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THE DYNAMICS OF SIS EPIDEMIC MODEL WITH EFFECT OF TREATMENT

K. A. HASAN[1], M. F. HAMA[1,*], M. B. MRAKHAN[2]

[1]University of Sulaimani, Faculty of science and science Educations, School of Science, Department of Mathematics, Sulaimani, Iraq

[2]University of Garmian, Faculty of Educations- Department of Mathematics, Kalar, Iraq

Abstract. The study of SIS epidemic model mainly concerns global asymptotic stability and it is one of the most basic and most important model in decreasing the spread of many disease. In this paper, an SIS epidemic model with treatment and without treatment is studied. The incidence rate of the model, which can include the standard incidence rate $\frac{\beta IS}{1+aS+bI}$, is a nonlinear incidence rate. The global stability of the disease-free equilibrium, and the existence and global stability of the endemic equilibrium are proved and then we can understand the effect of the capacity for treatment. According to different recovery rates, we use differential stability theory and qualitative theory to analyze the various kinds of endemic equilibria and disease-free equilibrium.

Keywords: epidemic model; incidence rate; with treatment; globally asymptotically stable.

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1. Introduction:

The study of SIS epidemic model mainly concerns global asymptotic stability and it is one of the most basic and most important model in decreasing the spread of many disease. In 1927 Kermack and Mckerdick [3] proposed a simple SIS model with infective immigrants. Gao and Hethcote (1995) [2] considered the SIS model with a standard disease incidence and density – Pendant demographics. In [5] Li and Ma study the SIS model with vaccination and Temporary immunity. Zhou and Liu [14] considered an SIS model with pule vaccination. Treatment including isolation or quarantine is an important method to prevent and control the spread of various infectious diseases. Many mathematicians ([1,4,6-13]) have begun to investigate the rule

[*]Corresponding author

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of treatment function in epidemiological models. In classical epidemic models, the treatment function of the invectives individuals is assumed to be proportional to the number of the infective individual. Because of every community should have a maximal capacity for the treatment of a disease and the resources for treatment should be every large. In (2004) Wany and Ruan [7] considered the maximal treatment capacity to cure infective individuals so that the disease can be eradicated. Recently, saturated treatment function has been widely applied in many epidemic models. For example [9] Zhang and Liu took a continuous and differentiable saturated treatment function $T(I) = \frac{rI}{1+\alpha I}$ where $\alpha \geq 0\,, r > 0$, Further, a piecewise linear treatment function was considered, that is,

$$T(I) = \begin{cases} KI & 0 \leq I \leq I_0 \\ m & I > I_0 \end{cases}$$

Where $m = KI_0$ and K and I_0 are positive constants. This means that the treatment function is proportional to the number of the infective individuals when the capacity of treatment has not been reached; otherwise it takes the maximal capacity of treatment KI_0. The treatment function has been used by some other researchers. For example Zhang and Liu [9] studied a model with a general incidence $\lambda SI(I+S)^{n-1}$ $(0 \leq n \leq 1)$ and the treatment function. Hu et al. [8] considered an epidemic model with standard incidence rate $\frac{\beta SI}{N}$ and the treatment function. Li et al. [12] studied an epidemic model with nonlinear incidence rate $(\frac{\beta I}{1+\alpha I})$ with the treatment function and analyzed the stability and bifurcation of the system. In this paper we introduce the global dynamics of SIS model with saturated incidence $\frac{\beta SI}{1+aS+bI}$ and saturated treatment function. Sufficient condition for the existence of equation points is obtained and the dynamical behavior of the model is saturated. Also, the global asymptotic stability of the disease – free and endemic equilibria is studied.

2. Model formulation:

In this section, we study an SIS epidemic model with saturated incidence rate $\frac{\beta IS}{1+aS+bI}$ and treatment, an SIS epidemic model with consists of the susceptible individuals $S(t)$, the infectious individuals $I(t)$ and the total population $N(t)$ at time t, which represented in the block diagram given by Figure (1) can be represented by the following system of non-linear ordinary differential equations.

