameters, for example, one-dimensional discrete dynamical systems of the form

$$x_{n+1} = f(\mu, x_n), n = 0, 1, \cdots, x_n \in I, \qquad (1)$$

where I is a subset of $R = (-\infty, +\infty)$ and μ is a fixed parameter. However, systems with variable parameters have attracted great interest in various scientific communities due to broad applications in many areas including vibration [5] and synchronization [6], just to name a few. In 2006, Tian and Chen proposed the notion of variable parametric discrete systems in [9, 10], i.e. systems of the form

$$x_{n+1} = g(\mu_n, x_n) = f(n+1, x_n), n = 0, 1, \cdots, \qquad (2)$$

we call it generalized discrete dynamical system in this paper. As system (1) is the special case of system (2), we think that whether the basic theories of system (1) can be popularized to study the corresponding theories of system (2)?

In 1965, Adler et al. [11] introduced the concept of topological entropy of continuous functions for a compact dynamical system on general topological spaces; subsequently, this was generalized by Liu et al. [12] for an arbitrary dynamical system. This concept is of great interest among mathematicians, especially topologists. Already, many researchers have studied the concept of topological entropy. Some of the research works are found in the papers of Bowen [13], Cánovas and Rodríguez [14], Kwietniak and Oprocha [15], Bowen [16] generalized the concept of topological entropy due to Adler et al. [1] that is metric dependent. Pena and López [17] introduced the concept of topological entropy for induced hyperspace maps. However the notion of the topological entropy for so called generalized discrete dynamical system has never been studied in this setting before.

In the present paper, we will restrict our attention to generalized discrete dynamical system. The remainder of the paper is organized as follows. Section 2 introduces some basic notations. Section 3 is devoted to give the topological entropy of generalized discrete dynamical system, based on Bowen's definition. Also we prove some properties for the entropy defined in our paper. Finally, we further discuss the relations of the generalized spanning sets and generalized separated sets between systems and their product system. These results coincide with the situation in the classic discrete dynamical system and further extend the scope of the research on discrete system.

2. Preliminaries

Throughout this paper, space (X,d) represents a compact metric space, i.e. X is compact and d is a metric on X. Let $F=\{f_k\}_{k=1}^{\infty}$ be a sequence of continuous maps on X. For a point $x_0\in X$, define a sequence as follows:

$$x_1=f_1(x_0), x_2=f_2(x_1)\ldots,\quad x_{n+1}=f_{n+1}(x_n),\quad \ldots n=0,1,2,\ldots$$

Then the system (X,F) is said to be a generalized discrete dynamical system.

For convenience, denote maps $F_k:X\to X$ by

$$F_k(x)=f_k(f_{k-1}(\ldots f_1(x)))=f_k\circ f_{k-1}\circ\cdots\circ f_1(x). k=1,2,\ldots$$

Then, it's obvious that

$$x_1=f_1(x_0)=F_1(x_0),\quad x_2=f_2\circ f_1(x_0)=F_2(x_0),\quad \cdots,\ x_n=f_n\circ f_{n-1}\circ\cdots\circ f_1(x_0)=F_n(x_0).$$

In this paper, we will use (X,F) to denote a generalized discrete dynamical system with a metric d and a sequence of continuous maps $F=\{f_k\}_{k=1}^{\infty}$ on X.

3. Main results

For $n>0$ and $\varepsilon>0$, we give the following definition:

Definition 3.1. *A set* $E\subset X$ *is generalized* (n,ε)*-separated if for any* $x\neq y$ *in* E*, there is a* $0\le i<n$ *for which* $d(F_i(x),F_i(y))>\varepsilon$*. Dually, a set* $K\subset X$ *is generalized* (n,ε)*-spanning if for every* $x\in X$*, there is a* $y\in K$ *for which* $d(F_i(x),F_i(y))\le\varepsilon$, $i=0,1,\cdots,n-1$.

Let $r_n(\varepsilon,X,F)$ represent the minimum cardinality of any (n,ε)-spanning set, and let $S_n(\varepsilon,X,F)$ be the maximum cardinality of any (n,ε)-separated set.

Lemma 3.1: (a) $r_n(\varepsilon,X,F)\le S_n(\varepsilon,X,F)\le r_n(\frac{\varepsilon}{2},X,F)<\infty.$

(b) *If* $0<\varepsilon_1<\varepsilon_2$*, then* $r_n(\varepsilon_1,X,F)\ge r_n(\varepsilon_2,X,F)$, $S_n(\varepsilon_1,X,F)\le S_n(\varepsilon_2,X,F)$.

