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ON SOME SLOPE-LIMITER METHODS FOR THE LINEAR ADVECTION EQUATION

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Abstract: In this paper, we propose two slope-limiter methods for solving hyperbolic conservation laws. The methods are developed through flux formulation with piecewise linear construction and applied to solve the Linear Advection Equation using two initial conditions. The results which were compared with those of the Lax-Wendroff method, and the minmod method demonstrate the accuracy of the proposed methods.
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INTRODUCTION

Conservation laws arise in many models in science and engineering. They are applied in fluid and gas dynamics, relativity theory, quantum mechanics, aerodynamics, meteorology and astrophysics (Eymard *et al*, 2003). Numerical methods for solving conservation laws include the finite difference method, finite element method and finite volume method. The finite volume method is now a popular choice for solving conservation laws because of its accuracy and ability to handle complex geometries as well as good approximations of boundary conditions (LeVeque, 2004, Hu & Joseph, 1990, Grigoryan, 2010, Moroney, 2006).

According to LeVeque (2004), finite volume methods for solving hyperbolic conservation laws include Fromm’s method, Beam-Warming method and Lax-Wendrroff method. These methods generate good approximations for smooth solutions but fail near discontinuities – they generate oscillatory approximations to discontinuous solutions. Slope-limiter methods are high resolution

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methods that use slope-limiters to avoid the spurious oscillations that occur with high order spatial discretization schemes due to shocks, discontinuities or sharp changes in the solution domain (Mazzia, 2010).

In this paper, we propose two slope-limiters for solving hyperbolic conservation laws and have applied them to solve the Linear Advection Equation, a type of conservation law.

METHODS

Formulation of Finite Volume Methods for Conservation Laws (LeVeque, 2004).

Consider the flow of gas in a tube where properties of the gas such as density and velocity are constant. If $u(x,t)$ and $v(x,t)$ are the density and velocity of the gas respectively, the rate of change of mass in $[x_1, x_2]$ is given as

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx = u(x_1,t)v(x_1,t) - u(x_2,t)v(x_2,t) \tag{1}$$

Equation (1) is the integral form of conservation laws. If

$$C_i = \left(x_{i-1/2}, x_{i+1/2}\right)$$

denotes the ith grid cell, then from equation (1) we have that

$$\frac{d}{dt}\int_{C_i} u(x,t)dx = f\left(u\left(x_{i-1/2},t\right)\right) - f\left(u\left(x_{i+1/2},t\right)\right) \tag{2}$$

Integrating equation (2) in time from t_n to t_{n+1}, rearranging and dividing by Δx gives

$$\frac{1}{\Delta x}\int_{C_i} u(x,t_{n+1})dx = \frac{1}{\Delta x}\int_{C_i} u(x,t_n)dx - \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} f\left(u\left(x_{i+1/2},t\right)\right)dt - \int_{t_n}^{t_{n+1}} f\left(u\left(x_{i-1/2},t\right)\right)dt\right] \tag{3}$$

This suggests numerical methods of the form

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right) \tag{4}$$

where $F_{i-1/2}^n$ is some approximation to the average flux along $x = x_{i-1/2}$ at $t = t_n$ given as

$$F_{i-1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f\left(u\left(x_{i-1/2},t\right)\right)dt.$$

Equation (4) is the general form of the finite volume methods.

For the linear advection equation

$$F_{i-1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} a\tilde{u}^n\left(x_{i-1/2},t\right)dt.$$

From the cell average $\bar{u}_i^n$, we can construct a piecewise linear function of the form

$$\tilde{u}^n(x,t_n) = \bar{u}_i^n + \sigma_i^n(x - x_i) \qquad \text{for} \qquad x_{i-1/2} \le x \le x_{i+1/2} \tag{5 }$$

where

$$x_i = \frac{1}{2}\left(x_{i-1/2} + x_{i+1/2}\right) = x_{i-1/2} + \frac{1}{2}\Delta x.$$

The expression for the flux $F_{i-1/2}^n$ becomes

$$\begin{aligned} F_{i-1/2}^n &= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} a\tilde{u}^n\left(x_{i-1/2}, t\right)dt \\ &= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} a\tilde{u}^n\left(x_{i-1/2} - a(t - t_n), t_n\right)dt \\ &= a\bar{u}_{i-1}^n + \frac{1}{2}a(\Delta x - a\Delta t)\sigma_{i-1}^n. \end{aligned}$$

