Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 4, 1052-1072
ISSN: 1927-5307

THE DYNAMICS OF PREY-PREDATOR MODEL WITH DISEASE IN PREY

RAID KAMEL NAJI[1], KAWA AHMED HASAN[2,*]

[1]Department of Mathematics, College of Science, University of Baghdad, Iraq

[2]Department of Mathematics, School of Science, Faculty of Science and Science of Education ,University of Sulaimani, Iraq

Abstract: In this paper an eco-epidemiological model, consisting of a Crowley–Martin prey-predator with disease in prey ,is investigated analytically as well as numerically. The conditions for the existence and local stability of equilibrium points are obtained. The global dynamics is studied numerically for different sets of initial values and for different sets of parameters values.

Keywords: An eco-epidemiological model; Functional response; equilibrium points; global stability; numerical simulation.

2000 AMS Subject Classification: 47H17; 47H05; 47H09

1. Introduction: Ecology and epidemiology are two major and distinct fields of study. However, there are situations where some diseases, which are responsible for an epidemic, have a strong impact heavily on the dynamics of ecological (prey-predator or competition) systems. In fact mathematical models became important tools in analyzing the effect of spreading and controlling infections diseases on coexistence and the dynamical behavior of ecological systems. For instance, Hethcote et al [11] showed that how the presence of parasites can change the demographic behavior of population. Indeed, such diseases regulate the host population density and some times help the coexistence of species [2,8]. The

*Corresponding author
E-mail addresses: rknaji@gmail.com (R.K. Naji), kawa79b@gmail.com (K.A. Hasan).

Received March 7, 2012

mathematical models which describe the dynamics of the coupling of an ecological model and an epidemiological SI , SIS , or SIR model are known as eco-epidemiological models. Such models have received much attention from scientists in recent years.

Anderson and May [1] who were the first to propose an eco-epidemiological model by merging the Lotka–volterra prey–predator model and the epidemiological SIR model was introduced by Kermack and Mckendrick. Many works have been devoted to study of the effects of a disease on a prey-predator system [3, 4,5,6,7, 9, 11,14,15, 16].Most of these studies focused on the dynamical behavior of prey-dependent prey-predator model in company with SI or SIR epidemiological model. In this chapter, an eco-epidemiological model consisting of Crowley–Martin prey-predator model with SIS epidemiological model was proposed and analyzed.

2.The mathematical model

Let X(t) be the total population density of the prey species and Y(t) be the population density of the predator species. Now, in order to formulate our eco-epidemiologic model, we make the following assumptions.

H_1**)** In the absence of disease, the prey population density grows according to a logistic curve with carrying capacity K (K>0) and an intrinsic growth rate constant R(R>0)

H_2) In the presence of disease, we assume that the total prey population X is composed of two population classes , the first is the class of susceptible prey denoted by S, and the other is the class of the infected prey, denoted by I. Therefore, at any time t, the total density of prey population is X(t)=S(t)+I(t).

H_3) It is assumed that, only susceptible prey S is capable of reproducing with logistic law, while the infected prey I is removed by death with a positive death rate constant d_1 or by predation before having the possibility of reproducing. Further, the infective population I still contributes with S to the population growth toward the carrying capacity.

H_4) The disease spreads among the population only and the disease is not genetically inherited. The infected population may recover and return to the susceptible class with a positive recover rate constant α. Moreover, the incidence is assumed to be the simple mass action incidence λSI, where $\lambda > 0$ is called infected rate constant or transmission coefficient.

H_5) The predator has a death rate constant d_2 ($d_2 > 0$), and it mainly eats the infected prey according to Crowley –Martin type of functional response, this is due to the fact that the infected individuals are less active and can be caught more easily. The coefficient in conversing prey into predator is e ($e > 0$). According to the above assumptions, the dynamics of the eco-epidemiological model consisting of Crowley –Martin prey-predator model with SIS epidemic model can be represented by the following set of differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= RS\left(1-\frac{S+I}{K}\right)-\lambda SI+\alpha I\\
\frac{dI}{dt} &= \lambda SI-\frac{aIY}{(1+bI)(1+cY)}-(d_1+\alpha)I\\
\frac{dY}{dt} &= \frac{eaIY}{(1+bI)(1+cY)}-d_2Y
\end{aligned} \tag{1}$$

3. Main results

Theorem (3.1): *All solutions of the system (1) which initiate in R_+^3, are uniformly bounded provided the following condition holds*

$$\frac{\alpha}{d_1+\alpha} < e < 1 \tag{2}$$

Proof. Consider the following function $W(t) = S + eI + Y$

Then time derivative of $W(t)$ along the trajectory of the system (1) can be written as: $\frac{dW}{dt} = RS - \frac{RS^2}{K} - \frac{RSI}{K} - \lambda SI + \alpha I + \lambda eSI - (d_1+\alpha)eI - d_2Y$

Since d_2 and $d_1 + \alpha - \frac{\alpha}{e}$ are positive constant, then there exists a positive constant namely

μ with $\mu \le \min\{ d_2, d_1 + \alpha - \frac{\alpha}{e} \}$.Therefore, the following inequality

$$\frac{dW}{dt} + \mu W \le S(R + \mu - \frac{RS}{K}) - \lambda(1-e)SI - e((d_1 + \alpha) - \frac{\alpha}{e} - \mu)I - (d_2 - \mu)Y \tag{3a}$$

Now since the maximum value of the expression $S(R + \mu - \frac{RS}{K})$ is $\frac{K(R+\mu)^2}{4R}$ and the condition (2) holds, then the inequality (3a) gives:

$$\frac{dW}{dt} + \mu W \le \frac{K(R+\mu)^2}{4R} \tag{3b}$$

It is clear that the right –hand side of the Eq. (3b) is constant. Then we can find a constant $M > 0$ such that $\frac{dW}{dt} + \mu W \le M$ with $\frac{K(R+\mu)^2}{4R} \le M$

Applying the theory of differential inequalities [10], we obtain

$0 < W(S,I,Y) \le \frac{M}{\mu}(1 - e^{-\mu t}) + W(S(0), I(0), Y(0))e^{-\mu t}$.Thus for $t \longrightarrow \infty$, we have

$0 < W \le \frac{M}{\mu}$.Hence all the solutions of the system (1) that initiate in R_+^3 are eventually

confined in the region $B = \{(S,I,Y) \in R_+^3 : S + eI + Y < \frac{M}{\mu} + \varepsilon \;\; \forall \varepsilon > 0\}$.

