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ON THE $\mathbb{Z}_q$-MACDONALD CODE AND ITS WEIGHT DISTRIBUTION OF DIMENSION 3

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Abstract. In this paper, we determine the parameters of $\mathbb{Z}_q$-MacDonald Code of dimension k for any positive integer $q \geq 2$. Further, we have obtained the weight distribution of $\mathbb{Z}_q$-MacDonald code of dimension 3 and furthermore, we have given the weight distribution of $\mathbb{Z}_q$-Simplex code of dimension 3 for any positive integer $q \geq 2$.

Keywords: $\mathbb{Z}_q$-linear code; Codes over finite rings; $\mathbb{Z}_q$-Simplex code; $\mathbb{Z}_q$-MacDonald code; Minimum Hamming distance.

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1. Introduction

A code C is a subset of $\mathbb{Z}_q^n$ where $\mathbb{Z}_q$ is the set of all integers modulo q and n is any positive integer. Let $x, y \in \mathbb{Z}_q^n$. Then the *Hamming distance* between x and y is the number of coordinates in which they differ. It is denoted by $d(x,y)$. Vividly $d(x,y) = wt(x-y)$, the number of non-zero coordinates in $x-y$ is called the *Hamming weight* of $x-y$. The *minimum Hamming distance d*

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of C is defined as

$$d = \min\{d(x,y) \mid x,y \in C \text{ and } x \neq y\} = \min\{wt(x-y) \mid x,y \in C \text{ and } x \neq y\}$$

and the *minimum Hamming weight* of C is $\min\{wt(c) \mid c \in C \text{ and } c \neq 0\}$. Hereafter we simply call the minimum Hamming distance and the minimum Hamming weight, the minimum distance and the minimum weight respectively. A code over $\mathbb{Z}_q$ of length n, cardinality M with the minimum distance d is called an $(n,\ M,\ d)$ $\mathbb{Z}_q$-code. Let C be an $(n,\ M,\ d)$ $\mathbb{Z}_q$-code. For $0 \leq i \leq n$, let A_i be the number of codewords of the Hamming weight i. Then $\{A_i\}_{i=0}^{n}$ is called the *weight distribution* of the code C.

We know pretty well that $\mathbb{Z}_q$ is a group under the addition modulo q. Then $\mathbb{Z}_q^n$ is a group under coordinate-wise addition modulo q. C is said to be a $\mathbb{Z}_q$*-linear code* if C is a subgroup of $\mathbb{Z}_q^n$. In fact, it is a free $\mathbb{Z}_q$-module. Since $\mathbb{Z}_q^n$ is a free $\mathbb{Z}_q$-module, it has a basis. Therefore, every $\mathbb{Z}_q$-linear code has a basis. Since $\mathbb{Z}_q^n$ has a finite basis, $\mathbb{Z}_q$-linear code has a finite dimension. Since $\mathbb{Z}_q^n$ is finitely generated $\mathbb{Z}_q$-module, it implies that C is a finitely generated submodule of $\mathbb{Z}_q^n$. The cardinality of a minimal generating set of C is called the rank of the code C [15]. A generator matrix of C is a matrix the rows of which generate C. Any linear code C over $\mathbb{Z}_q$ with generator matrix G is permutation-equivalent to a code with generator matrix of the form

$$\begin{bmatrix} I_k & A_{01} & A_{02} & \cdots & A_{0s-1} & A_{0s} \\ 0 & z_1 I_{k_1} & z_1 A_{12} & \cdots & z_1 A_{1s-1} & z_1 A_{1s} \\ 0 & 0 & z_2 I_{k_2} & \cdots & z_2 A_{2s-1} & z_2 A_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z_{s-1} I_{k_{s-1}} & z_{s-1} A_{s-1s} \end{bmatrix},$$

where A_{ij} are matrices over $\mathbb{Z}_q$, $\{z_1,\ z_2,\ \cdots,\ z_{s-1}\}$ are the zero-divisors in $\mathbb{Z}_q$ and the columns are grouped into blocks of sizes $k,\ k_1,\ \cdots,\ k_{s-1}$ respectively. Then $|C| = q^k (\frac{q}{z_1})^{k_1} (\frac{q}{z_2})^{k_2} \cdots (\frac{q}{z_{s-1}})^{k_{s-1}}$. If $k_1 = k_2 = \cdots = k_{s-1} = 0$, then the code C is called k-dimensional code. Every k dimension $\mathbb{Z}_q$-linear code with length n and the minimum distance d is called an $[n,\ k,\ d]$ $\mathbb{Z}_q$*-linear code.*

There are many researchers doing research on codes over finite rings [1], [4], [8], [13] and [16]. In the last decade, there have been many number of researchers doing research on codes over $\mathbb{Z}_4$ and $\mathbb{Z}_q$ [3], [7], [9], [10] and [14]. Further, in [11], they have determined the parameters of $\mathbb{Z}_q$-Simplex codes of dimension k and in [12], they have obtained the weight distribution of $\mathbb{Z}_q$-Simplex codes of dimension 2 for any positive integer $q \geq 2$.