Similarly,

$$F_{i+1/2}^n = a\bar{u}_i^n - \frac{1}{2}a(\Delta x - a\Delta t)\sigma_i^n.$$

Using the expressions for $F_{i-1/2}^n$ and $F_{i+1/2}^n$ in (4) gives

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{a\Delta t}{\Delta x}(\bar{u}_i^n - \bar{u}_{i-1}^n) - \frac{1}{2}\frac{a\Delta t}{\Delta x}(\Delta x - a\Delta t)(\sigma_i^n - \sigma_{i-1}^n) \tag{6}$$

where σ_i^n is the slope in the ith grid cell C_i.

The finite volume method (6) depends on the choice of slope. Choosing the downwind slope (7) gives the Lax-Wendroff method.

$$\sigma_i^n = \frac{\bar{u}_{i+1}^n - \bar{u}_i^n}{\Delta x}. \tag{7}$$

But this slope is defined based on the assumption that the solution is smooth. Near a discontinuity there is no reason to believe that introducing this slope will improve the accuracy.

Slope Limiters

Slope limiters are defined with the aim of limiting the solution gradient to avoid oscillations. Accuracy is therefore expected even at discontinuities. Example of an existing slope-limiter is the minmod slope defined as

$$\sigma_i^n = \text{minmod}\left(\frac{\bar{u}_i^n - \bar{u}_{i-1}^n}{\Delta x}, \frac{\bar{u}_{i+1}^n - \bar{u}_i^n}{\Delta x}\right)$$

where the minmod function of two arguments is defined as

$$\text{minmod}(a,b) = \begin{cases} a \text{ if } |a| < |b| \text{ and } ab > 0 \\ b \text{ if } |b| < |a| \text{ and } ab > 0. \\ 0 \text{ if } ab \le 0 \end{cases}$$

Proposed Slope Limiters

We propose two slope-limiters which we call 'amod' and 'bmod', defined as

$$\text{amod:} \qquad \sigma_i^n = \frac{1}{2}\left(\frac{\bar{u}_{i+1}^n - \bar{u}_{i-1}^n}{2\Delta x}\right) + 2\left(\text{minmod}\left(\frac{1}{2}\left(\frac{\bar{u}_{i+1}^n - \bar{u}_{i-1}^n}{2\Delta x}\right), \left(\frac{\bar{u}_{i+1}^n - \bar{u}_i^n}{\Delta x}\right)\right)\right).$$

$$\text{bmod:} \qquad \sigma_i^n = \text{mean}(\mathrm{V}, \mathrm{K})$$

where

$$\mathrm{V} = \text{minmod}\left(2\left(\frac{\bar{u}_{i+1}^n - \bar{u}_i^n}{\Delta x}\right), \left(\frac{\bar{u}_i^n - \bar{u}_{i-1}^n}{\Delta x}\right)\right),$$

$$\mathrm{K} = \text{minmod}\left(\left(\frac{\bar{u}_{i+1}^n - \bar{u}_i^n}{\Delta x}\right), 2\left(\frac{\bar{u}_i^n - \bar{u}_{i-1}^n}{\Delta x}\right)\right),$$

and

$$\text{mean}\ (a,b) = \frac{a+b}{2}.$$

NUMERICAL EXPERIMENTS

In this section, we will solve the linear advection equation (8) with unit velocity subject to two initial conditions.

$$u_t + u_x = 0, \qquad x \in [-1, 1] \tag{8}$$

Solutions are obtained using the Lax-Wendroff method, the minmod method and the proposed methods. We will solve for $T = 2$. On the graphs, the red thick line represents the exact solution while the blue dotted line represents the approximate solution. The minimum and maximum values of the solutions – a test of accuracy of the methods, are obtained and tabulated.

Example One

Solve Equation (8) subject to the initial condition

$$u(x,0) = \sin(2\pi x). \tag{9}$$

This is a smooth solution and the results are thus, presented in terms of errors, and the errors are obtained using the 2-norm.

Table 1: Errors in 2-norm obtained from Solution of Equation (8) subject to initial condition (9) by the downwind slope, minmod, 'amod' and 'bmod' limiters.

