Available online at http://scik.org
J. Math. Comput. Sci. 6 (2016), No. 1, 22-38
ISSN: 1927-5307

ASPECT OF HARMONIC ANALYSIS ON PERMUTATIONS GROUP AND APPLICATIONS

OMAR EL FOURCHI[1,*], ADIL ECHCHELH[2]

[1]Department of Mathematics, CRMEF of Rabat, Rabat, Morocco

[2]Department of Physics,University Ibn Tofail, Faculty of Sciences, Kenitra, Morocco

Abstract. The principal aim of the present paper is to develop the theory of Gelfand pairs on the symmetric group in order to define and study the horocyclic Radon transform on this group. We also find a simple inversion formula for the Radon transform of the solution to the heat equation associated to this group.

Keywords: Harmonic analysis and PDE; Radon transform; Heat equation.

2010 AMS Subject Classification: 42B37, 44A12 , 35K05.

1. Introduction and preliminaries

Radon transform play a critical role in subjects as diverse as application to partial differential equations, X-ray technology and radioastronomy. Like much of mathematics in the field of harmonic analysis and integral geometry on homogeneous space has some of its application in the work of Helgason [12]. Although the permutation group form one of the oldest parts of group theory and the harmonic analysis in this work may be regarded as a trivial of compact case. Through the ubiquity of group actions and representations theory, permutations group

*Corresponding author

Received July 28, 2015

continue to be lively topic of research in their own right, see [14], [17-18]. Working on a single spherical analysis and developing the theory of Gelfand pairs on permutations group we essentially want to define the horocyclic Radon and construct its inverse. As an example of the use Radon transforms, we give a simple solution to the Radon transform of the heat equation on permutations group, all done very explicitly.

A bijective function from $Z_n = \{1,2,3,....,n\}$ onto itself is called a permutation of n numbers; the set of all permutations of n numbers, together with the usual composition of functions, is called the symmetric group of degree n. This group will be denoted by $\mathscr{S}_n$. Note that $\mathscr{S}_n$ is defined for $n \geq 0$, and $\mathscr{S}_n$ has $n!$ elements (where $0! = 1$). If Y is a subset of Z_n, we shall write $\mathscr{S}_Y$ for the subgroup of $\mathscr{S}_n$ which fixes every number outside Y.

A permutation $\sigma \in \mathscr{S}_n$, which interchanges two distinct numbers i and j and leaves all other numbers fixed, is called a transposition and is written as $\sigma = \tau_{i,j}$. The function $\varepsilon : \mathscr{S}_n \to \{\pm 1\}$, such that $\varepsilon(\sigma) = (-1)^N$ if σ is a product of N transpositions, is well-defined, called the signature of σ. The number $\varepsilon(\sigma)$ depends only on the parity of N and we have $\varepsilon(\sigma \cdot \zeta) = \varepsilon(\sigma).\varepsilon(\zeta)$.

The normalized Haar measure in $\mathscr{S}_n$ is given by $\nu = \frac{1}{n!} \sum_{\sigma \in \mathscr{S}_n} \delta_\sigma$, where δ_σ is Dirac measure.

The complex group algebra of the group $\mathscr{S}_n$ is $\mathbb{C}(\mathscr{S}_n) = \left\{ \sum_{\sigma \in \mathscr{S}_n} \lambda_\sigma \delta_\sigma : \lambda_\sigma \in \mathbb{C} \right\}$.

This is a vector space over $\mathbb{C}$, for which the set $\{\delta_\sigma : \sigma \in \mathscr{S}_n\}$ is a basis. We note that the algebras $\mathbb{C}(\mathscr{S}_n)$, $\mathbb{L}^1(\mathscr{S}_n)$ and $\mathbb{L}^2(\mathscr{S}_n)$ (the space of integrable functions resp the square integrable functions) are all equal, see [4-5].

The structure of this paper is the following. In Section 2, We recall briefly the main definitions and results of the representation theory on the symmetric group. In Section 3, we give a characterization of the set right cosets $\mathscr{S}_{n+1}/\mathscr{S}_n$ and those of double cosets $\mathscr{S}_n \backslash \mathscr{S}_{n+1}/\mathscr{S}_n$. The main goal of this section will be devoted to spherical function of the Gelfand pair $(\mathscr{S}_{n+1}, \mathscr{S}_n)$. In Sections 4 and 5 we introduce and invert the spherical Fourier transform in permuatation group. In Section 6 we introduce and investigate the Radon transform on permutations group. In Section 7 we establish the connection between this transform and the solution to the heat equations associated to this group which is the technical heart of the paper.

2. Irreducible representations of permutations groups

We begin this section by recalling a few facts from the representation theory of permutation groups, see [5], [14-17].

A subset S of Z_n is invariant under a permutation $\sigma \in \mathscr{S}_n$ if $\sigma(S) = S$. A permutation is said to be circular when it admits Ø and Z_n as its only invariant subsets.

An invariant subset S of Z_n is called a cycle of σ if $\sigma|_S$ is circular, where we are writing $\sigma\ |_S$ for the permutation which coincides with σ on S, and which is the identity outside of S.

Lemma 2.1. Let $\sigma \in \mathscr{S}_n$, the cycles of σ, denoted $S_1, S_2, ..., S_k$, form a partition of Z_n. Furthermore for all i, j, $\sigma\ |_{S_i}$ and $\sigma\ |_{S_j}$ commute and we have

$$\sigma = \prod_{i=1}^{k} \sigma\ |_{S_i}\ .$$

Since the conjugacy class of an element $\sigma \in \mathscr{S}_n$ is characterized by the lengths of the cycles of σ (with repetitions), the number of conjugacy classes in $\mathscr{S}_n$ is equal to the number of partitions of n. As [4] the number of inequivalent complex irreducible representations of $\mathscr{S}_n$ is equal to the number of conjugacy classes of $\mathscr{S}_n$. Therefore the number of inequivalent complex irreducible representations of $\mathscr{S}_n$ is equal to the number of partitions of n.