Let

$$G_2(q) = \left[\begin{array}{c|c|cccc} 0 & 1 & 1 & 2 & \cdots & q-1 \\ \hline 1 & 0 & 1 & 1 & \cdots & 1 \end{array}\right].$$

Then the code generated by this matrix is called *2-dimensional $\mathbb{Z}_q$-Simplex code.* In [11], they have given the parameters of $\mathbb{Z}_q$-Simplex codes of dimension 2 and it is stated below.

Theorem 1.1. [11] *Let $q = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}$, where $p_1, p_2, \cdots p_r$ are distinct primes. Let $p = \min\{p_i \mid 1 \leq i \leq r\}$, then the code generated by the matrix $G_2(q)$ is $[q+1, 2, \frac{q(p-1)}{p} + 1]$ $\mathbb{Z}_q$-linear code.*

Now, we define inductively

$$G_{k+1}(q) = \left[\begin{array}{c|c|c|c|c|c} 00\cdots0 & 1 & 11\cdots1 & 22\cdots2 & \cdots & q-1q-1\cdots q-1 \\ \hline & 0 & & & & \\ G_k(q) & \vdots & G_k(q) & G_k(q) & \cdots & G_k(q) \\ & 0 & & & & \end{array}\right]$$

for $k \geq 2$.

Clearly, this $G_{k+1}(q)$ matrix generates $[n_{k+1} = \frac{q^{k+1}-1}{q-1},\ k+1,\ d]$ $\mathbb{Z}_q$-linear code of dimension $k+1$. The code generated by the matrix $G_k(q)$ is called *$\mathbb{Z}_q$-Simplex code of dimension k.* It is denoted by $S_k(q)$. In [11], they have obtained the parameters of $\mathbb{Z}_q$-Simplex codes of dimension k and it is given below.

Theorem 1.2. [11] *The $\mathbb{Z}_q$-Simplex code of dimension k is an $[n_k = \frac{q^k-1}{q-1},\ k,\ d_k = \frac{q}{p}(p-1)n_{k-1}+1]$ $\mathbb{Z}_q$-linear code where $p > 1$ is the smallest divisor of q.*

In [5], they have defined a $\mathbb{Z}_q$-linear code which is similar to the MacDonald code over finite field. But it gives different weight distribution. In the generator matrix $G_k(q)$ of $\mathbb{Z}_q$-Simplex code $S_k(q)$ of dimension k, by deleting the matrix

$$\begin{bmatrix} O \\ G_u(q) \end{bmatrix}$$

where $2 \leq u \leq k-1$ and O is $(k-u) \times \frac{q^u-1}{q-1}$ zero matrix, they have obtained

$$G_{k,u}(q) = \left(G_k(q) \setminus \begin{pmatrix} 0 \\ G_u(q) \end{pmatrix} \right) \tag{1}$$

for $2 \leq u \leq k-1$ and $\left(A \setminus B\right)$ is a matrix obtained from the matrix A by removing the matrix B. A code generated by the matrix $G_{k,u}(q)$ is called $\mathbb{Z}_q$-*MacDonald* code. It is denoted by $M_{k,u}(q)$. It is clear that the dimension of this code is k. The Quaternary MacDonald codes were discussed in [6] and the MacDonald codes over finite field were discussed in [2].

In this correspondence, we concentrate on $\mathbb{Z}_q$-MacDonald Code. In Section 2, we determine the parameters of $\mathbb{Z}_q$-MacDonald code of dimension k and in Section 3, we obtain the weight distribution of $\mathbb{Z}_q$-MacDonald code of dimension 3, for any positive integer $q \geq 2$. In Section 4, we find the weight distribution of $\mathbb{Z}_q$-Simplex code of dimension 3, for any positive integer $q \geq 2$.