3.1 SIS epidemic model

The eco-epidemiological population model given by the system (1) is a simple Lotka-Volterra prey-predator system with logistic growth for the prey and Crowley-Martin type of functional response. Obviously, in the absence of predator species the system (1) will be reduced to the following SIS epidemic model

$$\begin{aligned} \frac{dS}{dt} &= RS(1 - \frac{S+I}{K}) - \lambda SI + \alpha I \\ \frac{dI}{dt} &= \lambda SI - (d_1 + \alpha)I \end{aligned} \tag{4}$$

Clearly (4) is a subsystem of the system (1), which represents a mathematical model that describes the dynamics of SIS epidemic model. It possesses three biologically relevant equilibria $(0,0)$, $(K,0)$ and $(\bar{S},\bar{I})$. The equilibrium points $(0,0)$ and $(K,0)$ are always exist. However, the positive equilibrium $(\bar{S},\bar{I})$ exists in the Int. R_+^2 of the SI –plane if there is a positive solution to the following set of algebraic equations

$$RS(1-\frac{S+I}{K})-\lambda SI+\alpha I=0 \tag{5a}$$

$$\lambda S-(d_1+\alpha)=0 \tag{5b}$$

Straight forward computations give that $\bar{S}=\frac{(d_1+\alpha)}{\lambda}$, $\bar{I}=\frac{R(d_1+\alpha)[\lambda K-(d_1+\alpha)]}{R\lambda(d_1+\alpha)+d_1\lambda^2 K}$ (6)

Clearly $\bar{I}>0$ under the following condition

$$\lambda>\frac{d_1+\alpha}{K} \tag{7}$$

Now, in order to discuss the stability of the system (4), the Variational matrix for the system (4) at the point (S,I) is determined.

$$V(S,I)=\begin{pmatrix} R[1-\frac{2S}{K}]-I(\frac{R}{K}+\lambda) & \alpha-\frac{RS}{K}-\lambda S \\ \lambda I & \lambda S-(d_1+\alpha) \end{pmatrix} \tag{8}$$

Accordingly, the Variational matrix about $(0,0)$ is $\begin{pmatrix} R & \alpha \\ 0 & -(d_1+\alpha) \end{pmatrix}$

Hence $(0,0)$ is a hyperbolic saddle point with locally unstable manifold in the S-direction and locally stable manifold in the I-direction. Further, the Variational matrix about $(K,0)$ can be written as: $\begin{pmatrix} -R & \alpha-R-\lambda K \\ 0 & \lambda K-(d_1+\alpha) \end{pmatrix}$

Obviously, $(K,0)$ is locally asymptotically stable if and only if the condition (7) has been violated or $(\bar{S},\bar{I})$ does not exist, otherwise, it's a hyperbolic saddle point with locally stable manifold in the S-direction and with locally unstable manifold in the I-direction. Finally, the Variational matrix about $(\bar{S},\bar{I})$ is:

$$V(\bar{S},\bar{I}) = \begin{pmatrix} R[1-\frac{2\bar{S}}{K}]-\bar{I}(\frac{R}{K}+\lambda) & \alpha-\frac{R\bar{S}}{K}-\lambda\bar{S} \\ \lambda\bar{I} & 0 \end{pmatrix}$$

Consequently, $\text{tr}(V((\bar{S},\bar{I}))) = R[1-\frac{2\bar{S}}{K}]-\bar{I}(\frac{R}{K}+\lambda) = \frac{1}{\bar{S}}[R\bar{S}(1-\frac{\bar{S}+\bar{I}}{K})-\frac{R(\bar{S})^2}{K}-\lambda\bar{I}\,\bar{S}]$

$=\frac{-1}{\bar{S}}[\alpha\bar{I}+\frac{R(\bar{S})^2}{K}]<0$ and $\det V((\bar{S},\bar{I})) = -\lambda\bar{I}[\alpha-\frac{R\bar{S}}{K}-\lambda\bar{S}] = \lambda\bar{I}[d_1+\frac{R\bar{S}}{K}]>0$

Hence $(\bar{S},\bar{I})$ is always locally asymptotically stable in the $Int.R_+^2$ of the $SI-$plane whenever it is feasible.

Now, in the following theorem, the global dynamics of the system (4) is discussed

Theorem (3.2): *Assume that the positive equilibrium point* $(\bar{S},\bar{I})$ *is feasible, then it is globally asymptotically stable in the* $Int.R_+^2$ *of the* $SI-$ *plane.*

Proof. Let $J(S,I)=\frac{1}{SI}$, clearly $J(S,I)$ is a continuously differentiable scalar function in the $Int.R_+^2$ of the $SI-$ plane and since

$$\Delta(S,I)=\frac{\partial}{\partial S}(J\frac{dS}{dt})+\frac{\partial}{\partial I}(J\frac{dI}{dt})=\frac{-R}{KI}+\frac{-\alpha}{S^2}<0$$

then $\Delta(S,I)$ does not change the sign and is not identically zero in the $Int.R_+^2$ of the $SI-$ plane. Therefore by Bendixson Dulac criterion the system (4) has no non-trivial positive periodic solutions. Hence $(\bar{S},\bar{I})$ is globally asymptotically stable and the proof is complete. ■

3.2 The stability analysis of 3D system

In this section, we try to find all the biologically feasible equilibria admitted by the system (1) and study the dynamics of them around each equilibrium points. Now there are at most four non-negative equilibrium points.

1) The trivial equilibrium point $E_0 = (0,0,0)$ always exists

2) The axial equilibrium point $E_1 = (K, 0, 0)$ always exists.

3) The planar equilibrium point $E_2 = (\bar{S}, \bar{I}, 0)$, where $\bar{S}$ and $\bar{I}$ are given by the Eq.(1.6a) and (1.6b) respectively, exists under the condition (7).

4) The positive equilibrium point $E_4 = (S^*, I^*, Y^*)$ exists in the $Int.R_+^3$ if and only if S^*, I^* and Y^* represent a positive solution of the following set of nonlinear algebraic equations:

$$RS(1-\frac{S+I}{K}) - \lambda SI + \alpha I = 0 \tag{9a}$$

$$\lambda S - \frac{aY}{(1+bI)(1+cY)} - (d_1+\alpha) = 0 \tag{9b}$$

$$\frac{eaI}{(1+bI)(1+cY)} - d_2 = 0 \tag{9c}$$

From Eq. (9c), we get that
$$I^* = \frac{d_2(1+cY^*)}{ea - bd_2(1+cY^*)} \tag{10a}$$

Clearly for the positivity of I^*, we should have:

$$0 < Y^* < \frac{ea - bd_2}{cbd_2} \tag{10b}$$

By substituting Eq. (10a) in Eq. (9a) and then solving for S, we obtain that:

$$S^{*2} + [(\frac{\lambda K}{R}+1)I^* - K]S^* - \frac{\alpha K}{R} I^* = 0 \tag{11a}$$

Obviously the Eq.(11a) has a positive root given by:

$$S^* = -\frac{B}{2} + \frac{1}{2}\sqrt{B^2 + 4\frac{\alpha K}{R} I^*} \tag{11b}$$

Where $B = (\frac{\lambda K}{R}+1)I^* - K$

Consequently, by substituting the Eq.s (10a) and (11b) in Eq.(1.9 b) and solving for Y, E_4 exists uniquely in the $Int.R_+^3$ if and only if Y^* is a positive root for Eq. (9b) which satisfies Eq. (10b).