We should therefore aim to construct a regular representation of $\mathscr{S}_n$ for each partition of n. This is made easier by knowing the primitive idempotents [4].

A non-zero element e of $\mathbb{C}(\mathscr{S}_n)$ is said idempotent if $e * e = e$. More generally e is said essentially idempotent if $e * e = \lambda e$ for some $\lambda \neq 0$. An idempotent e is said to be primitive if e decomposes as the sum of two idempotents: $e = e' + e''$ with $e' * e'' = e'' * e' = 0$. An idempotent which does not decompose in this way is called a primitive idempotent.

In order to describe the primitive idempotents of $\mathbb{C}(\mathscr{S}_n)$, we will need to recall some definitions.

If $\lambda = (n_1, n_2, n_3, \dots, n_k)$ is a partition of n (i.e., $n_1 \geq n_2 \geq n_3 \geq .... \geq n_k \geq 1$ and $n = n_1 + n_2 + n_3 + ... + n_k$), we associates to this partition a tableau $[\lambda] = \{(i, j) : i, j \in \mathbb{Z}; 1 \leq i; 1 \leq j \leq n_i\}$ (here $\mathbb{Z}$ denotes the set of integers).

If $(i,j) \in [\lambda]$, then (i,j) is called a node of $[\lambda]$. The k^{th} row (respectively column) of tableau consists of those nodes whose first (respectively second) coordinate is k.

A Young-diagram is the one of the $n!$ arrays of integers obtained by replacing each node in $[\lambda]$ by one of the integers $1,2,3,4,....,n$ allowing no repeats.

To the Young-diagram t, we associate its row-stabilizer, $\mathscr{P}_t$, is the subgroup of $\mathscr{S}_n$ keeping the rows of t fixed setwise. i.e.,

$$\mathscr{P}_t = \{\sigma \in \mathscr{S}_n : \text{for all } i \in Z_n,\, i \text{ and } \sigma(i) \text{ belong to the same row of } t\}.$$

The column-stabilizer $\mathscr{Q}_t$, of t is defined similarly.

Proposition 2.1. [5] *Let t a λ- tableau, if $\pi \in \mathscr{S}_n$ then $\mathscr{P}_{\pi t} = \pi \mathscr{P}_t \pi^{-1}$ $\mathscr{Q}_{\pi t} = \pi \mathscr{Q}_t \pi^{-1}$.*

We define a relation of equivalence in the set of λ-tableau by
$t_1 \sim t_2$ if and only if there exists $\pi \in \mathscr{P}_{t_1}$ such that $\pi t_1 = t_2$.

The conjugacy class of tableau modulo this relation of equivalence are called tabloids and the conjugacy class of tableau t is the tabloid noted by $\{t\}$.

Proposition 2.2. [5] *The permutations group acts on the set of λ-tableau in the following way: If $\pi \in \mathscr{S}_n$ and t a λ-tableau then $\pi\{t\} = \{\pi t\}$.*

To each partition μ of n, $\mu = (p_1, p_2, ..., p_k)$, we associate the Young sub-group S_μ of $\mathscr{S}_n$ defined as product of the following sub-group

$$S_{p_1+...p_{i-1}+1, p_1+...p_{i-1}+2, ..., p_1+p_2+...+p_{i-1}+p_i}, \quad i = 1,2,3,...,k.$$

we have then

$$S_{1,...p_1} \times S_{p_1+1,...p_1+p_2,} \times S_{p_1+p_2+1,...,p_1+p_2+p_3} \times ....$$

If μ is a partition of n, notice M^μ the $\mathbb{C}$vector space whose basis are the μ distinct tabloids, so M^μ is $\mathbb{C}[\mathscr{S}_n]$-module
For any Young-diagram t, we associate the element of $\mathbb{C}(\mathscr{S}_n)$ defined as follows

$$e_t = \sum_{q \in \mathscr{Q}_t} \sum_{p \in \mathscr{P}_t} \varepsilon(q) \delta_p * \delta_q.$$

We note [4] that e_t is essentially idempotent and $\frac{1}{\lambda_t} e_t$ is a primitive idempotent ($\lambda_t \neq 0$). In some notation e_t is called polytabloid

Lemma 2.2. *If $\pi \in \mathscr{S}_n$ and if e_t a polytabloid then $\pi e_t = e_{\pi t}$.*

Definition 2.1. The specht module S^{μ} is $\mathbb{C}[\mathscr{S}_n]$-module monogene generated by any μ-tabloids.

Remark 2.1. Every result interpreted via the Specht module is the same via the left ideal $\mathscr{O}_t$ of $\mathbb{C}(\mathscr{S}_n)$ generated by e_t [14,p: 17].

Theorem 2.1. [4,p:67] *Let t be a Young-diagram, $\mathscr{P}_t$ its row-stabilizer, $\mathscr{Q}_t$ its column-stabilizer, and let e_t be the element of $\mathbb{C}(\mathscr{S}_n)$ defined by*

$$e_t = \sum_{q \in \mathscr{Q}_t} \sum_{p \in \mathscr{P}_t} \varepsilon(q) \delta_p * \delta_q.$$

We shall write $\mathscr{O}_t$ for the left ideal of $\mathbb{C}(\mathscr{S}_n)$ generated by e_t and $\mathscr{R}_t$ for the associated representation of $\mathscr{S}_n$. Then

- *$\mathscr{R}_t$ is irreducible;*
- *two such representations $\mathscr{R}_s$ and $\mathscr{R}_t$ are equivalent if and only if s and t are Young diagrams for the same partition λ.*

As the number of partitions is equal to the number of irreducible complex representations, we may obtain a representative $\mathscr{R}_t$ for each equivalence class of irreducible representations by choosing for each partition λ a Young diagram t.