2. Minimum distance of $\mathbb{Z}_q$-MacDonald code of dimension k

In Equation (1), if we put $u = k-1$, then a generator matrix of k-dimensional $\mathbb{Z}_q$-MacDonald code is

$$G_{k,k-1}(q) = \left[\begin{array}{c|c|c|c|c} 1 & 11\cdots1 & 22\cdots2 & \cdots & q-1q-1\cdots q-1 \\ \hline 0 & & & & \\ \vdots & G_{k-1}(q) & G_{k-1}(q) & \cdots & G_{k-1}(q) \\ 0 & & & & \end{array}\right],$$

where $G_{k-1}(q)$ is a generator matrix of $\mathbb{Z}_q$-Simplex code of dimension $k-1$. Then this matrix generates the code

$$M_{k,k-1}(q) = \{(0cc\cdots c) + \alpha(1\mathbf{1}\mathbf{2}\cdots\mathbf{q-1}) \mid \alpha \in \mathbb{Z}_q,\ c \in S_{k-1}(q)\},$$

where $\mathbf{i} = ii\cdots i \in \mathbb{Z}_q^n$ and $n = \frac{q^{k-1}-1}{q-1} = n_{k-1}$. The code generated by the Matrix $G_{k,k-1}(q)$ is a $[q^{k-1},\ k,\ d(M_{k,k-1}(q))]$ $\mathbb{Z}_q$-linear code.

Case (i). Let $\alpha = 0$. Then

$$\min\{wt(0cc\cdots c) \mid c \in S_{k-1}(q)\} = (q-1)d(S_{k-1}(q)) = (q-1)\left(\frac{q}{p}(p-1)n_{k-2}+1\right) \tag{2}$$

where $p > 1$ is the smallest divisor of q.

Case (ii). Let $\alpha \neq 0$.

Subcase (i). Let $\alpha \in \mathbb{Z}_q$ with $(\alpha, q) = 1$. If $\alpha i = \alpha j$, then $\alpha(i-j) = 0$. Since α is a unit, it implies $i = j$. Therefore $\{\alpha\mathbf{1},\ \alpha\mathbf{2},\ \cdots,\ \alpha(\mathbf{q-1})\} = \{\mathbf{1},\ \mathbf{2}, \cdots,\ \mathbf{q-1}\}$.

Consider

$$\begin{aligned}
wt\big((0cc\cdots c) + \alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) &= wt\big((0cc\cdots c) + (\alpha\ \alpha\mathbf{1}\ \alpha\mathbf{2}\ \cdots\ \alpha(\mathbf{q-1}))\big) \\
&= wt\big((0cc\cdots c) + (\alpha\ \mathbf{1}\ \mathbf{2}\ \cdots\ \mathbf{q-1})\big) \\
&= 1 + \sum_{i=1}^{q-1} wt(c + \mathbf{i}) \\
wt\big((0cc\cdots c) + \alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) &= 1 + \sum_{i=1}^{q-1} wt(-c + \mathbf{i}) \text{ since } S_{k-1}(q) \text{ is } \mathbb{Z}_q\text{-linear.}
\end{aligned}$$

Let $n(i)$ be the number of i coordinates in $c \in S_{k-1}(q)$ where $i = 0,\ 1,\ 2,\ \cdots,\ q-1$. Then for $0 \leq i \leq q-1$, $wt(-c+\mathbf{i}) = n - n(i)$, where n is the length of $S_{k-1}(q)$. Therefore,

$$\begin{aligned}
wt\big((0cc\cdots c) + \alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) &= 1 + \sum_{i=1}^{q-1} (n - n(i)) \\
&= 1 + (q-1)n - \sum_{i=1}^{q-1} n(i) \\
&= 1 + (q-1)n - (n - n(0)) \\
wt\big((0cc\cdots c) + \alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) &= 1 + (q-2)n + n(0) \text{ for all } c \in S_{k-1}(q).
\end{aligned}$$

Therefore,

$$\min_{c \in S_{k-1}(q)} \big\{wt\big((0cc\cdots c) + \alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) \mid (\alpha,\ q) = 1\big\} = 1 + (q-2)n + \min_{c \in S_{k-1}(q)} \{n(0)\} \tag{3}$$

The largest weight codeword of $S_{k-1}(q)$ gives the minimum value of the above Equation (3).

Subcase (ii). Let $(\alpha, q) \neq 1$ and $o(\alpha) = d$. Then, $\{\alpha 1,\ \alpha 2,\ \cdots,\ \alpha(q-1)\} = \{\alpha 1,\ \alpha 2,\ \cdots,\ \alpha(d-1),\ 0\}$. Clearly, in $\{\alpha 1,\ \alpha 2,\ \cdots,\ \alpha(q-1)\}$, each non-zero αi appears $\frac{q}{d}$ times and zero appears $(\frac{q}{d} - 1)$ times.