Proof. (a) If K is a generalized (n,ε)-separated set with maximum cardinality of X, then K is a generalized (n,ε)-spanning set obviously. For the second inequality, let E be a generalized (n,ε)-separated set and K be a generalized $(n,\frac{\varepsilon}{2})$-spanning set, by definition 3.1, for every $x\in X$, there is a point $y\in K$ for which

$$d(F_i(x),F_i(y))\le\frac{\varepsilon}{2},\quad i=0,1,\cdots,n-1.$$

Define a map $g: E \to K$ as following, for each $x \in E$, pick a point $g(x) \in K$ with the property that $d(F_i(x), F_i(y)) \leq \frac{\varepsilon}{2}$. Then g is injective, so $|E| \leq |K|$, which completes the proof.

(b) This is clear.

Definition 3.2. *Define*

$$r(\varepsilon, X, F) = \lim_{x\to\infty}\sup \frac{1}{n}\log r_n(\varepsilon, X, F) \quad \textit{and} \quad S(\varepsilon, X, F) = \lim_{x\to\infty}\sup \frac{1}{n}\log S_n(\varepsilon, X, F).$$

Then by lemma 3.1, the following remark is clear.

Remark 3.1. *(1) Suppose* $0 < \varepsilon_1 < \varepsilon_2$, *we have*

$$r(\varepsilon_1, X, F) \geq r(\varepsilon_2, X, F) \quad \textit{and} \quad S(\varepsilon_1, X, F) \leq S(\varepsilon_2, X, F).$$

(2) $r(\varepsilon, X, F) \leq S(\varepsilon, X, F) \leq r(\frac{\varepsilon}{2}, X, F)$.

It follows that the definition below makes sense.

Definition 3.3. *Let* (X, F) *be a generalized discrete dynamical system, and* d *is a metric on* X *. Then the generalized topological entropy of* $F = \{f_k\}_{k=1}^{\infty}$ *is defined by*

$$h(F) = \lim_{\varepsilon\to 0} r(\varepsilon, X, F) = \lim_{\varepsilon\to 0} S(\varepsilon, X, F).$$

Theorem 3.1. *Let* (X, F) *be a generalized discrete dynamical system, where* $F = \{f_k\}_{k=1}^{\infty}$ *is a sequence of continuous maps on* X *, then* $h(F_m) = m \cdot h(F)$ *for any* $m > 0$.

Proof. Suppose $m > 0$ **,** $n > 0$ and $\varepsilon > 0$, it is clear from definition that the (mn,ε) -spanning set of F is an (n,ε) -spanning set of F_m , then $r_n(\varepsilon,X,F_m)\le r_{m\cdot n}(\varepsilon,X,F)$. So that $\frac{1}{n}\log r_n(\varepsilon,X,F_m)\le\frac{m}{m\cdot n}\log r_n(\varepsilon,X,F)$ .This implies $h(F_m)\le m\cdot h(F)$. We can know $F=\{f_k\}_{k=1}^{\infty}$ is uniformly continuous due to the continuity of $F=\{f_k\}_{k=1}^{\infty}$ and the compactness of X, so for any $\varepsilon>0$, we can choose a $\delta>0$ for which

$$d(F_i(x),F_i(y))\le\varepsilon,\ \ i=0,1,\cdots m-1\ \text{if}\ d(x,y)<\delta.$$

It follows that a generalized (n,ε)-spanning set with respect to F_m automatically a generalized $(m\cdot n,\varepsilon)$-spanning set with respect to F. So $r_n(\varepsilon,X,F_m)\ge r_{m\cdot n}(\varepsilon,X,F)$, and therefore $h(F_m)\ge m\cdot h(F)$. Thus $h(F_m)=m\cdot h(F)$.

Remark 3.2. *The classical topological entropy due to Bowen is a special case of the topological entropy defined above.*

Let (X,F) *and* (Y,G) *be generalized discrete dynamical systems and* $X\times Y$ *be product space with metric* $d^{"}((x_1,y_1),(x_2,y_2))=\max\{d_X(x_1,x_2),d_Y(y_1,y_2)\}$, *where* d_X *and* d_Y *are, respectively, metrics for* X *and* Y. *Let the sequence of maps* $F\times G$ *be defined by*

$$\forall(x,y)\in X\times Y,\ \ (f_i\times g_i)(x,y)=(f_i(x),g_i(x)),i=1,2,\cdots.$$

And it is verified that

$$(F\times G)^i(x,y)=(f_i\times g_i)\circ(f_{i-1}\times g_{i-1})\circ\cdots\circ(f_1\times g_1)(x,y)=(F_i\times G_i)(x,y).$$

Now we obtain the product system $(X\times Y,F\times G)$ *with metric* $d^{"}$ *and the sequence of continuous maps* $F\times G$.