Remark 2.2. F. Scarabotti [17] has given a short proof of a characterization of James [14]of the irreducible modules as the intersection of kernels of certain invariant operators using the class sum of transpositions and a collection of related transform for the complex representation of the permutations group

3. Harmonic analysis of the pair $(\mathscr{S}_{n+1}, \mathscr{S}_n)$

We may regard $\mathscr{S}_n = \mathscr{S}(\mathrm{Z}_n)$ as a subgroup of $\mathscr{S}_{n+1} = \mathscr{S}(\mathrm{Z}_{n+1})$. More precisely for $\sigma \in \mathscr{S}_n$, the map $\overline{\sigma} : \mathrm{Z}_{n+1} \to \mathrm{Z}_{n+1}$, defined by $x \mapsto \sigma(x)$ for $x \in \mathrm{Z}_n$ and $n+1 \mapsto n+1$, is an element of $\mathscr{S}_{n+1}$. We note that $\mathscr{S}_n$ acts transitively on Z_n via the map $\mathscr{S}_n \times \mathrm{Z}_n \to \mathrm{Z}_n$: $(\sigma, i) \mapsto \sigma.i = \sigma(i)$. Then our objective in below is to establish the spherical transform of the pair $(\mathscr{S}_{n+1}, \mathscr{S}_n)$. Indeed, we will characterize the right cosets $\mathscr{S}_{n+1}/\mathscr{S}_n$ and the double cosets $\mathscr{S}_n \backslash \mathscr{S}_{n+1}/\mathscr{S}_n$.

3.1. Realization of $\mathscr{S}_{n+1}/\mathscr{S}_n$ of right cosets

We now study the set of right cosets $\mathscr{S}_{n+1}/\mathscr{S}_n = \{\sigma.\mathscr{S}_n; \sigma \in \mathscr{S}_{n+1}\}$. Consider the function $f : \mathscr{S}_{n+1} \to \mathrm{Z}_{n+1}$ defined by $f(\sigma) = \sigma(n+1)$. As $\mathscr{S}_{n+1}$ acts transitively on Z_{n+1} it follows that f is surjective.
For $\sigma, \sigma' \in \mathscr{S}_{n+1}$ we have $f(\sigma) = f(\sigma')$ if and only if $\sigma^{-1} \circ \sigma'(n+1) = n+1$, or equivalently if $\sigma\mathscr{S}_n = \sigma'\mathscr{S}_n$. Thus f induces a bijection $\overline{f} : \mathscr{S}_{n+1}/\mathscr{S}_n \to \mathrm{Z}_{n+1}$ given by $\overline{f}(\sigma\mathscr{S}_n) = \sigma(n+1)$. Thus $\mathscr{S}_{n+1}/\mathscr{S}_n = \mathrm{Z}_{n+1}$.

3.2. Realization of $\mathscr{S}_n\backslash\mathscr{S}_{n+1}/\mathscr{S}_n$ of double cosets

We now consider the set of double cosets:

$$\mathscr{S}_n\backslash\mathscr{S}_{n+1}/\mathscr{S}_n = \{\mathscr{S}_n\sigma\mathscr{S}_n : \sigma \in \mathscr{S}_{n+1}\}.$$

We shall calculate for $\sigma \in \mathscr{S}_{n+1}$ the double coset $\mathscr{S}_n\sigma\mathscr{S}_n$.

If $\sigma \in \mathscr{S}_n$ then $\mathscr{S}_n\sigma\mathscr{S}_n = \mathscr{S}_n$ so we assume $\sigma \notin \mathscr{S}_n$. Thus $\sigma(n+1) \neq n+1$ so we must have $\sigma(n+1) \in \mathrm{Z}_n$. As $\mathscr{S}_n$ acts transitively on Z_n there is a $\sigma' \in \mathscr{S}_n$ such that $\sigma'\sigma(n+1) = 1$. This means $\sigma'\sigma(n+1) = \tau_{1,n+1}(n+1)$, so by the discussion above we have

$$\sigma'\sigma\mathscr{S}_n = \tau_{1,n+1}\mathscr{S}_n.$$

This implies

$$\mathscr{S}_n\sigma'\sigma\mathscr{S}_n = \mathscr{S}_n\tau_{1,n+1}\mathscr{S}_n.$$

However since $\sigma' \in \mathscr{S}_n$, we have

$$\mathscr{S}_n\sigma\mathscr{S}_n = \mathscr{S}_n\tau_{1,n+1}\mathscr{S}_n.$$

Therefore there are only two double cosets. The result follows since a group may always be expressed as the disjoint union of its double cosets with respect to any subgroup. Then

$$\mathscr{S}_{n+1} = \mathscr{S}_n \bigcup \mathscr{S}_n.\tau_{1,n+1}.\mathscr{S}_n,$$

and the union is disjoint. We therefore have $\mathscr{S}_n\backslash\mathscr{S}_{n+1}/\mathscr{S}_n = \{\mathscr{S}_n; \mathscr{S}_n.\tau_{1,n+1}.\mathscr{S}_n\}$ so the space of radial and integrable function $\mathbb{L}^{1,\#}(\mathscr{S}_{n+1})$ is equal to $\mathbb{L}^1(\mathscr{S}_n\backslash\mathscr{S}_{n+1}/\mathscr{S}_n) = \mathbb{L}^1(\{\mathscr{S}_n; \mathscr{S}_n.\tau_{1,n+1}.\mathscr{S}_n\})$.

We say that $(\mathscr{S}_{n+1}, \mathscr{S}_n)$ is a Gelfand pair when the convolution algebra $\mathbb{L}^{1,\#}(\mathscr{S}_{n+1}) = \mathbb{L}^1(\mathscr{S}_n \backslash \mathscr{S}_{n+1} / \mathscr{S}_n)$ of integrable and $\mathscr{S}_n$-biinvariante functions on $\mathscr{S}_{n+1}$ is abelian.