Consider

$$wt\big((0cc\cdots c) + \alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) = wt\big((0cc\cdots c) + (\alpha\ \alpha\mathbf{1}\ \alpha\mathbf{2}\ \cdots\ \alpha(\mathbf{q-1}))\big)$$

$$wt\big((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) = 1+\frac{q}{d}\Big\{wt(\alpha\mathbf{1}+c)+wt(\alpha\mathbf{2}+c)+\cdots+wt(\alpha(\mathbf{d-1})+c)\Big\} +\big(\frac{q}{d}-1\big)wt(c). \tag{4}$$

If there is a $c \in S_{k-1}(q)$ such that $c_i \in <\alpha>$ for all i, then $\sum_{i=1}^{d} n(c_i) = n$ and Equation (4) becomes

$$\begin{aligned} wt\big((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) &= 1+\frac{q}{d}\Big[(n-n(c_1))+(n-n(c_2))+\cdots+ \\ &\quad (n-n(c_{d-1}))\Big]+\big(\frac{q}{d}-1\big)wt(c) \\ &= 1+\frac{q}{d}\Big[(n-n(\alpha))+(n-n(2\alpha))+\cdots+ \\ &\quad (n-n((d-1)\alpha))\Big]+\big(\frac{q}{d}-1\big)wt(c) \\ wt\big((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) &= 1+\frac{q}{d}\Big[(d-1)n-\sum_{i=1}^{d-1}n(i\alpha)\Big]+\big(\frac{q}{d}-1\big)wt(c). \end{aligned}$$

Otherwise, $wt\big((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) \geq 1+\frac{q}{d}\Big[(d-1)n-\sum_{i=1}^{d-1}n(i\alpha)\Big]+\big(\frac{q}{d}-1\big)wt(c)$.
Since $n=\sum_{i=1}^{d-1}n(c_i)+n(0)$, we get

$$\begin{aligned} wt\big((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) &= 1+\frac{q}{d}\Big[(d-1)n-(n-n(0))\Big]+\Big(\frac{q}{d}-1\Big)wt(c) \\ &= 1+\frac{q}{d}\Big[(d-2)n+n(0)\Big]+\Big(\frac{q}{d}-1\Big)wt(c) \\ &= 1+\frac{q}{d}\Big[(d-2)n+n(0)\Big]+\Big(\frac{q}{d}-1\Big)\Big(n-n(0)\Big), \\ &\quad \text{where } c_i \in <\alpha> \\ &= 1+\frac{q}{d}(d-2)n+\frac{q}{d}n(0)+\frac{q}{d}n-n-\frac{q}{d}n(0)+n(0) \\ wt\big((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big) &= 1+(q-1)n-\frac{q}{d}n+n(0). \end{aligned} \tag{5}$$

If there exists $c=(c_1,c_2,\cdots,c_n)\in S_{k-1}(q)$ such that $c_i\in<\alpha>$ and d is smaller, then the Equation (5) gives the smaller value. That is,

(6)

$$\min_{c\in S_{k-1}(q)}\Big\{wt\big((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big)\mid(\alpha,\,q)\neq 1\Big\}=1+\big(q-\frac{q}{d}-1\big)n+\Big\{\min_{c\in S_{k-1}(q)}n(0)\Big\},$$

where $n(0)$ is the number of zeros in c and α must be smaller order element. Therefore,

$$\begin{aligned}&\min\Big\{wt\big((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big)\mid\alpha\in\mathbb{Z}_q,\ c\in S_{k-1}(q)\Big\}\\&=\min\Big\{(q-1)d(S_{k-1}(q)),1+\big(q-\frac{q}{d}-1\big)n+\min_{c\in S_{k-1}(q)}n(0)\Big\}.\end{aligned}$$

Let α be a least order non-zero element in $\mathbb{Z}_q$. Since $011\cdots1\in S_2(q)$, it implies that $c=0\alpha\alpha\cdots\alpha\in S_2(q)$. Therefore, for $k=3$, the above Equation (6) becomes

$$\min_{c\in S_2(q)}\{wt(0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\}=1+(q-\frac{q}{d}-1)n_2+n(0).$$

Since $n(0)=1$, it implies

$$\min_{c\in S_2(q)}\big\{wt\big((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big)\big\}=2+\big(q-\frac{q}{d}-1\big)n_2.$$

For $k=3$, Equation (2) gives $\min\limits_{c\in S_2(q)}\{wt(0cc\cdots c)\}=(q-1)d(S_2(q))$ and Equation (3) gives $\min\limits_{c\in S_2(q)}\{wt(0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\}=1+(q-2)n_2+1=2+(q-2)n_2$. Therefore, the minimum distance of $M_{3,2}(q)$ is

$$d(M_{3,2}(q))=\min_{c\in S_2(q)}\big\{(q-1)d(S_2(q)),\ 2+\big(q-\frac{q}{d}-1\big)n_2\big\}.$$

Since $d(S_2(q))=\frac{q}{p}(p-1)+1$, it follows that $d(M_{3,2}(q))=2+\big(q-\frac{q}{d}-1\big)n_2$.