The local stability conditions of these equilibrium points are established below. The general Variational matrix of the system (1) at (S, I, Y) is computed by:

$$V(S,I,Y)=\begin{pmatrix} S\frac{\partial f_1}{\partial S}+f_1(S,I,Y) & \frac{\partial f_1}{\partial I}S+\alpha & \frac{\partial f_1}{\partial Y}S \\ \frac{\partial g_1}{\partial S}I & \frac{\partial g_1}{\partial I}I+g_1(S,I,Y) & \frac{\partial g_1}{\partial Y}I \\ \frac{\partial h_1}{\partial S}Y & \frac{\partial h_1}{\partial I}Y & \frac{\partial h_1}{\partial Y}Y+h_1(S,I,Y) \end{pmatrix} \quad (12)$$

Here $\frac{\partial f_1}{\partial S}=\frac{-R}{K}<0$, $\frac{\partial f_1}{\partial I}=\frac{-R}{K}-\lambda<0$, $\frac{\partial f_1}{\partial Y}=0$, $\frac{\partial g_1}{\partial S}=\lambda>0$,

$\frac{\partial g_1}{\partial I}=\frac{abY}{(1+bI)^2(1+cY)}>0$, $\frac{\partial g_1}{\partial Y}=\frac{-a}{(1+bI)(1+cY)^2}<0$, $\frac{\partial h_1}{\partial S}=0$

, $\frac{\partial h_1}{\partial I}=\frac{ea}{(1+bI)^2(1+cY)}>0$, $\frac{\partial h_1}{\partial Y}=\frac{-eacI}{(1+bI)(1+cY)^2}<0$

Now the Variational matrix about the equilibrium points E_0 is given below:

$$V(E_0)=\begin{pmatrix} R & \alpha & 0 \\ 0 & -(\alpha+d_1) & 0 \\ 0 & 0 & -d_2 \end{pmatrix},$$

Now the eigenvalues of $V(E_0)$ are: $\lambda_{01}=R>0,\ \lambda_{02}=-(\alpha+d_2)<0$ and $\lambda_{03}=-d_2<0$

Hence the equilibrium point E_0 is a hyperbolic saddle point with locally stable manifold in the IY-plane and with locally unstable manifold in the S-direction. Also the Variational matrix about equilibrium point E_1 is

$$V(E_1)=\begin{pmatrix} -R & -K(\frac{R}{K}-\lambda)+\alpha & 0 \\ 0 & \lambda K-(d_1+\alpha) & 0 \\ 0 & 0 & -d_2 \end{pmatrix}$$

Again, the eigenvalues of $V(E_1)$ are given by $\lambda_{11}=-R<0,\ \lambda_{12}=\lambda K-(d_1+\alpha)$ and $\lambda_{13}=-d_2<0$. Hence the equilibrium point E_1 is locally asymptotically stable in the $Int.R_+^3$

provided that $\lambda < \frac{d_1+\alpha}{K}$ or E_2 does not exist. While it is a hyperbolic saddle point with locally stable manifold in the SY-plane and with locally unstable manifold in the I-direction under the condition (7).

Further, in the following theorem, the local stability near the plane SI is discussed.

Theorem (3.3): *Assume that the planar equilibrium point $E_2=(\bar{S},\bar{I},0)$ of the system (1) exists, then it is locally asymptotically stable in the $Int.R^3_+$ providing that the following condition holds:*

$$ea\bar{I} < d_2(1+b\bar{I}). \tag{13}$$

Proof. By substituting E_2 in the general Variational matrix (12), we get that:

$$V(E_2)=\begin{pmatrix} \frac{-1}{\bar{S}}[\alpha\bar{I}+\frac{R(\bar{S})^2}{K}] & \alpha-\frac{R\bar{S}}{K}-\lambda\bar{S} & 0 \\ \lambda\bar{I} & 0 & \frac{-a\bar{I}}{(1+b\bar{I})} \\ 0 & 0 & \frac{ea\bar{I}}{(1+b\bar{I})}-d_2 \end{pmatrix}$$

Accordingly, the characteristic polynomial of $V(E_2)$ can be written as:

$$\left((\frac{ea\bar{I}}{(1+b\bar{I})}-d_2)-\lambda\right)\left(\lambda^2+\left[\frac{1}{\bar{S}}[\alpha\bar{I}+\frac{R(\bar{S})^2}{K}]\right]\lambda-\left[(\alpha-\frac{R\bar{S}}{K}-\lambda\bar{S})\lambda\bar{I}\right]\right)=0$$

Therefore, it is easy to verify that the roots of this characteristic polynomial satisfy the following relations:

$$\lambda_{21}+\lambda_{22}=-\left[\frac{1}{\bar{S}}[\alpha\bar{I}+\frac{R(\bar{S})^2}{K}]\right]<0 \tag{14a}$$

$$\lambda_{21}\lambda_{22}=(d_1+\frac{R(\alpha+d_1)}{\lambda K})\lambda\bar{I}>0 \tag{14b}$$

$$\lambda_{23}=\frac{ea\bar{I}}{(1+b\bar{I})}-d_2 \tag{14c}$$

Hence, according to Routh-Hurwitz criterion, E_2 is locally asymptotically stable in the $Int.R^3_+$ if and only if the condition (13) holds, otherwise E_2 is a hyperbolic saddle point with unstable manifold in the positive direction orthogonal to the $SI-$ plane (i.e. Y -direction) and with locally stable manifold in SY -plane ■

Further, the global stability analysis of E_1 in the $Int.R_+^3$ is investigated in the following theorem.

Theorem (3.4): *Assume that* E_2 *does not exist, then* E_1 *is globally asymptotically stable in the* $Int.R_+^3$ *if the following condition holds.*

$$S < \frac{\alpha K}{R} \tag{15}$$

Proof. Consider the following function

$$V(S,I,Y) = C_1(S - K - K\,Ln\frac{S}{K}) + C_2 I + C_3 Y$$

It is easy to see that $V(S,I,Y) \in C^1\left(R_+^3, R\right)$ and $V(K,0,0)=0$, while $V(S,I,Y)>0$ for all $(S,I,Y) \in R_+^3/\{(K,0,0)\}$ and C_i for $(i=1,2,3)$ are positive constants to be determined.