Remark 3.1.

1) As $\mathscr{S}_n \backslash \mathscr{S}_{n+1} / \mathscr{S}_n$ is finite, it is easy to see that the convolution algebra $\mathbb{L}^{1,\#}(\mathscr{S}_{n+1}) = \mathbb{L}^1(\mathscr{S}_n \backslash \mathscr{S}_{n+1} / \mathscr{S}_n)$ is Abelian, thus $(\mathscr{S}_{n+1}, \mathscr{S}_n)$ is a Gelfand pair.

2) As the cardinal of $\mathscr{S}_n \backslash \mathscr{S}_{n+1} / \mathscr{S}_n$ is equal to number of $\mathscr{S}_{n+1}$-orbits in $\mathscr{S}_n \backslash \mathscr{S}_{n+1} \times \mathscr{S}_{n+1} / \mathscr{S}_n$ ($\mathscr{S}_{n+1}$ acts via $(\mathscr{S}_n \sigma, \tau \mathscr{S}_n) \longrightarrow (\mathscr{S}_n \sigma \zeta, \zeta^{-1} \tau \mathscr{S}_n)$), then we have two $\mathscr{S}_{n+1}$-orbits in $\mathscr{S}_n \backslash \mathscr{S}_{n+1} \times \mathscr{S}_{n+1} / \mathscr{S}_n$.

For any subset $A \subset \mathscr{S}_{n+1}$, we define χ_A be the characteristic function of A

Corollary 3.1. *The element* $\mathbb{I}_{\mathscr{S}_{n+1}} = (n+1)\chi_{\mathscr{S}_n}$ *is an identity element of* $\mathbb{L}^{1,\#}(\mathscr{S}_{n+1})$.

Proof. Choose any $f \in \mathbb{L}^{1,\#}(\mathscr{S}_{n+1})$, we must show that $\chi_{\mathscr{S}_n} * f = f * \chi_{\mathscr{S}_n} = \frac{1}{n+1} f$. By definition of convolutions we have

$$(\chi_{\mathscr{S}_n} * f)(x) = \int_{\mathscr{S}_{n+1}} \chi_{\mathscr{S}_n}(y) f(xy^{-1}) d\mu(y) = \int_{\mathscr{S}_n} f(xy^{-1}) d\mu(y).$$

As f is right-invariant we have

$$(\chi_{\mathscr{S}_n} * f)(x) = \int_{\mathscr{S}_n} f(x) d\mu(y) = f(x)\mu(\mathscr{S}_n).$$

On the other hand $\mu(\mathscr{S}_n) = [\mathscr{S}_{n+1} : \mathscr{S}_n]^{-1} = \frac{1}{n+1}$, so we have $\chi_{\mathscr{S}_n} * f = \frac{1}{n+1} f$. The formula $\chi_{\mathscr{S}_n} * f = \frac{1}{n+1} f$ follows in the same way but using left-invariance rather than right-invariance of f.This completes the proof.

For any subset $A \subseteq \mathscr{S}_{n+1}$ we define χ_A to be the characteristic function of A. We denote by χ_σ the characteristic function of the set $\{\sigma\}$ and by δ_σ the Dirac measure at the point σ. Note that for $\sigma, \tau \in \mathscr{S}_{n+1}$ we have $\delta_\sigma * \delta_\tau = \delta_{\sigma\tau}$. From our normalization of the Haar measure dk on $\mathscr{S}_{n+1}$ it follows that $\chi_\sigma(dk) = \frac{1}{(n+1)!}\delta_\sigma$. We therefore have $\chi_\sigma * \chi_\tau = \frac{1}{(n+1)!}\chi_{\sigma\tau}$.

Let $\sigma \in \mathscr{S}_{n+1}$, we have

$$\begin{aligned} \chi^{\#}_\sigma(x) &= \int_{\mathscr{S}_n} \int_{\mathscr{S}_n} \chi_\sigma(\tau.x.h)\, d\tau dh \\ &= \chi_{\mathscr{S}_n.\sigma.\mathscr{S}_n}(x). \end{aligned}$$

So, for $\sigma = id_{\mathscr{S}_n}$

$$\chi^{\#}_{id(\mathscr{S}_n)} = \chi_{\mathscr{S}_n}$$

By virtue of $\tau_{1,n+1} = \tau_{1,i} \circ \tau_{i,n+1} \circ \tau_{i,1}$ for all $i \leq n$, we will have

$$\chi^{\#}_{1,n+1} = \chi^{\#}_{i,n+1} = \chi_{\mathscr{S}_n.\tau_{1,n+1}.\mathscr{S}_n}$$

and

$$\begin{aligned} \chi^{\#}_{n+1,n+1} &= \chi^{\#}_{id(\mathscr{S}_n)} \\ &= \chi_{\mathscr{S}_n}. \end{aligned}$$

Remark 3.2. From Corollary 3.1, we have

$$\chi^{\#}_{1,n+1} * \chi^{\#}_{id(\mathscr{S}_n)}(x) = \frac{1}{n+1}\chi^{\#}_{1,n+1}(x) = \chi^{\#}_{id(\mathscr{S}_n)} * \chi^{\#}_{1,n+1}(x),$$

so any two basis elements of the convolutions algebra $\mathbb{L}^{1,\#}(\mathscr{S}_{n+1})$ commute, which is another way to see that $(\mathscr{S}_n, \mathscr{S}_{n+1})$ is a Gelfand Pair. Also we have

$$\chi^{\#}_{id(\mathscr{S}_n)} * \chi^{\#}_{id(\mathscr{S}_n)} = \frac{1}{n+1}\chi_{id(\mathscr{S}_n)}.$$