For $k=4$, in $S_3(q)$, the codeword $c=0\alpha\alpha\cdots\alpha\in S_2(q)$ is repeated q times and $c^{'}=c0c\cdots c$ is a codeword in $S_3(q)$ which gives the minimum number of zeros, and all coordinates of $c^{'}$ are in $<\alpha>$. The number of zeros in $c^{'}$ is $q+1$. Hence, Equation (6) becomes

$$\min_{c\in S_3(q)}\big\{wt\big((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big)\big\}=1+\big(q-\frac{q}{d}-1\big)n_3+(q+1).$$

For $k=5$, the codeword $c^{'}\in S_3(q)$ is repeated q times in $S_4(q)$ and hence $c^{''}=c^{'}0c^{'}\cdots c^{'}$ is codeword in $S_4(q)$ which gives the minimum number of zeros, and its coordinates are in $<\alpha>$.

The number of zeros in $c^{''}$ is $[q(q+1)]+1$. Hence, Equation (6) becomes

$$\min_{c\in S_4(q)}\{wt\big((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big)\}=1+\big(q-\frac{q}{d}-1\big)n_4+\big[q(q+1)+1\big].$$

In general, for any k, in $S_{k-1}(q)$, there is a codeword $c\in S_{k-2}(q)$ the coordinates of which are in $<\alpha>$ with minimum number of zeros $\frac{q^{k-3}-1}{q-1}$ and hence $c_1=c0c\cdots c$ is a codeword in $S_{k-1}(q)$ which gives the minimum number of zeros, and its coordinates are in $<\alpha>$. The number of zeros in c_1 is $\frac{q^{k-2}-1}{q-1}$. Hence, Equation (6) becomes

$$\min_{c_1\in S_{k-1}(q)}\{wt\big((0c_1c_1\cdots c_1)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\big)\}=1+\big(q-\frac{q}{d}-1\big)n_{k-1}+\frac{q^{k-2}-1}{q-1}.$$

Therefore,

$$d(M_{k,k-1}(q))=1+\big(q-\frac{q}{d}-1\big)n_{k-1}+\frac{q^{k-2}-1}{q-1}.$$

Thus, we have the following.

Theorem 2.1. *The $\mathbb{Z}_q$-MacDonald code $M_{k,k-1}(q)$ is a $\big[q^{k-1},\ k,\ 1+\big(q-\frac{q}{d}-1\big)\big(\frac{q^{k-1}-1}{q-1}\big)+\frac{q^{k-2}-1}{q-1}\big]$ $\mathbb{Z}_q$-linear code where $d>1$ is the smallest divisor of q.*

3. Weight distribution of $\mathbb{Z}_q$-MacDonald code of dimension 3

Let

$$G_{3,2}(q)=\left[\begin{array}{c|c|c|c|c} 1 & 1\,1\,1\,1\cdots\ 1 & 2\,2\,2\,2\cdots\ 2 & \cdots & q-1\,q-1\,q-1\,q-1\ \cdots\ q-1\\ \hline 0 & 0\,1\,1\,2\,\cdots\,q-1 & 0\,1\,1\,2\,\cdots\,q-1 & \cdots & 0\,1\,1\,2\,\cdots\,q-1\\ 0 & 1\,0\,1\,1\,\cdots\ 1 & 1\,0\,1\,1\,\cdots\ 1 & \cdots & 1\,0\,1\,1\,\cdots\ 1 \end{array}\right].$$

Then by Theorem 2.1, this matrix generates $\big[q^2,\ 3,\ 2+\big(q-\frac{q}{d}-1\big)\big(q+1\big)\big]$ $\mathbb{Z}_q$-linear code where $d>1$ is the smallest divisor of q. It is $M_{3,2}(q)=\{(0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})\mid\alpha\in\mathbb{Z}_q\}$. In [12], they have given the weight distribution of 2-dimensional $\mathbb{Z}_q$-Simplex code as the following.