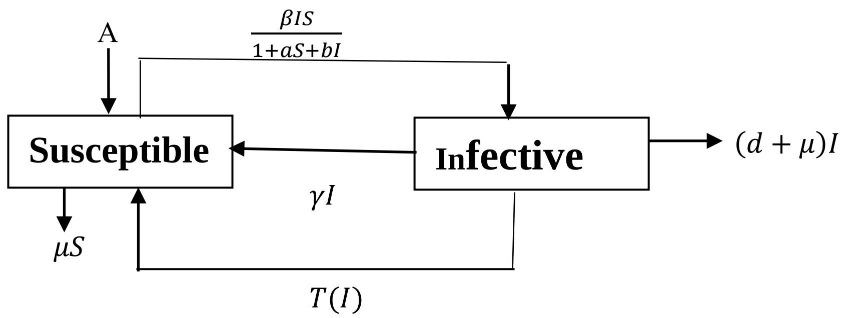

Figure (1)

$$\frac{dS}{dt} = A - \mu S - \frac{\beta IS}{1+aS+bI} + \gamma I + T(I)$$
$$\frac{dI}{dt} = \frac{\beta IS}{1+aS+bI} - (d+\mu+\gamma)I - T(I) \qquad (1)$$
$$N(t) = S(t) + I(t)$$

where

$$T(I) = \begin{cases} \delta I & if\ 0 \le I \le I_0 \\ K & if \quad I > I_0 \end{cases}$$

(K=δI_0) is the rate at which infected individuals are treated, A is the recruitment rate of individuals (including newborns and immigrants) into the susceptible population, μ is the natural death rate in each class, γ is the nature recovery rate of infected individuals, d is the disease-related death rate, β is the infection coefficient. A, μ, γ, d, β, a, and b are all positive numbers. If $0 \le I \le I_0$, then model (1) implies:

$$\frac{dS}{dt} = A - \mu S - \frac{\beta IS}{1+aS+bI} + (\gamma+\delta)I$$
$$\frac{dI}{dt} = \frac{\beta IS}{1+aS+bI} - (d+\mu+\gamma+\delta)I. \qquad (2)$$

If $I > I_0$, then model (1) implies:

$$\frac{dS}{dt} = A - \mu S - \frac{\beta IS}{1+aS+bI} + \gamma I + \mathrm{K}$$
$$\frac{dS}{dI} = \frac{\beta IS}{1+aS+bI} - (d+\mu+\gamma)I - K. \qquad (3)$$

Obviously, due to the biological meaning of the components $S(t)$ and $I(t)$ we focus on the model in the domain $R_+^2 = \{(S,I) \in R^2 : S \ge 0, I \ge 0\}$ which is positively invariant for system (1).

Theorem (2.1): All solutions of system (1) which initiate in R_+^2 are uniformly bounded.

Proof: Let$(S(t), I(t))$ be any solution of system (1) with non-negative condition $(S(0), I(0))$. Consider the function$N(t) = S(t) + I(t)$, time derivative of $N(t)$ along the trajectory of system

(1) gives the following differential equation:

$\frac{dN}{dt} + \mu N(t) \le A.$

Now, by solving the above linear differential equation, we get that the total population is asymptotically constant by:

$N(t) = \frac{A}{\mu}.$

Hence all solutions of system (1) that initiate in the region R_+^2 are eventually confined in the region

$B = \left\{(S,I) \in R_+^2 : N = \frac{A}{\mu} + \epsilon, for\ any\ \epsilon > 0\right\}.$

3. Stability analysis of SIS epidemic model without treatment

In this section we study the dynamics of SIS epidemic model (1) without treatment, system (1) can be rewritten in the following form:

$$\begin{aligned} \frac{dS}{dt} &= A - \mu S - \frac{\beta IS}{1+aS+bI} + \gamma I \\ \frac{dI}{dt} &= \frac{\beta IS}{1+aS+bI} - (d+\mu+\gamma)I. \end{aligned} \tag{4}$$

Existence of equilibrium points and Stability analysisof system (4)

- The disease free equilibrium point $E\left(\frac{A}{\mu}, 0\right)$ always exist and it is locally asymptotically stable provided that the following condition holds:$\frac{\beta A}{\mu + aA} < (d + \mu + \gamma)$.
- The endemic equilibrium point $\breve{E}(\breve{S}, \breve{I})$ exist in the region $IntR_+^2$,

where $\breve{S} = \frac{(d+\mu+\gamma)(1+b\breve{I})}{\beta - a(d+\mu+\gamma)}$, $\breve{I} = \frac{A-P_0}{bP_0+d+\mu}$, and $P_0 = \frac{\mu(d+\mu+\gamma)}{\beta - a(d+\mu+\gamma)}$ provided that the following condition holds $\beta > a(d+\mu+\gamma) \quad and \ \ A > P_0$and it is always locally asymptotically stable.