Lemma 3.1. *We have*

$$\chi^{\#}_{1,n+1} * \chi^{\#}_{1,n+1} = \frac{n-1}{n+1}\chi^{\#}_{1,n+1} + \frac{n}{n+1}\chi^{\#}_{id(\mathscr{S}_n)}.$$

Proof. To prove this lemma, we use the fact that $\mathscr{S}_{n+1} = \mathscr{S}_n \bigcup \mathscr{S}_n.\tau_{1,n+1}.\mathscr{S}_n$ is disjoint union. Then the identity function on $\mathscr{S}_{n+1}$ noted by $\mathbb{I}_{\mathscr{S}_{n+1}}$ may be expressed as

$$\mathbb{I}_{\mathscr{S}_{n+1}} = \chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}.$$

We use the fact that

$$\mathbb{I}_{\mathscr{S}_{n+1}} * \mathbb{I}_{\mathscr{S}_{n+1}} = \mathbb{I}_{\mathscr{S}_{n+1}}.$$

Therefore

$$(\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}) * (\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}) = (\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}).$$

Expanding the left hand side of the above equality we obtain:

$$\chi^{\#}_{1,n+1} * \chi^{\#}_{1,n+1} + 2(\chi^{\#}_{1,n+1} * \chi^{\#}_{id(\mathscr{S}_n)}) + \chi^{\#}_{id(\mathscr{S}_n)} * \chi^{\#}_{id(\mathscr{S}_n)}.$$

This is equal to:

$$\chi^{\#}_{1,n+1} * \chi^{\#}_{1,n+1} + \frac{2}{n+1}\chi^{\#}_{1,n+1} + \frac{1}{n+1}\chi^{\#}_{id(\mathscr{S}_n)}.$$

We therefore have:

$$\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)} = \chi^{\#}_{1,n+1} * \chi^{\#}_{1,n+1} + \frac{2}{n+1}\chi^{\#}_{1,n+1} + \frac{1}{n+1}\chi^{\#}_{id(\mathscr{S}_n)}.$$

Consequently

$$\chi^{\#}_{1,n+1} * \chi^{\#}_{1,n+1} = \frac{n-1}{n+1}\chi^{\#}_{1,n+1} + \frac{n}{n+1}\chi^{\#}_{id(\mathscr{S}_n)}.$$

This completes the proof.

3.3. Spherical function of the Gelfand pair $(\mathscr{S}_{n+1}, \mathscr{S}_n)$

A function ϕ is said to be a spherical function [7] if and only if ϕ is $\mathscr{S}_n$-biinvariante and ϕ is a character of $\mathbb{L}^{1,\#}(\mathscr{S}_{n+1})$. Then a function ϕ of $\mathscr{S}_{n+1}$ which is $\mathscr{S}_n$-biinvariante may be considered as a function of

$\mathscr{S}_n\backslash\mathscr{S}_{n+1}/\mathscr{S}_n = \{\mathscr{S}_n, \mathscr{S}_n\tau_{1,n+1}\mathscr{S}_n\}$. We shall use the following notation $\check{f}(x) = f(x^{-1})$ and $\widetilde{(f)}(x) = \overline{f(x^{-1})}$.

Theorem 3.1. *The spherical functions of the Gelfand pair $(\mathscr{S}_{n+1}, \mathscr{S}_n)$ are of the form*

1) *The characteristic function $\mathbb{I} = \chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}$ on $\mathscr{S}_{n+1}$ whose restriction to $\mathscr{S}_n$ is equal to $\chi_{id(\mathscr{S}_n)}$;*
2) *The function $\phi_n = \frac{-1}{n}\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}$.*

Proof. Let ϕ be a spherical function, then ϕ is an $\mathscr{S}_n$-biinvariante function such that $\phi(Id_{\mathscr{S}_{n+1}}) = 1$ and satisfying the following integral equation [7]

$$\phi(\sigma)\phi(\zeta) = \int_{\mathscr{S}_n} \phi(\sigma.\tau.\zeta)d\nu(\tau),$$

where σ and $\zeta \in \mathscr{S}_{n+1}$. Also we have, $\widetilde{\phi} = \phi$ and $\check{f} * \phi =< f, \phi > \phi, \quad \forall f \in \mathbb{L}^{1,\#}(\mathscr{S}_{n+1})$, with $< f, g >= \frac{1}{(n+1!)} \sum_{\sigma\in\mathscr{S}_{n+1}} f(\sigma)g(\sigma)$.

As ϕ is biinvariant we may express it in terms of our basis:

$$\phi = \alpha \chi^{\#}_{1,n+1} + \beta \chi^{\#}_{id(\mathscr{S}_n)}.$$

As $\phi(e) = 1$ we must have $\beta = 1$. Therefore

$$\phi = \alpha \chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}.$$

As $\check{\phi} * \phi = \overline{\phi} * \phi = <\phi,\phi> \phi$, we have, in the first hand

$$\begin{aligned}
\check{\phi} * \phi &= (\overline{\alpha}\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}) * (\alpha\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}) \\
&= |\alpha|^2(\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}) + 2Re(\alpha)(\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}) + (\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}) \\
&= |\alpha|^2(\frac{n-1}{1+n}\chi^{\#}_{1,n+1} + \frac{n}{1+n}\chi^{\#}_{id(\mathscr{S}_n)}) + 2Re(\alpha)\frac{1}{1+n}\chi^{\#}_{1,n+1} + \frac{1}{1+n}\chi^{\#}_{id(\mathscr{S}_n)} \\
&= \frac{(n-1)|\alpha|^2 + 2Re(\alpha)}{n+1}\chi^{\#}_{1,n+1} + (\frac{n|\alpha|^2+1}{n+1})\chi^{\#}_{id(\mathscr{S}_n)}.
\end{aligned}$$

In the second hand

$$<\phi,\phi> = <\alpha\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}, \alpha\chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}> .$$