Theorem 3.2. *If* E_X *and* E_Y *are, respectively, generalized* (n,ε) *-separated sets of* F *and* G *, then* $E_X \times E_Y$ *is a generalized* (n,ε) *-separated set of* $F \times G$ *.*

Proof. For any $x=(x_1,y_1), y=(x_2,y_2)\in(X,Y)$, and $x\neq y$, then either $x_1\neq x_2$ or $x_1\neq x_2$. Without lose of generality, let us suppose $x_1\neq x_2$, since E_X is a generalized (n,ε)-separated set of F, there is $0\le i<n$ for which $d_X(F_i(x_1),F_i(x_2))>\varepsilon$. Then $d^{"}((F\times G)^i(x),(F\times G)^i(y))=\max\{d_X(F_i(x_1),F_i(x_2)),d_Y(G_i(y_1),G_i(y_2))\}>\varepsilon, 0\le i<n.$ Therefore $E_X\times E_Y$ is a generalized (n,ε)-separated set of $F\times G$.

Theorem 3.3. *If* $E_X \times E_Y$ *is an* (n,ε)*-separated set of* $F\times G$*, then either* E_X *is an* (n,ε)*-separated set of* F*, or* E_Y *is an* (n,ε)*-separated set of* G *.*

Proof. (Reduction to absurdity) Assume neither E_X nor E_Y is a generalized (n,ε) -separated set of F and G respectively. Then by the definition of (n,ε)-separated set, there exist

$$x_1\neq x_2 \text{ in } X \text{, such that } d_X(F_i(x_1),F_i(x_1))\le\varepsilon \text{, for each } 0\le i<n.$$

and

$$y_1\neq y_2 \text{ in } Y \text{, such that } d_Y(G_i(y_1),G_i(y_1))\le\varepsilon \text{, for each } 0\le i<n.$$

Let $x=(x_1,y_1), y=(x_2,y_2)$, obviously $x,y\in X\times Y$. And therefore

$$d^{"}((F\times G)^i(x),(F\times G)^i(y))=\max\{d_X(F_i(x_1),F_i(x_2)),d_Y(G_i(y_1),G_i(y_2))\}\le\varepsilon,$$

for each $0\le i<n$, which contradicts with the condition given in this theorem.

Theorem 3.4. *$K_X \times K_Y$ is a generalized (n, ε)-spanning set of $F \times G$ if and only if K_X and K_Y are, respectively, generalized (n, ε)-spanning sets of F and G.*

Proof. Let us begin with the proof of necessity. For any $x \in X, y \in Y$, obviously $(x, y) \in X \times Y$, since $K_X \times K_Y$ is a generalized (n, ε)-spanning set of $F \times G$, then there exists $(z_1, z_2) \in K_X \times K_Y$ such that

$$d^{"}((F \times G)^i(x, y), (F \times G)^i(z_1, z_2)) = \max\{d_X(F_i(x), F_i(z_1)), d_Y(G_i(y), G_i(z_2))\} \leq \varepsilon,$$

where $i = 0, 1, \cdots, n-1$, thus

$$d_X(F_i(x), F_i(z_1)) \leq \varepsilon \text{ and } d_Y(G_i(y), G_i(z_2)) \leq \varepsilon. \quad i = 0, 1, \cdots, n-1.$$

Where $z_1 \in K_X, z_2 \in K_Y$, therefore K_X and K_Y are generalized (n, ε)-spanning sets of F and G respectively.

To prove the converse, suppose that K_X and K_Y are generalized (n, ε)-spanning sets of F and G respectively. For any $(x, y) \in X \times Y$, accordingly $x \in X, y \in Y$. Since K_X is ageneralized (n, ε)-spanning sets of F, there exists $z_1 \in K_X$ such that

$$d_X(F_i(x), F_i(z_1)) \leq \varepsilon, i = 0, 1, \cdots, n-1.$$

Similarly, there exists $z_2 \in K_Y$ such that

$$d_Y(G_i(y), G_i(z_2)) \leq \varepsilon. \quad i = 0, 1, \cdots, n-1.$$

Thus $(z_1, z_2) \in K_X \times K_Y$, and we have

$$d^{*}((F\times G)^{i}(x,y),(F\times G)^{i}(z_1,z_2))=\max\{d_X(F_i(x),F_i(z_1)),d_Y(G_i(y),G_i(z_2))\}\le\varepsilon,$$

For each $i=0,1,\cdots,n-1$. Hence $K_X\times K_Y$ is a generalized (n,ε)-spanning set of $F\times G$.

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