Theorem (3.1): Assume that the disease free-equilibrium point $E(\frac{A}{\mu}, 0)$ of the system (4) is locally asymptotically stable in the $IntR_+^2$ of the SI- plane with

$$\frac{A\beta}{\mu(1 + aS + bI)} < (d + \mu). \tag{5}$$

Then E is globally asymptotically stable in $IntR_+^2$ of the SI- plane.

Proof: Consider the following positive definite function about $E(\frac{A}{\mu}, 0)$

$V(S,I) = S + \frac{A}{\mu} - \frac{A}{\mu} \ln\frac{\mu S}{A} + I.$

By differentiating V with respect to t along the solution of the system (4) we get:

$$\frac{dV}{dt} = \frac{\left(S - \frac{A}{\mu}\right)}{S}\frac{dS}{dt} + \frac{dI}{dt}$$

$$\frac{dV}{dt} < -\frac{\mu}{S}\left(S - \frac{A}{\mu}\right)^2 - \left(d + \mu - \frac{A\beta}{\mu(1 + aS + bI)}\right)I < 0.$$

Hence $\frac{dV}{dt}$ is negative definite under the condition (5), and then V is Lyapunov function with respect to $E(\frac{A}{\mu}, 0)$.

Hence E is globally asymptotically stable in $IntR^2_+$ of SI- plane.

Theorem (3.2): Assume that the endemic equilibrium point $\check{E}(\check{S}, \check{I})$ of the system (4) is locally asymptotically stable in the $IntR^2_+$ of the SI- plane with

$$b > a \tag{6}$$

Then $\check{E}$ is globally asymptotically stable in $IntR^2_+$ of the SI- plane.

Proof: Consider a Dulac function $\mathrm{D} = \frac{1}{\mathrm{SI}}$ and assume that:

$$\frac{dS}{dt} = A - \mu S - \frac{\beta IS}{1 + aS + bI} + \gamma I$$

$$\frac{dI}{dt} = \frac{\beta IS}{1 + aS + bI} - (d + \mu + \gamma)I.$$

Hence

$$\Delta(S, I) = \frac{\partial(D\frac{dS}{dT})}{\partial S} + \frac{\partial(D\frac{dI}{dt})}{\partial I} = -\frac{1}{S^2}\left(\frac{A}{I} + \gamma\right) - \frac{\beta(b - a)}{(1 + aS + bI)^2}.$$

Note that $\Delta(S, I)$ dose not change sign and is not identically zero in the $IntR^2_+$ if $b > a$. Then according to Bendixon-Dulac criterion, there is no periodic solution in $IntR^2_+$. Now since all solutions of the system (4) are bounded and $\check{E}$ is a unique positive equilibrium point in $IntR^2_+$, hence by using the Pointcare-Bendixon theorem $\check{E}$ is globally asymptotically stable. ■

4. Stability analysis of an SIS epidemic model with treatment for system (2)

In this section the existence of all possible equilibrium points of system (2) and their locally and globally stability analysis are discussed. It is obvious that system (2) always has a unique disease free equilibrium point $E_0(\frac{A}{\mu}, 0)$, the endemic equilibrium of system (2) can be obtained by solving algebraic equations:

$$A - \mu S - \frac{\beta IS}{1+aS+bI} + (\gamma + \delta)I = 0$$
$$\frac{\beta IS}{1+aS+bI} - (d + \mu + \gamma + \delta)I = 0.$$

From two equations of system (2) we get that:

$$\hat{S} = \frac{A - (d + \mu)\hat{I}}{\mu} \tag{7}$$

By substituting Eq.(7) in the second equation of system (2), we obtain the following equations:

$$\beta A - \beta(d + \mu)\hat{I} - (d + \mu + \gamma + \delta)\left(\mu + aA + \mu b\hat{I} - a(d + \mu)\hat{I}\right) = 0$$

which implies that:

$$\hat{I} = \frac{(\mu + aA)(d + \mu + \delta + \gamma)(1 - \mathcal{T}_0)}{(d + \mu)[-(\beta + \mu bM) + a(d + \mu + \delta + \gamma)]},$$

where $\mathcal{T}_0 = \frac{\beta A}{(\mu+aA)(d+\mu+\delta+\gamma)} \neq 1$, and $M = 1 + \frac{\delta+\gamma}{d+\mu}$