The derivative of V along the trajectory of the system (1) is

$$\frac{dV}{dt} = C_1(S-K)\left[R(1-\frac{S+I}{K}) - \lambda I + \frac{\alpha I}{S}\right] + C_2 I\left[\lambda S - \frac{aY}{(1+bI)(1+cY)} - (\alpha+d_1)\right]$$
$$+ C_3 Y\left[\frac{eaI}{(1+bI)(1+cY)} - d_2\right]$$

Hence

$$\frac{dV}{dt} = -C_1\frac{(S-K)^2}{K}R + C_1(S-K)IM + C_2\lambda SI + \left[eC_3 - C_2\right]\frac{aIY}{(1+bI)(1+cY)}$$
$$-C_2(\alpha+d_1)I - C_3 d_2 Y$$

Here $M = \dfrac{\alpha K - \lambda SK - RS}{SK}$. So by choosing $C_1 = 1$, $C_2 = 1$,and $C_3 = \dfrac{1}{e}$, and then substituting these values in $\dfrac{dV}{dt}$, we get that:

$$\frac{dV}{dt} = -\frac{(S-K)^2}{K}R + I\,\frac{[(S-K)(\alpha K - RS) + SK(\lambda K - \alpha - d_1)}{SK} - \frac{d_2}{e}Y$$

Now from boundness logistic term, we have $S-K<0$ and since E_2 does not exist, $\lambda K - \alpha - d_1$<0. Therefore, $\dfrac{dV}{dt}$ is a negative definite function under the condition (15), and hence V is a Lyapunov function with respect to the positive equilibrium point E_1 of the system (1). Thus E_1 is globally asymptotically stable in the $Int.R_+^3$ and the proof is complete. ■

Theorem (3.5):

The positive equilibrium point $E_3=(S^*,I^*,Y^*)$ *is locally asymptotically stable in the* $Int.R_+^3$ *provided that the following condition holds:*

$$Y^* < \min\left\{\frac{\lambda RS^*(1+bI^*)^2(1+cY^*)}{abKL}, \frac{L(1+bI^*)^2(1+cY^*)}{abI^*}, \frac{ec(1+bI^*)-b}{bc}\right\} \quad (16)$$

Where $L=\dfrac{R(S^*)^2+\alpha KI^*}{KS^*}$

Proof. Substituting the positive equilibrium point $E_3=(S^*,I^*,Y^*)$ of then the system (1) in the general Variational matrix (12) gives that:

$$V(E_3)=\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} \quad (17)$$

Where $v_{11}=-[\frac{RS^*}{K}+\frac{\alpha I^*}{S^*}]<0$, $v_{12}=-[\frac{R}{K}+\lambda]S^*+\alpha$, $v_{13}=0$, $v_{21}=\lambda I^*>0$,

$v_{22}=\dfrac{abI^*Y^*}{(1+bI^*)^2(1+cY^*)}>0$, $v_{23}=\dfrac{-aI^*}{(1+bI^*)(1+cY^*)^2}<0$, $v_{31}=0$,

$v_{32}=\dfrac{eaY^*}{(1+bI^*)^2(1+cY^*)}>0$, $v_{33}=\dfrac{-eacI^*Y^*}{(1+bI^*)(1+cY^*)^2}<0$

Since the characteristic polynomial of V at E_3 can be written as

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0$$

Where $A_1=-(v_{11}+v_{22}+v_{33})$, $A_2=v_{11}v_{33}+v_{22}v_{33}+v_{11}v_{22}-v_{12}v_{21}-v_{23}v_{32}$

$$A_3=v_{33}(v_{12}v_{21}-v_{11}v_{22})+v_{11}v_{23}v_{32}.$$

Hence, from the condition (16), we get that $v_{11}+v_{22}<0$ and hence $A_1>0$

Now straightforward computations give that:

$$v_{12}v_{21}-v_{11}v_{22}=[\alpha-\frac{R}{K}S^*-\lambda S^*]\lambda I^*+\frac{LabI^*Y^*}{(1+bI^*)^2(1+cY^*)}$$

$$=-\lambda I^*[\frac{aY^*}{(1+bI^*)(1+cY^*)}+d_1]+I^*[\frac{LabY^*}{(1+bI^*)^2(1+cY^*)}-\frac{\lambda RS^*}{K}]$$

Now from the condition (16), we get that: $v_{12}v_{21}-v_{11}v_{22}<0$

Hence $A_3>0$. Note that, according to Routh-Hurwitz criterion, E_3 is locally asymptotically stable in the $Int.R^3_+$ if and only in addition to $A_1>0$ and $A_3>0$ we should have that $\Delta=A_1A_2-A_3>0$.

Straightforward computations show that:

$\Delta=(v_{11}+v_{22})[(v_{12}v_{21}-v_{11}v_{22})+A_1v_{33}]+v_{23}v_{32}(v_{22}+v_{33})$

Now it is easy to show that $(v_{11}+v_{22})[(v_{12}v_{21}-v_{11}v_{22})+A_1v_{33}]>0$ and

$v_{22}+v_{33}=\dfrac{aI^*Y^*[b(1+cY^*)-ec(1+bI^*)]}{(1+bI^*)^2(1+cY^*)^2}$ Again, according to the condition (16), we have $v_{22}+v_{33}<0$. Hence $\Delta>0$. Therefore, all the requirements of Routh-Hurwitz criterion are satisfied. Hence E_3 is locally asymptotically stable in the $Int.R^3_+$ and the proof is complete. ■

3.3 Numerical simulation

In this section, the global dynamics of the eco-epidemiological the system (1) is investigated numerically. In order to understand the impact of the disease on the dynamical behavior of the the system (1), we will choose the infected λ and the recover rate α as the control parameters in two different cases:

Case1. The system (1) has an asymptotically stable point.

For the following set of parameters values

$$R=3\,,K=100\,,\lambda=2,\alpha=1,a=2,\ b=0.1,\ c=0.2,d_1=2,\ e=0.4,d_2=1 \qquad (18)$$

The system (1) has a globally asymptotically stable point as shown in Fig.1(a-b)

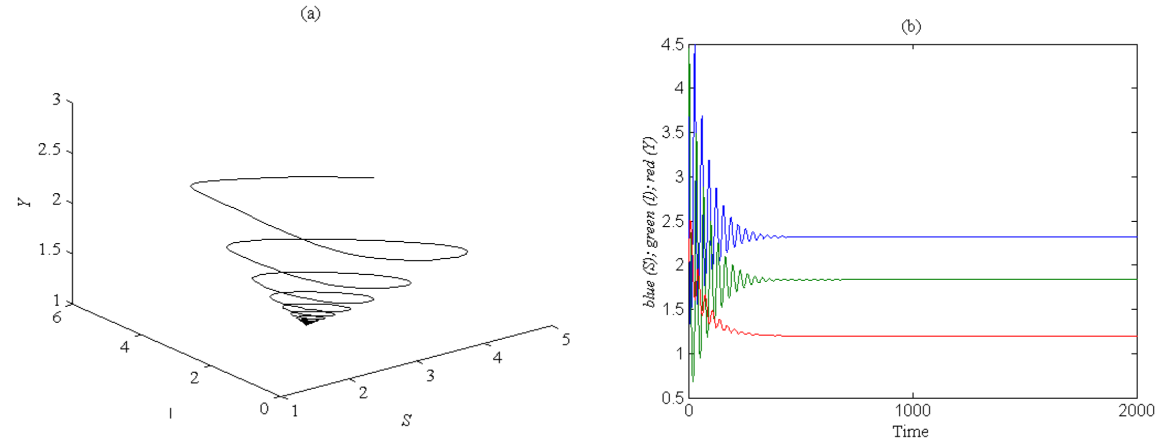


Fig. 1: (a) A globally stable point in the $Int.R_+^3$ of the system (1) at the parameters values given in Eq. (18). (b) Time series of the trajectory of the system (1) as given in (a)

Case 2. The system (1) has a stable limit cycle in the $Int.R_+^3$.