N	Downwind limiter (Lax-Wendroff method)	Minmod limiter (minmod method)	'amod' limiter ('amod' method)	bmod' limiter ('bmod' method)
50	4.9754×10^{-2}	5.0369×10^{-2}	5.0337×10^{-2}	5.0400×10^{-2}
100	2.5069×10^{-2}	2.5147×10^{-2}	2.5148×10^{-2}	2.5148×10^{-2}
200	1.2558×10^{-2}	1.2568×10^{-2}	1.2568×10^{-2}	1.2569×10^{-2}
400	6.2822×10^{-3}	6.2834×10^{-3}	6.2834×10^{-3}	6.2834×10^{-3}
800	3.1415×10^{-3}	3.1416×10^{-3}	3.1416×10^{-3}	3.1416×10^{-3}

Example Two

Consider Equation (8) subject to the initial condition

$$u(x,0) = \begin{cases} 1, & \text{if} \quad |x| < 0.1, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

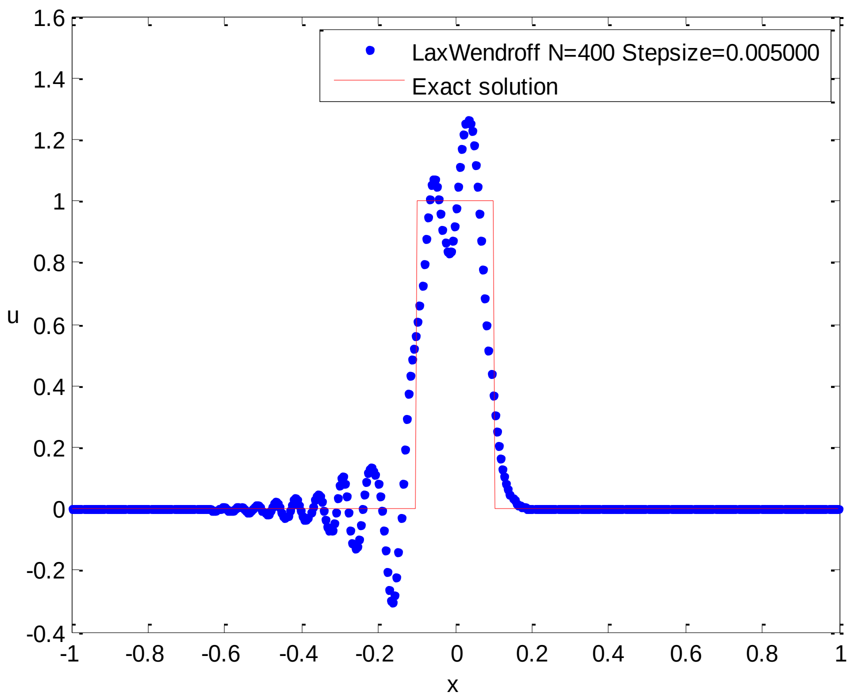

Fig. 1: Solution of Equation (8) subject to initial condition (10) using the Lax-Wendroff method with $N = 400$.

