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QUALITATIVE ANALYSIS IN TWO PREY-PREDATOR SYSTEM WITH PERSISTENCE

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Abstract. In this work, the system with two preys and one predator population is qualitatively analyzed. The predator exhibits a Holling type I response to one prey and a Holling type IV response to the other prey. The boundedness of the system is analyzed. We examine the occurrence of positive equilibrium points and stability of the system at those points. At trivial equilibrium $\left(E_0\right)$ and axial equilibrium $\left(E_1\right)$, the system is found to be unstable .Also; we obtain the necessary and sufficient conditions for existence of interior equilibrium point $\left(E^*\right)$ and local and global stability of the system at the interior equilibrium $\left(E^*\right)$. Depending upon the existence of limit cycle, the persistence condition is established for the system. The analytical findings are illustrated through computer simulations from which we observed that, using the parameter α_1 and c it is possible to break unstable behavior of system and drive it to a stable state.

Keywords: prey –predator system; functional response; local and global stability; persistence.
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1. Introduction

Mathematical modeling for interaction between species using differential equation is one of the most classical applications to biology. Analytical techniques with computer power paved a way for better understanding and development of these models. Prey-predator models are relatively

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well-studied example of interactions. The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in mathematical ecology due to its universal existence and importance. The most noteworthy component in prey-predator models is the “predator’s functional response on prey population”, it describes the amount of prey consumed by an average predator. The stability of prey-predator systems with such functional response has been the area of concentration for many theorists and experimentalists.

Two species models with functional responses are extensively studied in ecological literature [2, 9, 10, 14] Interactions on two species continuous time systems with a predator and a prey limited only to equilibrium point or to a limit cycle. Several ecological circumstances have been analyzed by interaction between two or more species. The system representing the interaction between three species shows complex dynamical behavior[3,4,6,7,8,12].The interaction of species involving persistence and extinction have been the area of interest for researchers [1,5,11,13]

This paper is organized as follows. We start in section 2 by defining the mathematical model of three species population which consists of two preys and one predator. The nonlinear system of differential equations governed this system is introduced. Section 3 deals with the determination of equilibrium points and their existence conditions. In section 4, we analyzed dynamical behavior of these equilibrium points. Global stability and persistence of the system is studied in section 5. In section 6 to deals with Numerical simulation and discuss the problem.

2. Mathematical model

Mathematical model considered is based on the predator-prey system with Holling type I and Holling type IV functional response .The predator exhibits a Holling type I response to one prey and a Holling type IV response to the other prey.

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - \alpha_1 xy - \lambda_1 xz$$
$$\frac{dy}{dt} = sy\left(1-\frac{y}{L}\right) - \alpha_2 xy - \frac{\lambda_2 yz}{m+y^2}$$
$$\frac{dz}{dt} = b_1\lambda_1 xz + b_2\frac{\lambda_2 yz}{m+y^2} - cz \tag{1}$$

Where x, y denote population densities of prey and z denote population density of the predator. In model(1) r and s are the intrinsic growth rate of two prey species, K and L are their carrying capacities, c is mortality rate of the predator, α_1 and α_2 are the interspecies interference co-efficient of two prey species λ_1 and λ_2 denote prey species searching efficiency of the predator, m is the half-saturation co-efficient, b_1 and b_2 are the conversion factors denoting the number of newly born predators for each captured of first and second prey respectively.

Theorem 1: The solutions $x(t)$, $y(t)$ and $z(t)$ of system (1) initiating in R_+^3 are positive and bounded for all $t \geq 0$.

Proof:

Since the densities of population can never be negative, obviously the solutions $x(t)$, $y(t)$ and $z(t)$ are positive for all $t \geq 0$.

From the first equation of model (1), we have

$$\frac{dx}{dt} \leq r(1-\frac{x}{K})$$

This gives $x(t) = \dfrac{1}{e^{-rt} + \dfrac{1}{K}}$

As $t \to \infty$ we get $x(t) \leq K$

Similarly, from equation (2) of model (1)

$$y(t) \leq L$$

Consider

$$w(t) = \xi_1 x(t) + \xi_2 y(t) + z(t)$$

For real positive number η,

$$\frac{dw}{dt} + \eta w(t) = \xi_1 \frac{dx}{dt} + \xi_2 \frac{dy}{dt} + \frac{dz}{dt} + \eta\left(\xi_1 x\left(t\right) + \xi_2 y\left(t\right) + z\left(t\right)\right) \tag{*}$$