If $a(d + \mu + \delta + \gamma) > \beta + \mu bM$ *and* $\mathcal{T}_0 < 1$

$$\hat{I} = \frac{(\mu + aA)(d + \mu + \delta + \gamma)(1 - \mathcal{T}_0)}{(d + \mu)[-(\beta + \mu bM) + a(d + \mu + \delta + \gamma)]} > 0.$$

If we put $\hat{I}$ in Eq. (7), then we get:

$$\hat{S} = -\frac{bAM + (d + \mu + \delta + \gamma)}{a(d + \mu + \delta + \gamma) - (\beta + \mu bM)} < 0.$$

So this case must be omitted.

If $a(d + \mu + \delta + \gamma) < \beta + \mu bM$ and $\mathcal{T}_0 > 1$ holds, then

$$\hat{I} = \frac{(\mu + aA)(d + \mu + \delta + \gamma)(\mathcal{T}_0 - 1)}{(d + \mu)[\beta + \mu bM - a(d + \mu + \delta + \gamma)} > 0$$

and

$$\hat{S} = \frac{bAM + (d + \mu + \delta + \gamma)}{\beta + \mu bM - a(d + \mu + \delta + \gamma)} > 0.$$

Then we get appositive equilibrium point $\hat{E}(\hat{S}, \hat{I})$ ofsystem (2).

But since $0 < \hat{I} \leq I_0$, then

$$\mathcal{T}_0 \leq 1 + \frac{(d + \mu)[\beta + \mu bM - a(d + \mu + \delta + \gamma)]I_0}{(\mu + aA)(d + \mu + \delta + \gamma)}$$

$$1 < \mathcal{T}_0 \leq 1 + \frac{(d + \mu)[\beta + \mu bM - a(d + \mu + \delta + \gamma)]I_0}{(\mu + aA)(d + \mu + \delta + \gamma)}.$$

Define

$$N_0 = 1 + \frac{(d+\mu)[\beta + \mu bM - a(d+\mu+\delta+\gamma)]I_0}{(\mu + aA)(d+\mu+\delta+\gamma)}.$$

So system (1) has an endemic equilibrium point $\hat{E}(\hat{S},\hat{I})$when $1 < \mathcal{T}_0 \le N_0$.

Theorem (4.1): If $\mathcal{T}_0 < 1$, then system (2) has only one disease – free equilibrium $E_0(\frac{A}{\mu},0)$. If $\mathcal{T}_0 > 1$, then system (2) has a unique endemic equilibrium $\hat{E}(\hat{S},\hat{I})$ except the disease – free equilibrium $E_0(\frac{A}{\mu},0)$.

Theorem (4.2): The disease – free equilibrium point E_0 of the system (2) is locally asymptotically stable point if $\mathcal{T}_0 < 1$ and it is saddle point if $\mathcal{T}_0 > 1$.

Theorem (4.3): The endemic equilibrium point $\widehat{\mathrm{E}}(\hat{\mathrm{S}},\hat{\mathrm{I}})$ of the system (2) is always locally asymptotically stable if it is exist.

Theorem (4.4): Assume that the disease free-equilibrium point $\mathrm{E}(\frac{\mathrm{A}}{\mu},0)$ of the system (2) is locally asymptotically stable in IntR^2_+ with $\frac{\mathrm{A}\beta}{\mu(1+\mathrm{aS}+\mathrm{bI})} < (\mathrm{d}+\mu)$. Then E is globally asymptotically stable in IntR^2_+ of the SI- plane.

Theorem (4.5): If the endemic equilibrium point $\widehat{\mathrm{E}} = (\hat{\mathrm{S}},\hat{\mathrm{I}})$ of the system (2) exists in the IntR^2_+, then it is globally asymptotically stable in $\mathrm{IntR}^2_+$of the SI- plane.