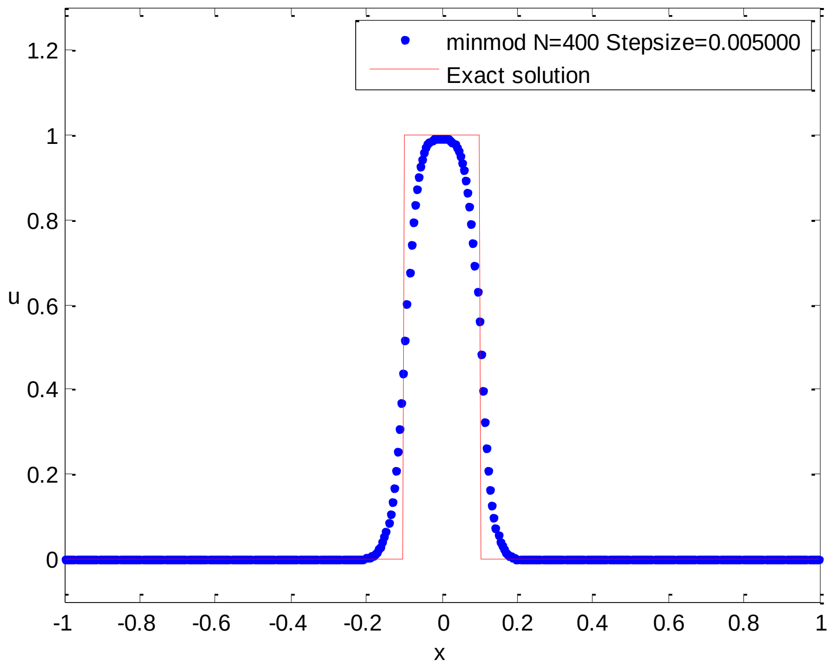

Fig. 2: Solution of Equation (8) subject to initial condition (10) using the minmod method with $N = 400$.

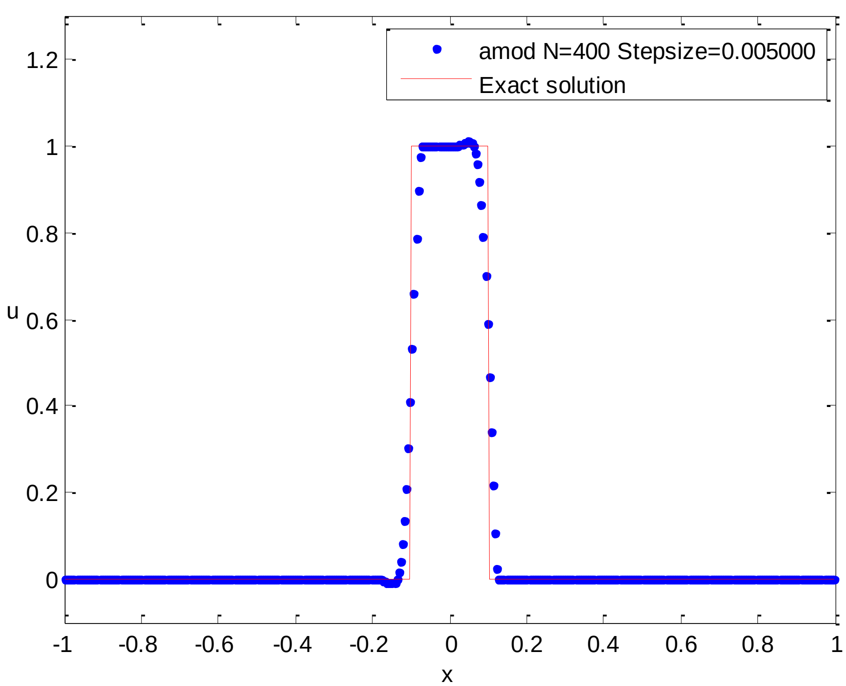

Fig. 3: Solution of Equation (8) subject to initial condition (10) using the ‘amod’ method with $N = 400$.