For the following set of parameters values:

$$R=3\,,K=100\,,\lambda=2,\alpha=1,a=3,\ b=0.1,\ c=0.2,d_1=2,\ e=0.4,d_2=1 \tag{19}$$

The system (1) has a stable limit cycle in the $Int.R_+^3$ as shown in Fig. 2 (a-b).

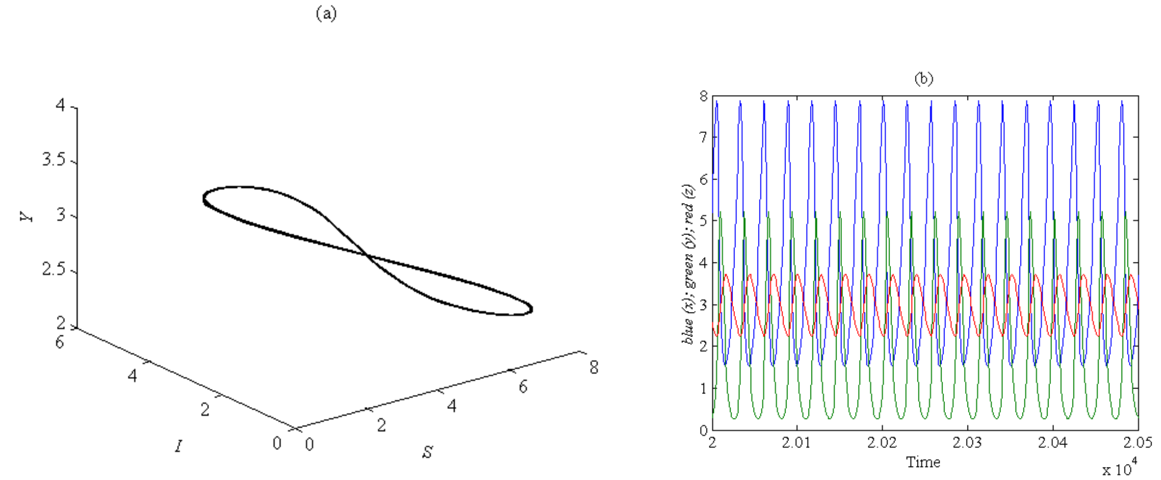


Fig. 2: (a) A stable limit cycle in the $Int.R_+^3$ of the system (1) at the parameters values given in Eq. (19). (b) Time series of the trajectory of the system (1) as given in (a)

Depending on what has been mentioned before, it has been observed that for different ranges of the control parameters, the system (1) has different types of attractors. The following two tables summarize our numerical results in the above two cases respectively, moreover, for more explanation, typical attracting sets along with their time series are also given.

Table (1): The effect of varying infected rates on the dynamical behavior of the system (1) in the first case.

Parameters kept fixed	Parameter varied	Dynamical behavior of the system (1)
As given in Eq. (18)	$0<\lambda\le 0.03$	The system (1) approaches asymptotically to (K,0,0)=(100,0,0)
	$0.03<\lambda\le 0.11$	The system (1) approaches asymptotically to a stable point in the $Int.R_{+}^{3}$
	$0.11<\lambda\le 1.26$	The system (1) approaches asymptotically to a stable limit cycle
	$1.26<\lambda\le 3$	The system (1) approaches asymptotically to a stable point in the $Int.R_{+}^{3}$ (see Fig.1)
	$3<\lambda$	The system (1) approaches asymptotically to a stable point in the $Int.R_{+}^{2}$ of the SI-plane

Moreover, as α varies in the range $0<\alpha<2.66$ with the rest of parameters fixed as given in Eq. (18), it has been observed that the system (1) still has a globally stable point in $Int.R_{+}^{3}$. Finally, typical attracting sets, which show the dynamical behavior of the system (1) as given in table 1, are shown in the following figures.

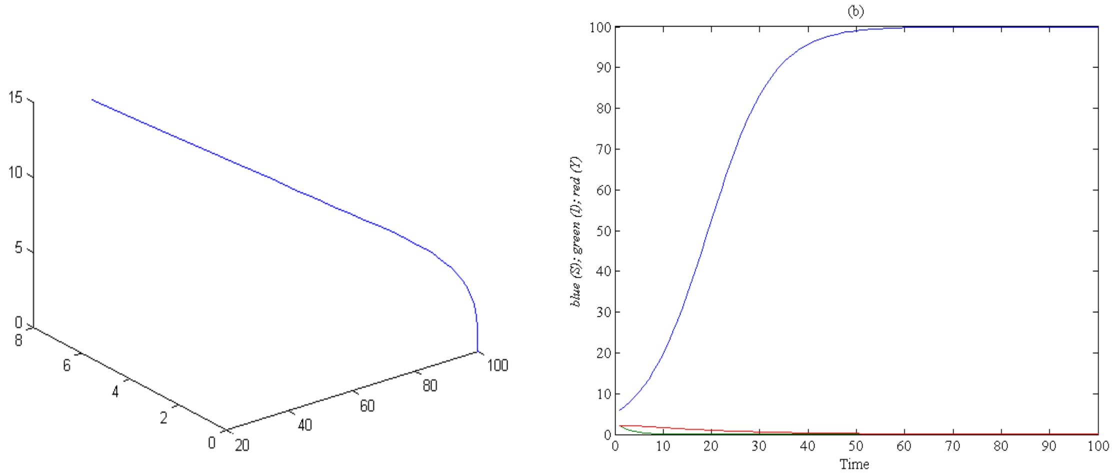


Fig. 3: (a) The system (1) approaches asymptotically to (100,0,0) at the parameters values given in Eq. (18) with $\lambda = 0.02$ *(b) Time series of the trajectory of the system (1.1) as given in (a)*