Proof: Consider a Dulac function $\mathrm{D} = \frac{1}{\mathrm{I}}$ and

$$\frac{dS}{dt} = A - \mu S - \frac{\beta IS}{1 + aS + bI} + (\gamma + \delta)I$$
$$\frac{dI}{dt} = \frac{\beta IS}{1 + aS + bI} - (d + \mu + \gamma + \delta)I.$$

Hence

$$\Delta(S,I) = \frac{\partial(D\frac{dS}{dt})}{\partial S} + \frac{\partial(D\frac{dI}{dt})}{\partial I} = -\frac{\mu}{I} - \frac{\beta(1+bS+bI)}{(1+aS+bI)^2} < 0.$$

Note that $\Delta(S,I)$ dose not change sign and is not identically zero in the $IntR^2_+$.

Then according to Bendixon-Dulac criterion, there is no periodic solution in the $IntR^2_+$. Now since all solutions of the system (2) are bounded and $\hat{E}$ is a unique positive equilibrium point in the$IntR^2_+$, hence by using the Pointcare-Bendixon theorem$\hat{E}$ is globally asymptotically stable.

5. Stability analysis of an SIS epidemic model with treatment for system (3)

The main goal of this section is study dynamics of endemic equilibrium of system (3) where it can be obtained by solving algebraic equations:

$$A - \mu S - \frac{\beta IS}{1+aS+bI} + \gamma I + \mathrm{K} = 0$$
$$\frac{\beta IS}{1+aS+bI} - (d+\mu+\gamma)I - K = 0.$$

By substituting Eq.(7) in to the second equation of the system (3), we obtain the following equation:

$$R_2 I^2 + R_1 I + R_0 = 0 \qquad (8)$$

where

$$R_2 = (d+\mu)[\beta + \mu bH - a(d+\mu+\gamma)]$$
$$R_1 = (d+\mu+\gamma)(\mu + aA) + \mu bK - \beta A - aK(d+\mu)$$
$$R_0 = K(\mu + aA)$$
$$H = 1 + \frac{\gamma}{d+\mu}.$$

We study Eq.(8) as follows:

If $\beta + \mu bH = a(d+\mu+\gamma)$, then Eq.(8) has a positive root if $R_1 < 0$, then

$$\breve{I} = \frac{K(\mu + aA)}{aK(d+\mu) - \mu b(HA+K) - \mu(d+\mu+\gamma)} > 0$$
$$\breve{S} = -\frac{bA(HA+K) + A(d+\mu+\gamma) + K(d+\mu)}{aK(d+\mu) - \mu b(HA+K) - \mu(d+\mu+\gamma)} < 0.$$

So this case must be omitted.

If $\beta + \mu bH < a(d+\mu+\gamma)$, it follows from Eq.(8) that:

$$(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]I^2 + [\beta A + aK(d+\mu) - (d+\mu+\gamma)(\mu + aA) - \mu bK]I - K(\mu + aA) = 0 \qquad (9)$$

Then

$$\xi_1 = [\beta A + aK(d+\mu) - (d+\mu+\gamma)(\mu + aA) - \mu bK]^2 + 4K(\mu + aA)(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)] > 0$$

Denoting two roots of Eq. (9) by I_1 and I_2 we have:

$$I_{1,2} = \frac{-[\beta A + aK(d+\mu) - (d+\mu+\gamma)(\mu + aA) - \mu bK] \mp \sqrt{\xi_1}}{2(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]}$$
$$I_1 + I_2 = -\frac{\beta A + aK(d+\mu) - (d+\mu+\gamma)(\mu + aA) - \mu bK}{(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]}$$

$$I_1 * I_2 = \frac{-K(\mu + aA)}{(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]}$$

So Eq.(9) has only one positive root denoted by I_1 and the other is negative root.

$$I_1 = \frac{R_1 + \sqrt{\xi_1}}{2(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]}$$

$$S_1 = \frac{1}{2\mu}\left[\frac{(d+\mu+\gamma)(aA-\mu) + aK(d+\mu) - \beta A - \mu b(2HA+K) - \sqrt{\xi_1}}{a(d+\mu+\gamma) - (\beta + \mu bH)}\right]$$

Then $S_1 > 0$ holds only if

$$\mathcal{T}_0 < \frac{(d+\mu+\gamma)(aA-\mu) + aK(d+\mu) - \mu b(2HA+K) - \sqrt{\xi_1}}{(\mu+aA)(d+\mu+\delta+\gamma)}.$$