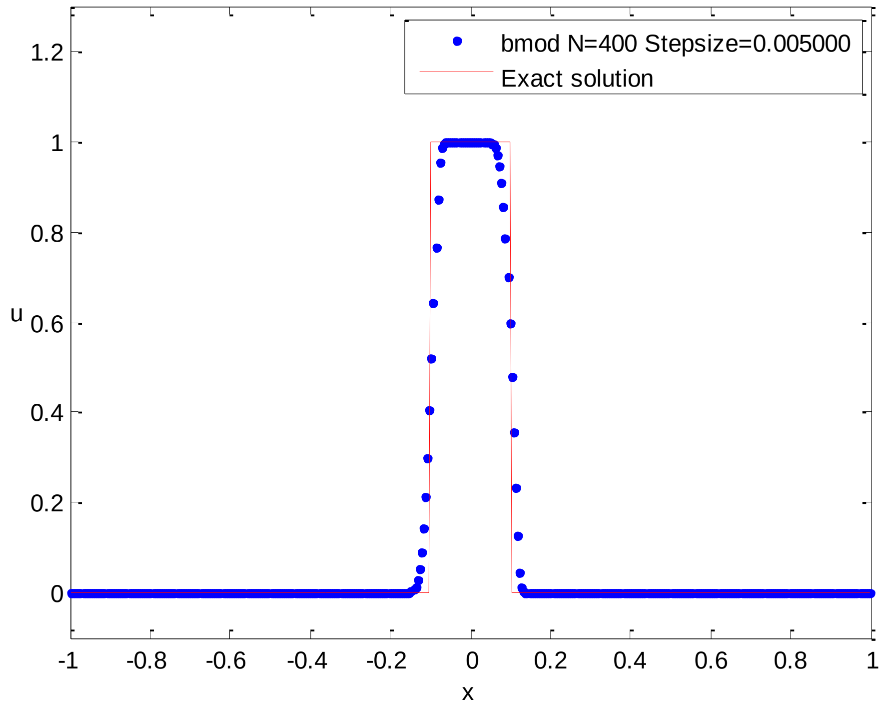

Fig. 4: Solution of Equation (8) subject to initial condition (10) using the 'bmod' method with $N = 400$.

Table 2: Minimum and Maximum values of the Exact Solution, and approximate solution of Equation (8) subject to initial condition (10) by Lax-Wendroff, Minmod, 'amod' and 'bmod' methods with $N = 400$.

Method	**Min(u)**	**Max(u)**
Exact	0.0000	1.0000
Lax-Wendroff	−0.3053	1.2618
Minmod	0.0000	0.9927
'amod'	−0.0095	1.0095
'bmod'	0.0000	1.0000

Example Three

Consider Equation (8) subject to the initial condition

$$u(x,0)=\begin{cases}\frac{1}{6}G(x,z-\delta)+G(x,z+\delta)+4G(x,z), & -0.8\le x\le -0.6\\ 1, & -0.4\le x\le -0.2\\ 1-|10(x-0.1)|, & 0\le x\le 0.2\\ \frac{1}{6}F(x,a-\delta)+F(x,a+\delta)+4F(x,a), & 0.4\le x\le 0.6\\ 0, & \text{otherwise}\end{cases} \qquad (11)$$

where $G(x,z)=\exp(-\beta(x-z)^2)$, $F(x,a)=\{\max(1-\alpha^2((x-z)^2,0\}^{\frac{1}{2}}$. The constants are taken as $a=0.5,\ z=-0.7,\ \delta=0.005,\ \alpha=10,$ and $\beta=(\log 2)/36\delta^2$.

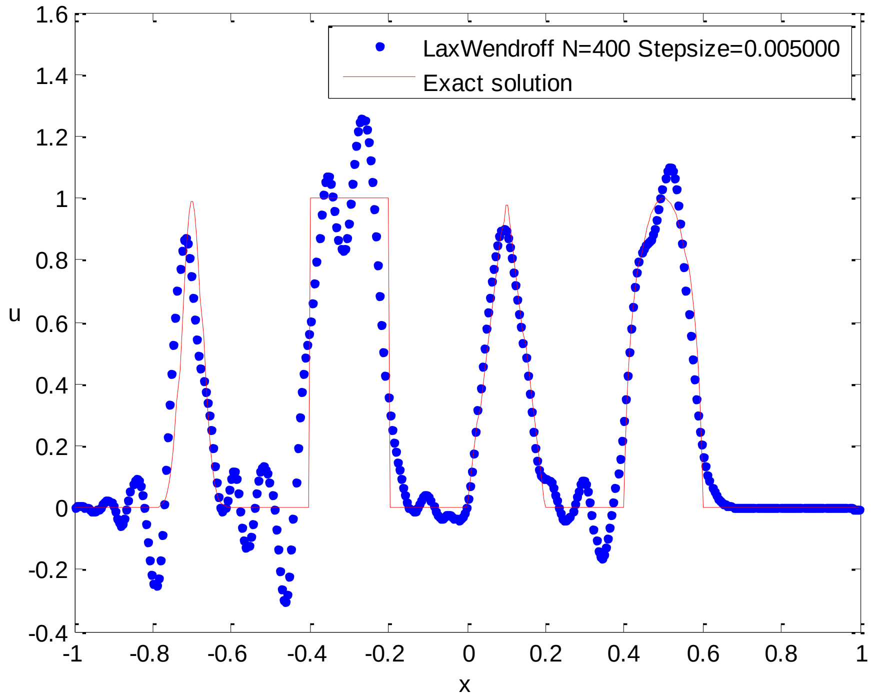

Fig. 5: Solution of Equation (8) subject to initial condition (11) using the Lax-Wendroff method with $N=400$.