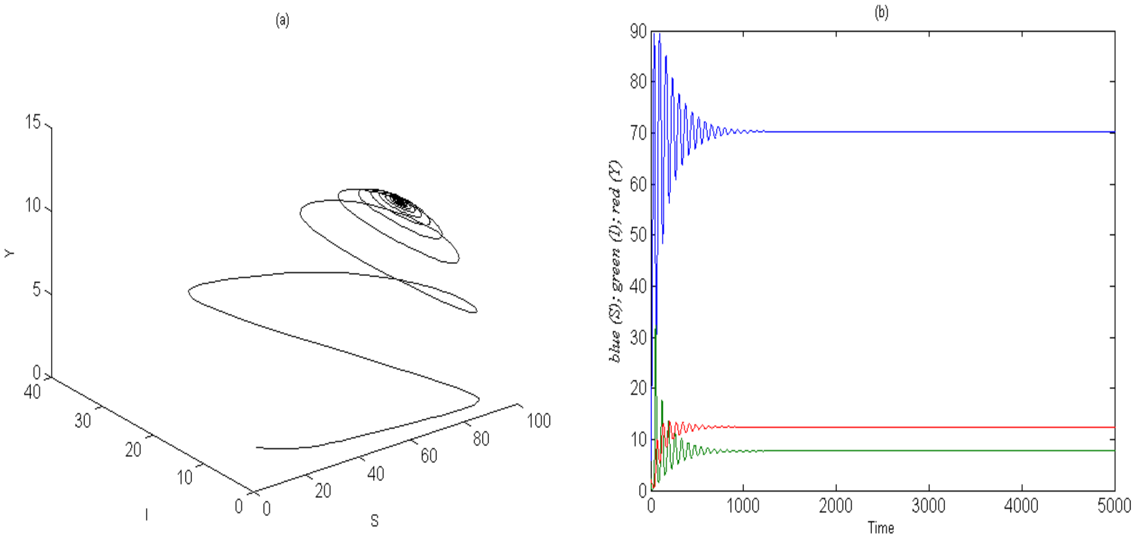


Fig.4: (a) The system (1) approaches asymptotically to a stable point at the parameters values given in Eq. (18) with $\lambda = 0.1$ *(b) Time series of the trajectory of the system (1) as given in (a).*

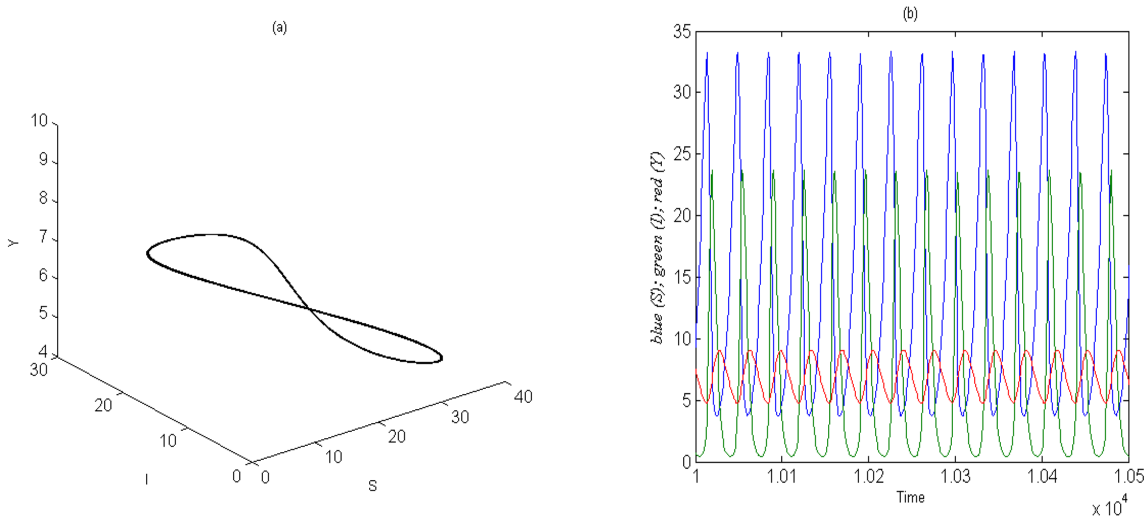


Fig.5: (a) The system (1) approaches asymptotically to a limit cycle at the parameters values given in Eq. (18) with $\lambda = 0.5$ *(b) Time series of the trajectory of the system (1) as given in (a).*

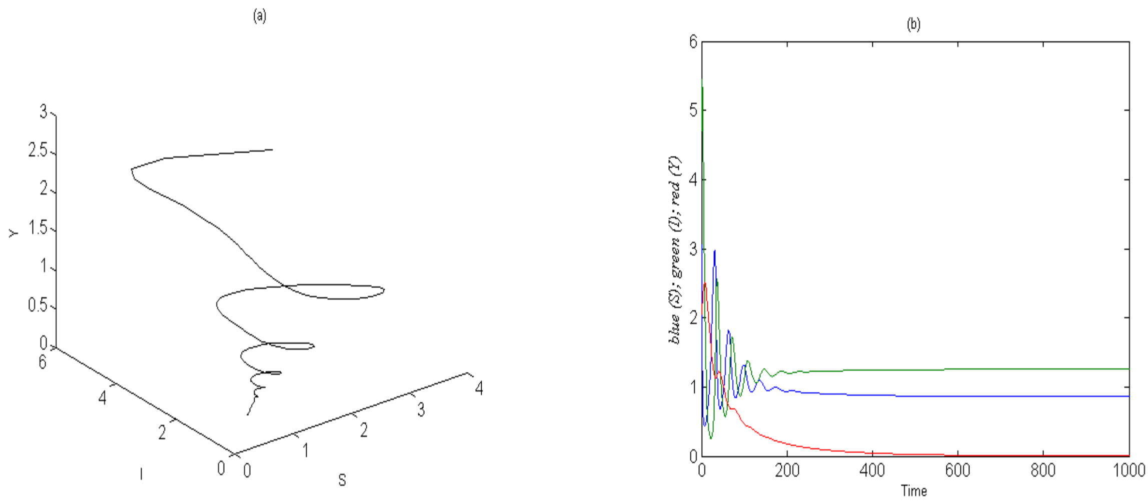


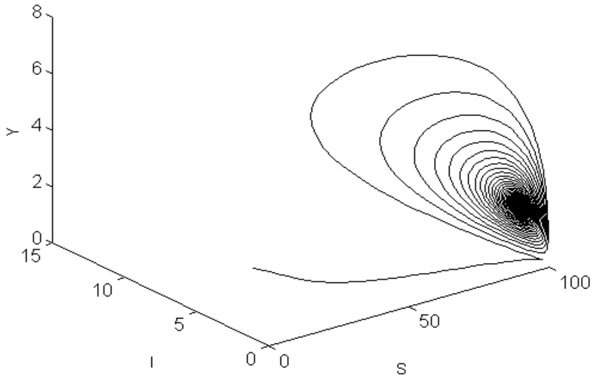
Fig. 6: (a) The system (1) approaches asymptotically to a stable point in the $Int.R_+^2$ *of the SI-plane at the parameters values given in Eq. (18) with* $\lambda = 3.5$ *(b) Time series of the trajectory of the system (1) as given in (a).*

Table (2): The effect of varying infected rate on the dynamical behavior of the system (1) in the second case.