Define

$$N_1 = \frac{(d+\mu+\gamma)(aA-\mu) + aK(d+\mu) - \mu b(2HA+K) - \sqrt{\xi_1}}{(\mu+aA)(d+\mu+\delta+\gamma)}.$$

The point $E_1(S_1, I_1)$ satisfies the system (3), that is,$I_1 > I_0$

$$\frac{R_1 + \sqrt{\xi_1}}{2(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]} > I_0.$$

We have

$$\sqrt{\xi_1} > -R_1 + 2(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]I_0. \qquad (10)$$

If

$-R_1 + 2(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]I_0 < 0$, then

$$\mathcal{T}_0 < 1 - \frac{\delta(\mu+aA) + aK(d+\mu)}{(\mu+aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu+aA)(d+\mu+\delta+\gamma)} - \frac{2(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]I_0}{(\mu+aA)(d+\mu+\delta+\gamma)}.$$

Define

$$N_2 = 1 - \frac{\delta(\mu+aA) + aK(d+\mu)}{(\mu+aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu+aA)(d+\mu+\delta+\gamma)} - \frac{2(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]I_0}{(\mu+aA)(d+\mu+\delta+\gamma)}$$

Then Eq.(10) holds only, if

$$\begin{cases} -R_2 + 2(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]I_0 > 0 \\ \xi_1 \geq [-R_2 + 2(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]I_0]^2. \end{cases}$$

Hence

$$N_2 < T_0 \leq 1 - \frac{\delta(\mu + aA) + aK(d+\mu)}{(\mu + aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu + aA)(d+\mu+\delta+\gamma)} + \frac{K}{(d+\mu+\delta+\gamma)I_0} - \frac{(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]I_0}{(\mu + aA)(d+\mu+\delta+\gamma)}.$$

Define

$$N_3 = 1 - \frac{\delta(\mu + aA) + aK(d+\mu)}{(\mu + aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu + aA)(d+\mu+\delta+\gamma)} + \frac{K}{(d+\mu+\delta+\gamma)I_0} - \frac{(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]I_0}{(\mu + aA)(d+\mu+\delta+\gamma)}.$$

So, if $\mathcal{T}_0 \leq N_3$ and $\mathcal{T}_0 < N_1$, then $E_1(S_1, I_1)$ is endemic equilibrium, where

$$I_1 = \frac{R_1 + \sqrt{\xi_1}}{2(d+\mu)[a(d+\mu+\gamma) - (\beta + \mu bH)]} \quad and\ S_1 = \frac{1}{\mu}[A - (d+\mu)I_1].$$

If $(\beta + \mu bH) > a(d+\gamma+\mu)$, then it is easy to see that Eq.(8) has no positive root if $R_1 \geq 0$. If $R_1 < 0$, then

$$\xi_2 = R_1^2 - 4K(\mu + aA)(d+\mu)[\beta + \mu bH - a(d+\mu+\gamma)]$$

and

$$R_1 = (d+\mu+\gamma)(\mu + aA) + \mu bK - \beta A - aK(d+\mu)$$

$$R_1 = -\mathcal{T}_0(d + aA)(d+\mu+\delta+\gamma) + (\mu + aA)(d+\mu+\delta+\gamma) - \delta(\mu + aA) + \mu bK - aK(d+\mu).$$

Thus $\xi_2 \geq 0$ implies $R_1^2 \geq 4K(\mu + aA)(d+\mu)[\beta + \mu bH - a(d+\mu+\gamma)]$, and we get that:

$$\mathcal{T}_0 \leq 1 - \frac{aK(d+\mu) + \delta(\mu + aA)}{(\mu + aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu + aA)(d+\mu+\delta+\gamma)} - \frac{2\sqrt{K(\mu + aA)(d+\mu)[\beta + \mu bH - a(d+\mu+\gamma)]}}{(\mu + aA)(d+\mu+\delta+\gamma)}$$

or

$$\mathcal{T}_0 \geq 1 - \frac{aK(d+\mu) + \delta(\mu + aA)}{(\mu + aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu + aA)(d+\mu+\delta+\gamma)} + \frac{2\sqrt{K(\mu + aA)(d+\mu)[\beta + \mu bH - a(d+\mu+\gamma)]}}{(\mu + aA)(d+\mu+\delta+\gamma)}.$$