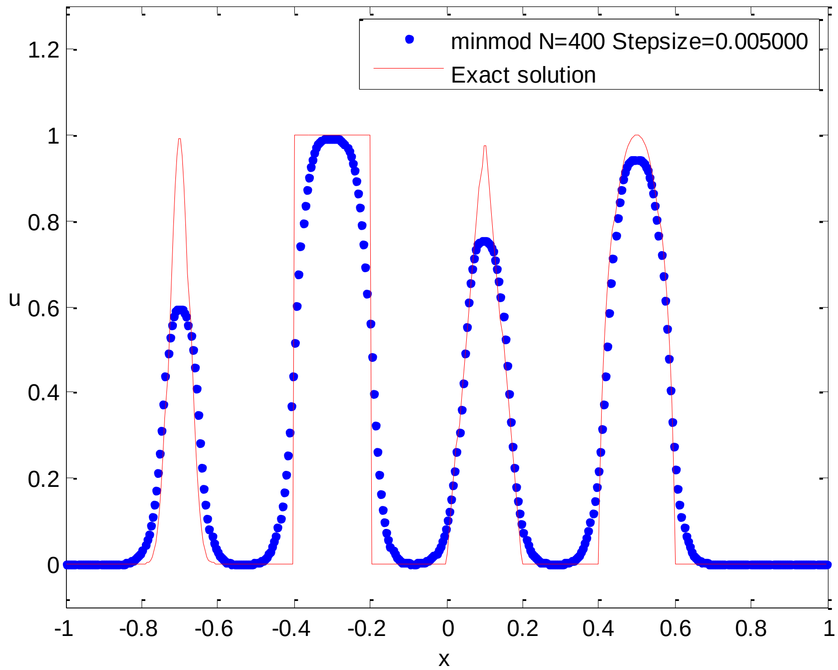

Fig. 6: Solution of Equation (8) subject to initial condition (11) using the minmod method with $N = 400$.

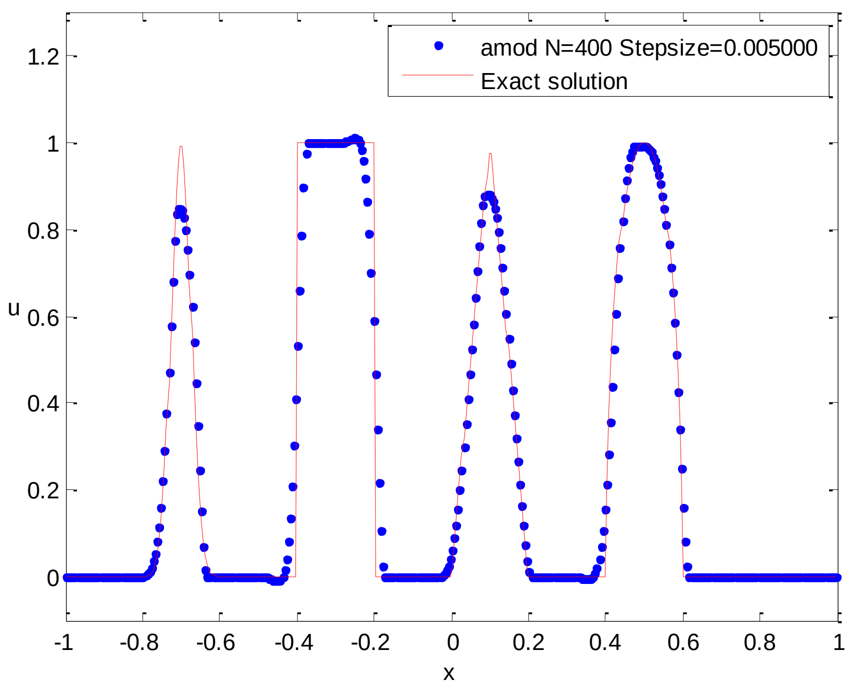

Fig. 7: Solution of Equation (8) subject to initial condition (11) using the 'amod' method with $N = 400$.