Parameters kept fixed	Parameter varied	Dynamical behavior of system (1)
AS given at Eq. (19)	$0<\lambda\le 0.03$	The system (1) approaches asymptotically to (K,0,0)=(100,0,0)
	$0.03<\lambda\le 0.08$	The system (1) approaches asymptotically to a stable point in the $Int.R_+^3$
	$0.08<\lambda\le 2.37$	The system (1) approaches asymptotically to a stable limit cycle (see Fig. 2)
	$2.37<\lambda<4.9$	The system (1) approaches asymptotically to a stable point in the $Int.R_+^3$
	$4.9\le\lambda$	The system (1) approaches asymptotically to a stable point in the $Int.R_+^2$ of SI-plane

Moreover, as α varies in the range $0<\alpha<2.15$ with the rest of parameters fixed as given in Eq. (19), the system (1) still has periodic dynamics. However, for $2.15\le\alpha<2.66$, the system (1) approaches asymptotically to a stable point in the $Int.R_+^3$. Again for more illustration, typical attracting sets, which show the dynamical behavior of the system (1) as given in table 2 are shown in the following figures.

(a)



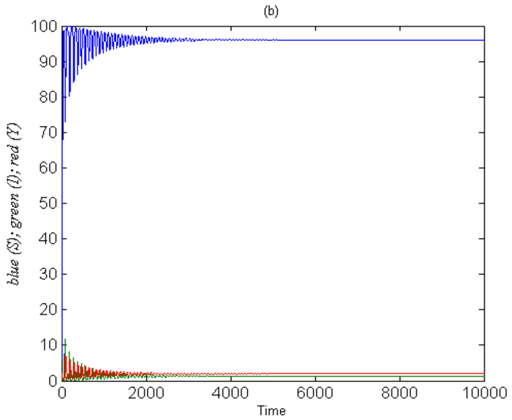


Fig. 7: (a) The system (1) approaches asymptotically to a stable point at the parameters values given in Eq. (19) with $\lambda = 0.07$*, and (b) Time series of the trajectory of the system (1) as given in (a)*

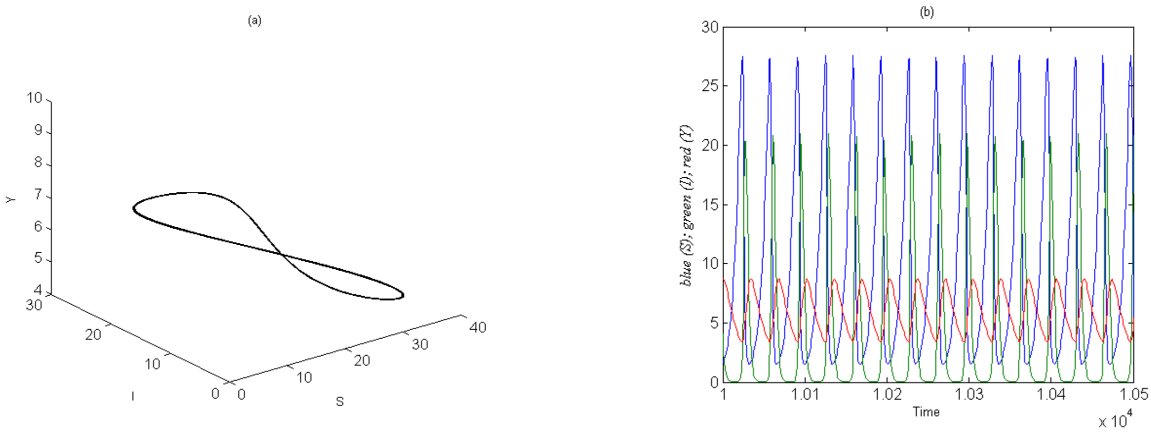


Fig. 8: (a) The system (1) approaches asymptotically to a limit cycle at the parameters values given in Eq. (19) with $\lambda = 1$ *(b) Time series of the trajectory of the system (1) as given in (a)*

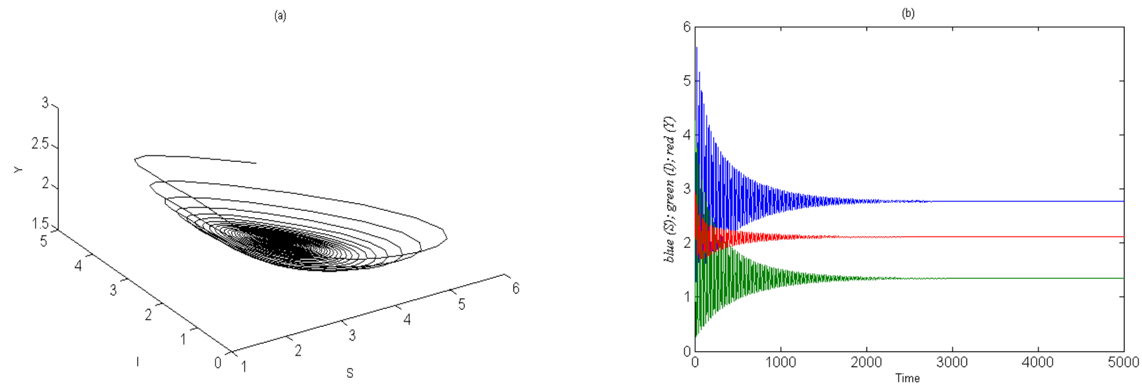


Fig. 9: (a) The system (1) approaches asymptotically to a stable point in the $Int.R^3_+$ *at the parameters values given in Eq. (19) with* $\lambda = 2.5$ *(b) Time series of the trajectory of the system (1) as given in (a).*

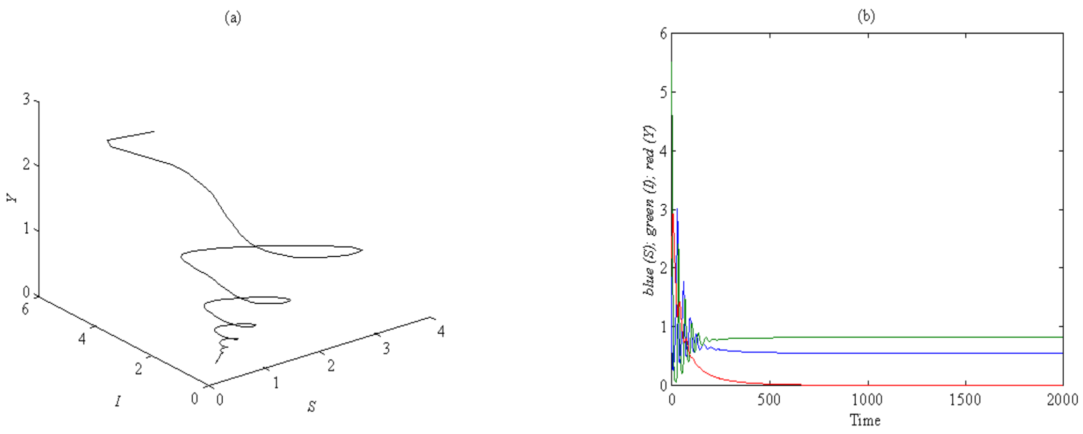


Fig. 10: (a) The system (1) approaches asymptotically to a stable point in the $Int.R_+^2$ *of the SI-plane at the parameters values given in Eq. (19) with* $\lambda = 5.5$. *(b) Time series of the trajectory of the system (1) as given in (a)*