Define

$$N_4 = 1 - \frac{aK(d+\mu)+\delta(\mu+aA)}{(\mu+aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu+aA)(d+\mu+\delta+\gamma)} - \frac{2\sqrt{K(\mu+aA)(d+\mu)[\beta+\mu bH-a(d+\mu+\gamma)]}}{(\mu+aA)(d+\mu+\delta+\gamma)}$$

$$N_5 = 1 - \frac{aK(d+\mu)+\delta(\mu+aA)}{(\mu+aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu+aA)(d+\mu+\delta+\gamma)} + \frac{2\sqrt{K(\mu+aA)(d+\mu)[\beta+\mu bH-a(d+\mu+\gamma)]}}{(\mu+aA)(d+\mu+\delta+\gamma)}.$$

At the same time, $R_1 < 0$ holds if and only if

$$\mathcal{T}_0 > 1 - \frac{aK(d+\mu)+\delta(\mu+aA)}{(\mu+aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu+aA)(d+\mu+\delta+\gamma)}.$$

Define

$$N_6 = 1 - \frac{aK(d+\mu)+\delta(\mu+aA)}{(\mu+aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu+aA)(d+\mu+\delta+\gamma)}$$

Therefore, if $\mathcal{T}_0 \geq N_5$, we have $R_1 < 0$ and $\xi_2 \geq 0$, then Eq.(8) has two positive roots I_2, I_3; where:

$$I_2 = \frac{-R_1-\sqrt{\xi_2}}{2(d+\mu)[\beta+\mu bH-a(d+\mu+\gamma)]} \quad , \quad I_3 = \frac{-R_1+\sqrt{\xi_2}}{2(d+\mu)[\beta+\mu bH-a(d+\mu+\gamma)]}.$$

Then $S_i = \frac{1}{\mu}[A-(d+\mu)I_i] > 0 \quad , \ (i = 2,3)$ holds if

$$\mathcal{T}_0 < 1 - \frac{\delta(\mu+aA)+aK(d+\mu)}{(\mu+aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu+aA)(d+\mu+\delta+\gamma)} + \frac{2A[\beta+\mu bH-a(d+\mu+\gamma)]+\sqrt{\xi_2}}{(\mu+aA)(d+\mu+\delta+\gamma)}$$

and

$$\mathcal{T}_0 < 1 - \frac{\delta(\mu+aA)+aK(d+\mu)+\sqrt{\xi_2}}{(\mu+aA)(d+\mu+\delta+\gamma)} + \frac{\mu bK}{(\mu+aA)(d+\mu+\delta+\gamma)} + \frac{2A[\beta+\mu bH-a(d+\mu+\gamma)]}{(\mu+aA)(d+\mu+\delta+\gamma)}.$$

Define

$$N_7 = 1 - \frac{\delta(\mu + aA) + aK(d + \mu)}{(\mu + aA)(d + \mu + \delta + \gamma)} + \frac{\mu bK}{(\mu + aA)(d + \mu + \delta + \gamma)} + \frac{2A[\beta + \mu bH - a(d + \mu + \gamma)] + \sqrt{\xi_2}}{(\mu + aA)(d + \mu + \delta + \gamma)}$$

$$N_8 = 1 - \frac{\delta(\mu + aA) + aK(d + \mu) + \sqrt{\xi_2}}{(\mu + aA)(d + \mu + \delta + \gamma)} + \frac{\mu bK}{(\mu + aA)(d + \mu + \delta + \gamma)} + \frac{2A[\beta + \mu bH - a(d + \mu + \gamma)]}{(\mu + aA)(d + \mu + \delta + \gamma)}.$$

It is easy to see that $N_8 < N_7$.

Which implies that Eq.(8) has two positive equilibrium points $E_2(S_2, I_2), E_3(S_3, I_3)$ if $\mathcal{T}_0 < N_8$, then Eq.(8) has no positive equilibrium point if $\mathcal{T}_0 \geq N_7$, and Eq.(8) has only one positive equilibrium point $E_2(S_2, I_2)$ if $N_8 < \mathcal{T}_0 < N_7$.