Theorem 3.1. [12] *For any integer $q \geq 2$, the weight distribution of $\mathbb{Z}_q$-Simplex code of dimension 2 is*

$$\begin{aligned} A_0 &= 1 \\ A_q &= q\phi(q) + q - 1 \\ A_{q-\frac{q}{d}+1} &= d\phi(d), \text{ for } d|q \text{ and } d \neq 1, d \neq q \\ A_{q+1} &= q(q-1) - \sum_{d|q, d\neq 1} d\phi(d). \end{aligned}$$

where $d > 1$ is the smallest divisor of q.

Note that there is only one codeword in $S_2(q)$ such that $n(0) = n$, $d\phi(d)$ codewords in $S_2(q)$ such that $n(0) = \frac{q}{d}$, for all $d|q$, $d \neq 1$ and $d \neq q$, $q\phi(q) + q - 1$ codewords in $S_2(q)$ such that $n(0) = 1$ and $q(q-1) - \sum_{d|q, d\neq 1} d\phi(d)$ codewords in $S_2(q)$ such that $n(0) = 0$.
Now, we consider the code $M_{3,2}(q)$.

Case (i). If $\alpha = 0$, then

$$\begin{aligned} wt\big((0cc\cdots c) + \alpha(1\mathbf{12}\cdots \mathbf{q-1})\big) &= wt(0cc\cdots c) \\ &= (q-1)wt(c), \text{ for all } c \in S_2(q). \end{aligned} \tag{7}$$

By Theorem 3.1, we get the following weights:

(8)
$$\begin{cases} \text{Number of zero weight codeword is } 1. \\ \text{Number of } (q-1)(q-\frac{q}{d}+1) \text{ weight codeword is } d\phi(d) \text{ where } d|q,\ d \neq 1 \text{ and } d \neq q. \\ \text{Number of } (q-1)q \text{ weight codeword is } q\phi(q)+q-1. \\ \text{Number of } (q-1)(q+1) = q^2-1 \text{ weight codeword is } q(q-1) - \sum_{d|q,d\neq 1} d\phi(d). \end{cases}$$

Case (ii). If $(\alpha, q) = 1$, then by Equation (3),

$$wt\big((0cc\cdots c) + \alpha(1\mathbf{12}\cdots \mathbf{q-1})\big) = 1 + (q-2)n + n(0), \tag{9}$$

for all $c \in S_2(q)$, and $n(0)$ is the number of zeros in c. By Theorem 3.1, in $M_{3,2}(q)$,

(10)

$$\begin{cases} \text{Number of } 1+(q-1)n \text{ weight codeword is } 1.\phi(q). \\ \text{Number of } 1+(q-2)n+\frac{q}{d} \text{ weight codeword is } (d\phi(d)).\phi(q), \text{ for all } d|q, d\neq 1 \text{ and } d\neq q. \\ \text{Number of } 2+(q-2)n \text{ weight codeword is } (q\phi(q)+q-1).\phi(q). \\ \text{Number of } 1+(q-2)n \text{ weight codeword is } (q(q-1)-\sum_{d|q,d\neq 1} d\phi(d)).\phi(q). \end{cases}$$

Case (iii). If α is not relatively prime to q, then by Equation (5), we have

$$wt((0cc\cdots c)+\alpha(1\mathbf{12}\cdots\mathbf{q-1})) = 1+(q-1)n-\frac{q}{d}\sum_{i=0}^{d-1} n(i\alpha)+n(0), \tag{11}$$

where $n(i\alpha)$ is the number of $i\alpha$'s in c.

If we know the details of coordinates in c, we can get the remaining weights of $M_{3,2}(q)$.

Example 3.1. For $q=4, k=3$, the Matrix

$$G_{3,2}(4) = \left[\begin{array}{c|c|c|c} 1 & 11111 & 22222 & 33333 \\ \hline 0 & 01123 & 01123 & 01123 \\ 0 & 10111 & 10111 & 10111 \end{array}\right]$$

generates the code

$$M_{3,2}(4) = \{(0ccc)+\alpha(1\mathbf{23}) \mid \alpha \in \mathbb{Z}_4\}.$$

By Theorem 3.1, the weight distribution of $S_2(4)$ is

$$A_0 = 1,\ A_4 = 11,\ A_3 = 2,\ A_5 = 2$$

and hence the $n(0)$s are such that 5, 1, 2 and 0 respectively.

Case (i). If $\alpha = 0$, then using Equation (7), we have

$$wt((0ccc)+\alpha(1\mathbf{23})) = 3wt(c),$$

for all $c \in S_2(4)$. Therefore, by Equation (8), there is only one codeword of weight zero, 2 codewords of weight 9, 11 codewords of weight 12 and 2 codewords of weight 15.