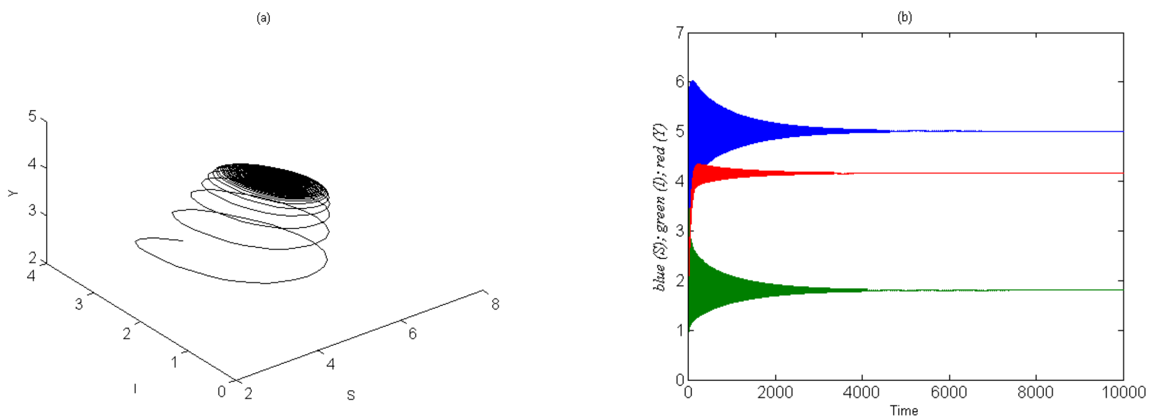


Fig. 11: (a) The system (1) approaches asymptotically to a stable point at the parameters values given in Eq. (19) with $\alpha = 2.25$ *(b) Time series of the trajectory of the system (1) as given in (a)*

3.4 Discussion and Conclusions

In this chapter, an eco-epidemiological model, consisting of a Crowley-Martin prey-predator with disease in prey, has been proposed and analyzed analytically as well as numerically. Analytically, it has been observed that, the system (1) is uniformly bounded under certain condition and has at most four non-negative equilibrium points. $E_0 = (0,0,0)$

always exists and it is unstable saddle point. $E_1 = (K,0,0)$ always exists and locally asymptotically stable provided that $E_2 = (\bar{S}, \bar{I}, 0)$ does not exist, otherwise, it is unstable saddle point. Further, E_1 is globally asymptotically stable provided that the number of susceptible prey S is less than a specific value $\left(\frac{\alpha K}{R}\right)$. The equilibrium point E_2 exists provided that the infection rate coefficient λ is greater than a specific value $\left(\frac{d_1+\alpha}{K}\right)$ and it is always globally asymptotically stable in the $Int.R_+^2$ of the $SI-$plane. Moreover, E_2 is locally asymptotically stable for any initial value in the $Int.R_+^3$ near the $SI-$plane provided that the natural death rate of predator d_2 is greater than a specific value $\left(\frac{ea\bar{I}}{1+b\bar{I}}\right)$,otherwise, it is unstable saddle point. Finally, the positive equilibrium point $E_4 = (S^*, I^*, Y^*)$ exists uniquely in the $Int.R_+^3$ under certain conditions and it is locally asymptotically stable if and only if Y^* is less than a specific value given in Eq. (17). Moreover, we cannot find Lyapunov function at E_4 so it may or may not be globally stable.

Numerically, the global dynamics of the system (1) is studied by solving it numerically for different sets of initial values and for different sets of parameters values. It is observed that, in the $Int.R_+^3$, the system (1) has only two basic patterns which approach to a stable point or a stable limit cycle. Further, the tables 1-2 show that the infection rate coefficient λ represents a bifurcation parameter for the system (1), indeed the dynamical behavior of the the system (1) exhibits a major change as the parameter λ passes through specific values, see tables 1-2. Moreover, the increases in the value of λ lead to extinction in the predator species. However, the recover rate coefficient α has stabilizing effect on the system (1)

REFERENCES

[1] Anderson, R.M., May, R.“The invasion, persistence and spread of infectious diseases within animal and plant communities” 314(1982), 533–570.

[2] Bairagi, N., Roy, P. and Chattopadhyay, J “Role of infection on the stability of a predator–prey system with

several response functions”. 248(2007), 10–25.

[3] Beltrami, E., Coarrol, T. “Modelling the role of viral disease in recurrent phytoplankton blooms. 32(1994), 857– 863.

[4] Beretta, E., Kuang, Y.“Modeling and analysis of a marine bacteriophage infection”. *Mathematical Bioscience,* 149(1998), 57–76.

[5] Chattopadhyay,J.,Bairagi,N.**”**Pelicans at risk in salton sea—an eco epide- miological study”*Ecol.Model* .136(2001)

[6] Chattopadhyay,J.,Srinivasu,P.and Bairagi,N**.**Pelicans at risk in salton sea —an ecoepidemiological model..*Ecol.Model.*167(2003),199–211.

[7] Freedman H.I. and Hongshun Q. “Interactions leading to Persistence in predator-prey system with group defense”. 50(1988), 517-530 .

[8]Getz, W.M., Pickering, J. “Epidemic models: thresholds and population regulation”. *American Naturalist,* 121(1983), 892–898.

[9] Hadeler, K.P., Freedman, H.I.“Predator–prey populations with parasitic infection” 27(1989), 609–631.

[10] Hall J.K. “Ordinary differential equation”. New York, Wiley-(1969) Interscience .

[11] Hethcote, H.W., Wang, W. , Han, L. and Ma, Z. “A predator–prey model with infected prey” 66(2004), 259–268.

[12] Hirsch M.W. and smale S.**”**Differential Equation,Dynamical System ,and Linear Algebra” (1974)New York ,Academic Press

[13]May R.M. “Biological populations with non-overlapping generations: stable point, stable cycles and chaos” 186(1974), 645-647.

[14] May R.M. “Simple mathematical models with very complicated dynamics” 261(1974) 459-467

[15]Mukherjee,D**.** “Uniform persistence in a generalized prey-predator system with parasitic infection ”47(1998),149-155.

[16] Mukherjee,D. “Persistence in a prey-predator system with disease in the prey” 11(2003),101-112

[17] Xiao,Y.,Chen,L**.”**Modeling and analysis of a predator-prey model with disease in the prey ” 171(2001) ,59-82.