So

$$\begin{aligned}
<\phi,\phi> &= |\alpha|^2 <\chi^{\#}_{1,n+1}, \chi^{\#}_{1,n+1}> + <\chi^{\#}_{id(\mathscr{S}_n)}, \chi^{\#}_{id(\mathscr{S}_n)}> \\
&= \frac{|\alpha|^2}{(n+1)!}\sum_{\sigma\in\mathscr{S}_{n+1}} \chi^{\#}_{1,n+1}(\sigma)\chi^{\#}_{1,n+1}(\sigma) \\
&+ \frac{1}{(n+1)!}\sum_{\sigma\in\mathscr{S}_{n+1}} \chi^{\#}_{id(\mathscr{S}_n)}(\sigma)\chi^{\#}_{id(\mathscr{S}_n)}(\sigma) \\
&= \frac{|\alpha|^2 nn!}{(n+1)!} + \frac{n!}{(n+1)!} \\
&= \frac{n|\alpha|^2+1}{(n+1)}.
\end{aligned}$$

Then

$$<\phi,\phi> \phi = (\frac{n|\alpha|^2+1}{(n+1)})\alpha\chi^{\#}_{1,n+1} + (\frac{n|\alpha|^2+1}{(n+1)})\chi^{\#}_{id(\mathscr{S}_n)}.$$

By virtue of $\check{\phi} * \phi =< \phi, \phi > \phi$, we will have

$$\begin{aligned}(\frac{(n-1)|\alpha|^2+2nRe(\alpha)}{n+1}) &= (\frac{n|\alpha|^2+1}{n+)})\alpha \ \ and \\ (\frac{n|\alpha|^2+1}{n+1}) &= (\frac{n|\alpha|^2+1}{n+1})\end{aligned}$$

Then $(n-1)|\alpha|^2+2Re(\alpha) = (n|\alpha|^2+1)\alpha$. From this equality, we deduce that α is real and $(n-1)\alpha^2+2\alpha = (n\alpha^2+1)\alpha$. Thus $\alpha(n\alpha^2+(n-1)\alpha+1-2n) = 0$ and this equality becomes $\alpha(\alpha-1)(\alpha+\frac{1}{n}) = 0$. The solutions of this are $\alpha = 0$, $\alpha = 1$ and $\alpha = -\frac{1}{n}$. This proves the theorem

Remark 3.3. The spherical functions on finite group are all of positive type [5, p:66]

4. The spherical Fourier transform on permutation groups

In this section, we consider the horocyclic Radon transform which turns functions defined on $\mathscr{S}_{n+1}$ into functions defined on the set of the horocycles.

We note that the limit inductive $\varinjlim \mathscr{S}_n = \lim\limits_{n\geq 1} \mathscr{S}_n$ ={σ bijection from $\mathbb{N}^* \to \mathbb{N}^*$ such that $supp(\sigma)$ is finite } (where $supp(\sigma)$={k such that$\sigma(k) \neq k$}), so $\lim\limits_{n\geq 1} \mathscr{S}_n$ ={σ bijection from $\mathbb{N}^* \to \mathbb{N}^*; \exists n_\sigma$ such that $\sigma(k) = k, \forall k \geq n_\sigma + 1$}.

The construction of a function from its Radon transform is a central point of study of this section and the short way to invert the Radon transform is to note the connection with the Fourier transform and the horocyclic Radon transform is defind by the following formula [12]

$$Rf = \mathscr{F}_1^{-1} \circ \widetilde{f},$$

where $\mathscr{F}_1^{-1}$ is the inverse of the finite Fourier transform and $\widetilde{f}$ is the spherical Fourier transform of the Gelfand pair $(\mathscr{S}_{n+1}, \mathscr{S}_n)$. Of course we must place hypotheses on f so that the Fourier inversion formula is valid in order to obtain the following result

$$f = R^{-1} \circ \mathscr{F}_1^{-1} \circ \widetilde{f}.$$

More explicitly: The spherical Fourier transform is

$$\begin{aligned}
\wedge f(n) = \widetilde{f}(n) &= \int_{\mathscr{S}_{n+1}} \phi_n(\sigma) f(\sigma) d\mu(\sigma) \\
&= \frac{1}{(n+1)!} \sum_{\sigma \in \mathscr{S}_{n+1}} \phi_n(\sigma) f(\sigma) \\
&= \frac{1}{(n+1)!} \sum_{\sigma \in \mathscr{S}_{n+1}} f(\sigma) (\frac{-1}{n} \chi^{\#}_{1,n+1} + \chi^{\#}_{id(\mathscr{S}_n)}) \\
&= \frac{1}{(n+1)!} \sum_{\sigma \in \mathscr{S}_n} f(\sigma) - \frac{1}{n(n+1)!} \sum_{\sigma \in \mathscr{S}_n \tau_{1,n+1} \mathscr{S}_n} f(\sigma) \\
&= \frac{1}{(n+1)} [\frac{1}{n!} \sum_{\sigma \in \mathscr{S}_n} f(\sigma) - \frac{1}{nn!} \sum_{\sigma \in \mathscr{S}_n \tau_{1,n+1} \mathscr{S}_n} f(\sigma)].
\end{aligned}$$

And we set

$$\begin{aligned}
\wedge_1 f(n) = \widetilde{f}_1(n) &= \frac{1}{n!} \sum_{\sigma \in \mathscr{S}_n} f(\sigma) = f * \nu(Id) \\
\wedge_2 f(n) = \widetilde{f}_2(n) &= \frac{1}{nn!} \sum_{\sigma \in \mathscr{S}_n \tau_{1,n+1} \mathscr{S}_n} f(\sigma).
\end{aligned}$$

We note that the values of two functions $\wedge_1 f(n)$ and $\wedge_2 f(n)$ are thus equal to the averages of the function f over $\mathscr{S}_n$ respectively over $\mathscr{S}_n \tau_{n+1} \mathscr{S}_n$: elements of the double cosets $\mathscr{S}_n \backslash \mathscr{S}_{n+1} / \mathscr{S}_n$.