Now, we consider the conditions for $I_i > I_0$, $(i = 2,3)$

$I_2 > I_0 \rightarrow -R_1 - \sqrt{\xi_2} > 2(d + \mu)[\beta + \mu bH - a|(d + \mu + \gamma)]I_0$

If $R_1 + 2(d + \mu)[\beta + \mu bH - a(d + \mu + \gamma)]I_0 < 0$, then

$$\mathcal{T}_0 > 1 - \frac{\delta(\mu + aA) + aK(d + \mu)}{(\mu + aA)(d + \mu + \delta + \gamma)} + \frac{\mu bK}{(\mu + aA)(d + \mu + \delta + \gamma)} + \frac{2(d + \mu)[\beta + \mu bH - a(d + \mu + \gamma)I_0)]}{(\mu + aA)(d + \mu + \delta + \gamma)}.$$

Define

$$N_9 = 1 - \frac{\delta(\mu + aA) + aK(d + \mu)}{(\mu + aA)(d + \mu + \delta + \gamma)} + \frac{\mu bK}{(\mu + aA)(d + \mu + \delta + \gamma)} + \frac{2(d + \mu)[\beta + \mu bH - a(d + \mu + \gamma)I_0)]}{(\mu + aA)(d + \mu + \delta + \gamma)}.$$

Furthermore, if $R_1 + 2(d + \mu)[\beta + \mu bH - a(d + \mu + \gamma)]I_0 > 0$, then

$\{ R_1 + 2(d + \mu)[\beta + \mu bH - a(d + \mu + \gamma)]I_0\}^2 > \xi_2$

i.e.

$$\mathcal{T}_0 < 1 - \frac{\delta(\mu + aA) + aK(d + \mu)}{(\mu + aA)(d + \mu + \delta + \gamma)} + \frac{\mu bK}{(\mu + aA)(d + \mu + \delta + \gamma)} + \frac{K}{(d + \mu + \delta + \gamma)I_0} + \frac{(d + \mu)[\beta + \mu bH - a(d + \mu + \gamma)]I_0}{(\mu + aA)(d + \mu + \delta + \gamma)}.$$

Therefore, if $N_9 < \mathcal{T}_0 < N_3$, then $I_2 > I_0$ holds

Similarly, if $I_3 > I_0$

$R_1 + 2(d+\mu)[\beta + \mu bH - a(d+\mu+\gamma)]I_0 < 0$

Or

$$\begin{cases} R_1 + 2(d+\mu)[\beta + \mu bH - a(d+\mu+\gamma)]I_0 > 0 \\ \quad \xi_2 > \{R_1 + 2(d+\mu)[\beta + \mu bH - a(d+\mu+\gamma)]I_0\}^2. \end{cases}$$

Thus we get that $\mathcal{T}_0 < N_9$ or $\mathcal{T}_0 > \max(N_3, N_9)$.

Theorem (5.1): From the above discussion, we get the following conclusions:

A. If $\beta + \mu bH < a(d+\mu+\gamma)$, then $E_1(S_1, I_1)$ is a unique endemic equilibrium of model (3) if $\mathcal{T}_0 < N_1$ and $E_1(S_1, I_1)$ is a unique endemic equilibrium of model (1) if $\mathcal{T}_0 < N_1$ and $\mathcal{T}_0 < N_3$.

B. If $\beta + \mu bH > a(d+\mu+\gamma)$, then model (3) has two positive equilibrium points $E_2(S_2, I_2), E_3(S_3, I_3)$ if $\mathcal{T}_0 < N_8$; model (3) has only one positive equilibrium point $E_2(S_2, I_2)$,if $N_8 < \mathcal{T}_0 < N_7$; model (3) has no positive equilibrium point if $\mathcal{T}_0 \geq N_7$; and $E_2(S_2, I_2)$ is an endemic equilibrium of model (1) if $N_9 < \mathcal{T}_0 < N_3$ and $E_3(S_3, I_3)$ is an endemic equilibrium point of model (1) if $\mathcal{T}_0 < N_9$ or $\mathcal{T}_0 > \max(N_3, N_9)$.

C. If $\beta + \mu bH = a(d+\mu+\gamma)$, then model (3) has no endemic equilibrium point.

Theorem (5.2): Assume that the endemic equilibrium point $E_1(S_1, I_1)$ of the system (3) is locally asymptotically stable in the $IntR_+^2$, then it is globally asymptotically stable in $IntR_+^2$ if $\delta < \mu$. (11) .

Theorem (5.3): The endemic equilibrium points E_2 and E_3 of system (3) are locally asymptotically stable under the condition (11) by replacing E_1 by E_2, E_3.

Conflict of Interests

The authors declare that there is no conflict of interests.

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