Case (ii). If $(\alpha, 4) = 1$, then $\alpha \in \{1,3\}$, by using Equation (9), we have,

$$wt((0ccc)+\alpha(1\mathbf{23})) = 1+(2)(5)+n(0) = 11+n(0).$$

By Equation (10), there are 2 codewords of weight 16, 4 codewords of weight 13, 22 codewords of weight 12 and 4 codewords of weight 11.

Case (iii). If α is not relatively prime to 4, then $\alpha \in \{2\}$ and by Equation (11), we get,

$$\begin{aligned} wt((0ccc)+2(1\mathbf{123})) &= 1+(4-1)(5)-\frac{4}{2}\sum_{i=0}^{1} n(i2)+n(0) \\ &= 1+15-2[n(0)+n(2)]+n(0) \\ wt((0ccc)+2(1\mathbf{123})) &= 16-n(0)-2n(2). \end{aligned}$$

Using the coordinates of $c \in S_2(q)$, there is only one codeword of weight 11, only one codeword of weight 7, 2 codewords of weight 8, 8 codewords of weight 13, 2 codewords of weight 14 and 2 codewords of weight 15.

By combining cases (i), (ii) and (iii), we have the following.

Theorem 3.2. *The weight distribution of $\mathbb{Z}_4$-MacDonald code $M_{3,2}(4)$ is*

$$A_0 = 1,\ A_7 = 1,\ A_8 = 2,\ A_9 = 2,\ A_{11} = 5,\ A_{12} = 33,\ A_{13} = 12,\ A_{14} = 2,\ A_{15} = 4,\ A_{16} = 2.$$

4. Weight distribution of $\mathbb{Z}_q$-simplex code of dimension 3, for any $q \geq 2$

Let

$$G_3(q) = \left[\begin{array}{c|c|c|c|c|c} 0\,0\,0\cdots 0 & 1 & 1\,1\,1\cdots 1 & 2\,2\,2\cdots 2 & \cdots & q-1\ q-1\ q-1\ \cdots\ q-1 \\ \hline 0\,1\,1\cdots q-1 & 0 & 0\,1\,1\cdots q-1 & 0\,1\,1\cdots q-1 & \cdots & 0\,1\,1\cdots q-1 \\ 1\,0\,1\cdots 1 & 0 & 1\,0\,1\cdots 1 & 1\,0\,1\cdots 1 & \cdots & 1\,0\,1\cdots 1 \end{array}\right].$$

Then this matrix generates the code $S_3(q) = \{(c0c\cdots c)+\alpha(\mathbf{0112}\cdots\mathbf{q-1}) \mid \alpha \in \mathbb{Z}_q\}$.

In [11], we have given the parameters of $S_k(q)$, and the weight distribution of $S_2(q)$ is given by Theorem 3.1.

Case (i). Let $\alpha = 0$. Then,

$$\begin{aligned} wt((c0cc\cdots c)+\alpha(\mathbf{0112}\cdots\mathbf{q-1})) &= wt(c0cc\cdots c) \\ &= (q)wt(c), \text{ for all } c \in S_2(q). \end{aligned} \tag{12}$$

In this way, we get the following weights.

(1) Number of zero weight codeword is 1.

(2) Number of $q\big(q-\frac{q}{d}+1\big)$ weight codeword is $d\phi(d)$, where $d|q$, $d\neq 1$ and $d\neq q$.

(3) Number of $qq=q^2$ weight codeword is $q\phi(q)+q-1$.

(4) Number of $q(q+1)$ weight codeword is $q(q-1)-\sum\limits_{d|q,d\neq 1} d\phi(d)$.

Case (ii). Let $(\alpha,q)=1$. Since $\{\alpha.1,\ \alpha.2,\ \cdots,\ \alpha.(q-1)\}=\{1,\ 2,\ \cdots,\ q-1\}$,

$$\begin{aligned} wt\big((c0cc\cdots c)+\alpha(\mathbf{0}1\mathbf{12}\cdots\mathbf{q-1})\big) &= wt\big((c0cc\cdots c)+(\mathbf{0}\alpha\mathbf{12}\cdots\mathbf{q-1}\big) \\ &= 1+\sum_{i=0}^{q-1} wt\big(c+\mathbf{i}\big) \\ &= 1+\sum_{i=0}^{q-1} wt\big(-c+\mathbf{i}\big) \\ &= 1+\sum_{i=0}^{q-1}\big[n-n(i)\big] \\ &= 1+qn-n \end{aligned}$$

$$wt\big((c0cc\cdots c)+\alpha(\mathbf{0}1\mathbf{12}\cdots\mathbf{q-1})\big) = 1+(q-1)n \text{ for all } c\in S_2(q). \tag{13}$$

From the above, the number of $1+(q-1)n$ weight codeword is $\#\big(S_2(q)\big)=q^2$ for all $c\in S_2(q)$.