5. Inversion formula

The solution of reconstruction function from its spherical Fourier transform can be written in a simple iterative form which is computationally very tractable. Notice that the identity

$Id = Id_{S_i}$ (for all $1 \leq i \leq n$) . Consider first how to find $f(Id)$ from $\widetilde{f}$.

$$\begin{aligned}
\widetilde{f}(1) &= \frac{1}{2}f(Id) - \frac{1}{2}f(\tau_{1,2}) \\
\widetilde{f}(2) &= \frac{1}{3!}(f(Id)+f(\tau_{1,2})) - \frac{1}{2.3!}[f(\tau_{1,3})+f(\tau_{1,3}\circ\tau_{1,2})+f(\tau_{1,2}\circ\tau_{1,3}) \\
&\quad +f(\tau_{1,2}\circ\tau_{1,3}\circ\tau_{1,2})] \\
&= \frac{1}{3!}(f(Id)+f(\tau_{1,2})) - \frac{1}{2.3!}[f(\tau_{1,3})+f(\tau_{1,3}\circ\tau_{1,2})+f(\tau_{1,2}\circ\tau_{1,3}) \\
&\quad +f(\tau_{2,3})] \\
\widetilde{f}(3) &= \frac{1}{4!}[f(Id)+f(\tau_{1,2})+f(\tau_{1,3})+f(\tau_{2,3})+f(\tau_{1,3}\circ\tau_{1,2})+f(\tau_{1,2}\circ\tau_{1,3})] \\
&\quad -\frac{1}{3.4!}[f(\tau_{1,4})+f(\tau_{1,4}\circ\tau_{1,2})+f(\tau_{1,4}\circ\tau_{2,3})+f(\tau_{1,4}\circ\tau_{1,3}) \\
&\quad +f(\tau_{1,4}\circ\tau_{1,3}\circ\tau_{1,2})+f(\tau_{1,4}\circ\tau_{1,2}\circ\tau_{1,3})+\sum_{\sigma\in S_3\tau_{1,4}S_3} f(\sigma)] \\
\ldots\ldots &= \ldots.. \\
\widetilde{f}(n) &= \frac{1}{(n+1)!}[f(Id)+\sum_{\sigma\in\mathscr{S}_n\setminus Id} f(\sigma)] - \frac{1}{n.(n+1)!}\sum_{\sigma\in\mathscr{S}_n\tau_{1,2}\mathscr{S}_n} f(\sigma).
\end{aligned}$$

Since $f(Id)$ occurs in the sum for $\widetilde{f}(i)$ (for all $1 \leq i$) and $f(\tau_{1,2})$ occurs also in the sum for $\widetilde{f}(i)$ (for all $1 \leq i$) and continuing in this way in a simple iterative form , we have an explicit inversion formula for $f(Id)$ which can easily must be

$$f(Id) = \frac{1}{n}\sum_{k=1}^{k=n}(1+k)!\widetilde{f}(k) - \frac{1}{n}\sum_{k=2}^{k=n}(n-k)f(\tau_{1,k}) - \frac{1}{n^2}f(\tau_{1,n+1}) + \ldots$$

Applying the same reasoning for any $\tau_{1,i} \in \mathscr{S}_{n+1}$ in stead of Id leads to the full inversion formula

$$f(\tau_{1,i}) = \frac{1}{n-i}\sum_{k=1}^{k=n}(1+k)!\widetilde{f}(k) - \frac{n}{n-i}f(Id) + \ldots$$

Remark 5.1. In the case of permutations group, the difficulty arises for an excellent account of analysis and modeling of front propagation in permutation group and their utility in applications to the heat equations.

6. Horocyclic Radon transform and inversion formula

Given a function $f : \mathscr{S}_{n+1} \to \mathbb{C}$, the Radon transform of f is

$$\begin{aligned} R_{n+1}f(\sigma) &= \sum_{\sigma' \in \sigma \mathscr{S}_n} f(\sigma') \\ &= \frac{1}{n!} \sum_{h \in \mathscr{S}_n} f(\sigma.h) \end{aligned}$$

Remark 6.1. For any $h_0 \in \mathscr{S}_n$, we have

$$R_{n+1}f(\sigma) = R_{n+1}f(\sigma.h_0), \quad R_{n+1}f(Id_{\mathscr{S}_n}) = \frac{1}{n!} \sum_{h \in \mathscr{S}_n} f(\sigma.h) = \wedge_1 f(n)$$

This Radon transform is also defined for every $B \in \mathscr{S}_{n+1}/\mathscr{S}_n = \mathrm{Z}_{n+1}$ and every $f \in L(\mathrm{Z}_{n+1})$ by

$$Rf(B) = \sum_{A \in Z_{n+1}: A \subset B} f(A).$$

So for an absolutely summable function on $\lim\limits_{\rightarrow} Z_{n+1} = \mathbb{N}^*$, the Radon transform [16]

$$Rf(m) = \sum_{k=1}^{k=\infty} f(km),$$

for all $m \in \mathbb{N}^*$, assuming some more rapid decay (say $|f(n)| < cn^{-2-\varepsilon}$).