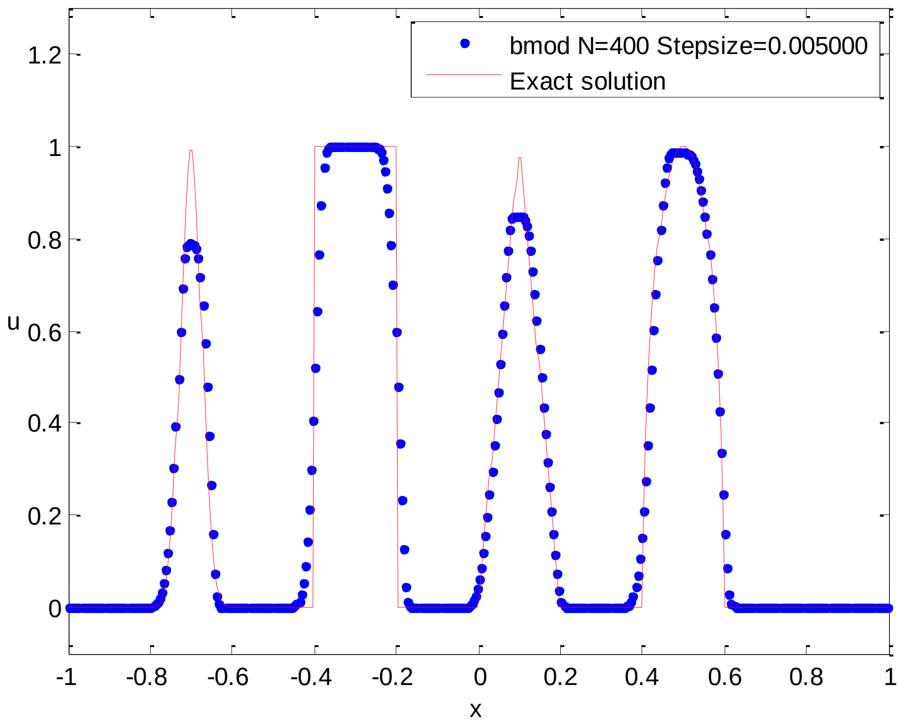

Fig. 8: Solution of Equation (8) subject to initial condition (11) using the ‘bmod’ method with $N = 400$.

Table 3: Minimum and Maximum values of the Exact Solution, and approximate solution of Equation (8) subject to initial condition (11) by Lax-Wendroff, Minmod, ‘amod’ and ‘bmod’ methods with $N = 400$.

Method	**Min(u)**	**Max(u)**
Exact	0.0000	1.0000
Lax-Wendroff	-0.3053	1.2618
Minmod	0.0000	0.9927
‘amod’	-0.0095	1.0095
‘bmod’	0.0000	1.0000

DISCUSSION

Table 1 shows result of Equation (8) subject to initial condition (9). The obtained result shows that the Lax-Wendroff method produced errors slightly less than the other methods hence, more accurate. This shows the efficiency of the Lax-Wendroff method for smooth solutions. Figures 1 and 5 are solutions obtained using the Lax-Wendroff method. The results clearly demonstrate the deficiency of finite volume methods that are not slope-limiter methods – near discontinuities they generate oscillations. Figures 2 and 6 are solutions by the existing slope-limiter method, the minmod method. Here, no oscillations are generated rather; the discontinuities that arose in the solution are resolved. However, the solution suffers from numerical diffusion. Figures 3 and 7 are solutions by the proposed 'amod' method. This method produced good results and resolves the discontinuities that arose in the solution. Nevertheless, slight oscillations are observed near discontinuities. Figures 4 and 8 are solutions by the proposed 'bmod' method. Results produced here are accurate and discontinuities that arose in the solution are properly resolved, even better than the minmod method. No case of oscillation is recorded even at discontinuities.

The results discussed above are evident in Tables 2 and 3. The tables record the minimum and maximum values of the solutions to demonstrate the accuracy of the methods. They show which methods produce oscillations and which do not.

CONCLUSION

The proposed methods produced good results compared to the existing ones. The methods should therefore be used as alternatives to the existing ones. In general, slope-limiter methods produced better results than finite volume methods that are not slope-limiter methods, therefore, slope-limiter methods should be applied to solve the linear advection equation in particular, and conservation laws in general.

Conflict of Interests

The authors declare that there is no conflict of interests.

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