Since there are $\phi(q)$ α's such that $(\alpha,q)=1$, it implies that the number of $1+(q-1)n$ weight codeword is $\phi(q).q^2$ and hence

$$A_{1+(q-1)n}=\phi(q).q^2. \tag{14}$$

Case (iii). If α is not relatively prime to q, then

$$\begin{aligned} wt\big((c0cc\cdots c)+\alpha(\mathbf{0}1\mathbf{12}\cdots\mathbf{q-1})\big) &= 1+wt(c+\alpha\mathbf{0})+wt(c+\alpha\mathbf{1})+\cdots+wt(c+\alpha(\mathbf{q-1})) \\ &= 1+wt(0\alpha-c)+wt(1\alpha-c)+\cdots+wt((d-1)\alpha-c)+\cdots \\ &= 1+\frac{q}{d}\sum_{i=0}^{d-1} wt\big(\mathbf{i}\alpha-c\big) \\ &= 1+\frac{q}{d}\sum_{i=0}^{d-1}\big[n-n(i\alpha)\big] \\ &= 1+\frac{q}{d}\big[dn\big]-\frac{q}{d}\sum_{i=0}^{d-1} n(i\alpha) \\ &= 1+qn-\frac{q}{d}\sum_{i=0}^{d-1} n(i\alpha). \end{aligned}$$

Therefore,

$$wt\big((c0cc\cdots c)+\alpha(\mathbf{0112}\cdots\mathbf{q-1})\big)=1+qn-\frac{q}{d}\sum_{i=0}^{d-1}n(i\alpha), \tag{15}$$

where $n(i\alpha)$ is the number of $i\alpha$'s in c. If we know the details of coordinates in c, we can get the remaining weights of $S_3(q)$.

Example 4.1. For $q=4, k=3$, the Matrix

$$G_3(4)=\left[\begin{array}{c|c|c|c|c} 00000 & 1 & 11111 & 22222 & 33333 \\ \hline 01123 & 0 & 01123 & 01123 & 01123 \\ 10111 & 0 & 10111 & 10111 & 10111 \end{array}\right]$$

generates the code

$$S_3(4)=\{(c0ccc)+\alpha(\mathbf{01123}) \mid \alpha\in\mathbb{Z}_4,\ c\in S_2(4)\} \text{ where } \mathbf{i}=ii\cdots i\in\mathbb{Z}_q^n$$

Case (i). Let $\alpha=0$. Then, using Equation (12), we have

$$wt\big((c0ccc)+\alpha(\mathbf{01123})\big)=wt(c0ccc)=4wt(c),$$

for all $c\in S_2(4)$. Therefore, by using the weight distribution of $S_2(4)$, there is only one codeword of weight zero, 11 codewords of weight 16, 2 codewords of weight 12 and 2 codewords of weight 20.

Case (ii). If $(\alpha,4)=1$, then $\alpha\in\{1,3\}$ and by Equation (13), we get

$$wt\big((c0ccc)+\alpha(\mathbf{01123})\big)=1+3n.$$

Then, by Equation (14), the number of $1+3n=16$ weight codeword is $\phi(4).4^2=32$. That is, there are 32 codewords of weight 16.

Case (iii). If α is not relatively prime to 4, then $\alpha\in\{2\}$ and by Equation (15), we get

$$\begin{aligned} wt\big((c0ccc)+2(\mathbf{01123})\big) &= 1+(4)(5)-\frac{4}{2}\sum_{i=0}^{1}n(i2) \\ &= 21-2[n(0)+n(2)]. \end{aligned}$$

Using the coordinates of $c\in S_2(4)$, we get, there are 4 codewords of weight 11, 8 codewords of weight 17 and 4 codewords of weight 19. Therefore, by combining cases (i), (ii) and (iii), we have the following.

Theorem 4.1. *The weight distribution of $\mathbb{Z}_4$-Simplex code of dimension 3 is*

$$A_0 = 1,\ A_{11} = 4,\ A_{12} = 2,\ A_{16} = 43,\ A_{17} = 8,\ A_{19} = 4,\ A_{20} = 2.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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