This transform can be easily inverted [16], First we find $f(1)$ from Rf of the form

$$f(1) = c_1 Rf(1) + c_2 Rf(2) + \ldots + c_n Rf(n) + \ldots$$

for certain coefficients c_n that are uniquely determined. The coefficients must be equal to the Mobius function $\mu(n)$, which is defined to be $(-1)^k$ if n has k distinct prime factors, and 0 if n is divisible by a square of a prime. The full inversion formula [16] is

$$f(n) = \sum_{k=1}^{\infty} \mu(k) Rf(nk)$$

7. Applications

In order to state the applications of the Radon transform, we consider functions $f : \mathscr{S}_{n+1} \times \mathbb{N} \longrightarrow \mathbb{C}$, whose values are denoted by $f(x,k) = f_k(x)$. In our setting, $x \in \mathscr{S}_{n+1}$ represents the space variable and $k \in \mathbb{N}$ the times variables. Let the sub-Laplacian of f with respect to the space variable defined in the following way

$$\triangle f(x) = \frac{1}{n!} \sum_{h \in \mathscr{S}_n} f(x.h) - f(x) = f * \nu(x) - f(x).$$

Now we consider the following boundary value problems, denoted $\mathscr{H}$, in analogy with the corresponding problems for the classical heat equation.

$$(\mathscr{H}) \qquad \triangle f_k(x) \quad = \quad f_{k+1}(x) - f_k(x) \qquad given \;\; f_0(x).$$

The heat equation associated to sub-Lapcian operator can be written in the following way

$$(\mathscr{H}) \;\; f_k(x) = \frac{1}{n!} \sum_{h \in \mathscr{S}_n} f_{k-1}(x.h) \;\; given \;\; f_0 = \delta_e,$$

Theorem 7.1. *The solution of the heat equation is given by*

$$f_k(x) = f_0 * \nu(x),$$

with $\nu = \frac{1}{n!} \sum\limits_{\sigma \in \mathscr{S}_n} \delta_\sigma$ *the Haar measure in* $\mathscr{S}_n$.
If $f_0 = \delta_e$, *we have* $f_k = \nu$.

Proof. The equation $(\mathscr{H})$ is therefore

$$f_k(x) = f_{k-1} * \nu(x) = f_0 * \nu^k(x),$$

with $\nu = \frac{1}{n!} \sum\limits_{\sigma \in \mathscr{S}_n} \delta_\sigma$ the Haar measure in $\mathscr{S}_n$ and $\nu^k = \nu * \nu * \nu * ... * \nu$ k-times. Using the fact that

$$(\sum_{\sigma \in \mathscr{S}_n} \delta_\sigma) * (\sum_{\sigma \in \mathscr{S}_n} \delta_\sigma) = n! \sum_{\sigma \in \mathscr{S}_n} \delta_\sigma.$$

Because $(\sum\limits_{\sigma \in \mathscr{S}_n} a_\sigma \delta_\sigma) * (\sum\limits_{\sigma \in \mathscr{S}_n} b_\sigma \delta_\sigma) = \sum\limits_{\sigma \in \mathscr{S}_n} (\sum\limits_{tu = \sigma} a_t b_u) \delta_\sigma$.
So $(\sum\limits_{\sigma \in \mathscr{S}_n} \delta_\sigma)^k = (n!)^{k-1} \sum\limits_{\sigma \in \mathscr{S}_n} \delta_\sigma$.
Then $\nu^k = \nu$ and we will have $f_k = f_0 * \nu = \nu$, with $f_0 = \delta_e$. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements

The authors would like to thank Professor Jacques Faraut for his comments and suggestions during the International Conference in honor of Professor Ahmed ABOUELAZand Professor Francois ROUVIERE at Casablanca on May 25-27, 2015.

REFERENCES

[1] O.E. Fourchi, A. Abouelaz, Horocyclic Radon Transform on Damek-Ricci Space, Bull. Polish Academy Sci. 49 (2001), 107-140

[2] O.E. Fourchi, A. Abouelaz, Spectral analysis on Damek-Ricci Space, preprint.

[3] E.C. Tarabusi, J. M. Cohen, F. Colonna, Characterization of the range of the Radon transform on Homogeneous trees, Elect. Res. Announcements of the Amer. Math. Soc. 5 (1999), 11-17.

[4] J. L. Clerc, Groupes compacts, C. I. M. P. A, Nice (1982).

[5] J. Dieudonne, Element d'analyse V, Gautier-Villars (1975).

[6] J. D. Dixon and B. Mortimer, GPermutations group, G.T. M, 163.

[7] J. Faraut,Analyse harmonique sur les paires de Gelfand et les espaces hyperboliques, C. I. M. P. A,Nice (1982).

[8] A. Figa-Talamanca and T. Steger,Harmonic analysis for anisotropic Random walks on homogeneous trees, Memoirs of Amer. Maths. Soc. (1994).

[9] A. Intissar and M. V. Ould Mustapha,Solution explicite de l'equation des ondes dans l'espace symmetrique de type non compact de rang un, C. R. Acad. Sc. Paris, 321 (1995), 77-81.

[10] S. Helgason, A duality for symmetric spaces with applications to group representations, Adv. Math. 5 (1970), 1-154.

[11] S. Helgason, Geometric analysis on symmetric spaces,Math. Surveys and Monogr. Amer. Math. Soc. Providence (R.I), 39 (1994).

[12] S. Helgason, The Radon transform, Boston Bassel Stuttgart,Birkhauser, 1980 (Progress in Mathematics: 5).

[13] M. P. Malliavin, Les groups finis et leurs representations complexes, Masson (1981).

[14] G. D. James,The representation theory of the symmetric groups,Springer lecture Notes 682, Spriger Verlag Heidelburg (1978).

[15] F. Matus and J. Flusser,Image representations via finite Radon transform , IEEC, Trans on Pattern analysis and machine Intelligence, 15, N 10, octobre (1993),996-1006.

[16] R. S. Strichartz, Radon inversion-variations on a theme, Amer. Math. Monthly 89, (1982), 377-384.

[17] F. Scarabotti,Radon transforms on the symmetric group and harmonic analysis of a class of invariant laplacians, Forum Math. 10 (1998), 407-411.

[18] F. Scarabotti, Fourier analysis of a class of finite Radon transforms, SIAM J. Discrete Math. 16 (1998), 545-554.