Substituting equation (1) in (*) and simplifying, we get

$$\frac{dw}{dt} + \eta w(t) = \xi_1 x\left(r+\eta\right) + \xi_2 y\left(s+\eta\right) - \xi_1 \alpha_1 xy - \xi_2 \alpha_2 xy - \frac{\xi r x^2}{k} - \frac{\xi s y^2}{l} + \left(\eta - e\right) z$$

If we choose $\eta \leq e$, the $\dfrac{dw}{dt} + \eta w(t) \leq \dfrac{\xi_1\left(r+\eta\right)}{K} + \dfrac{\xi_2\left(s+\eta\right)}{L}$

$$\leq \delta$$

Applying a Lemma on differential inequality we get,

$$0 \le w(x,y,z) \le \frac{\delta}{\eta}\left(1-e^{-\eta t}\right)+\frac{w\left(x(0),y(0),z(0)\right)}{e^{\eta t}}$$

And for $t \to \infty$

$$0 \le w \le \frac{\delta}{\eta}$$

Thus all solutions of system (1) enter into the region

$$B=\left\{(x,y,z): 0 \le x \le K, 0 \le y \le L, 0 \le w \le \frac{\delta}{\eta}+\varepsilon \, for any \varepsilon > 0\right\}$$

3. Equilibrium Analysis

It can be checked that the system (1) has seven non-negative equilibrium and three of them namely $E_0(0,0,0), E_1(K,0,0)$ and $E_2(0,L,0)$ always exists. We show that the existence of other equilibrium as follows

Existence of $E_3(\tilde{x},\tilde{y},0)$

Here $\tilde{x},\tilde{y}$ are the positive solutions of the following algebraic equations

$$r(1-\frac{x}{K})-\alpha_1 y=0 \tag{2}$$

$$s(1-\frac{y}{L})-\alpha_2 x=0 \tag{3}$$

Solving (2) and (3) we get

$$\tilde{x}=\frac{sK(r-\alpha_1 L)}{rs-\alpha_1\alpha_2 KL} \tag{4}$$

$$\tilde{y}=\frac{rL(s-\alpha_2 K)}{rs-\alpha_1\alpha_2 KL} \tag{5}$$

Thus the equilibrium $E_3(\tilde{x},\tilde{y},0)$ exists if $r-\alpha_1 L$ and $s-\alpha_2 K$ are of same sign.

That is either $r > \alpha_1 L$ and $s > \alpha_2 K$ (6)

$r < \alpha_1 L$ and $s < \alpha_2 K$ (7)

Existence of $E_4(\bar{x},0,\bar{z})$

Here $\bar{x},\bar{z}$ are the positive solutions of the following algebraic equations

$$r(1-\frac{x}{K})-\lambda_1 z=0 \tag{8}$$

$$b_1\lambda_1 x-c=0 \tag{9}$$

Solving (8) and (9) we get

$$\bar{x}=\frac{c}{b_1\lambda_1} \tag{10}$$

$$\bar{z}=\frac{r}{\lambda_1}(1-\frac{c}{Kb_1\lambda_1}) \tag{11}$$

It can be seen that $E_4(\bar{x},0,\bar{z})$ exists if $Kb_1\lambda_1>c$ (12)

Existence of $E_5(0,\hat{y},\hat{z})$

Here $\hat{y},\hat{z}$ are the positive solution of the following algebraic equations

$$s(1-\frac{y}{L})-\frac{\lambda_2 z}{m+y^2}=0 \tag{13}$$

$$\frac{b_2\lambda_2 y}{m+y^2}-c=0 \tag{14}$$

Solving (13) and (14) we get

$$\hat{y}=\frac{b_2\lambda_2\pm\sqrt{(b_2\lambda_2)^2-4c^2m}}{2c} \tag{15}$$

$$\hat{z}=\frac{s}{\lambda_2}(m+\hat{y}^2)(1-\frac{\hat{y}}{L}) \tag{16}$$

It can been seen that the equilibrium $E_5(0,\hat{y},\hat{z})$ exists if $b_2\lambda_2>c^2m$ (17)

Existence of $E_6(x^*,y^*,z^*)$

Here (x^*,y^*,z^*) is the positive solution of the system of algebraic equation given below:

$$r(1-\frac{x}{K})-\alpha_1 y-\lambda_1 z=0 \tag{18}$$

$$s(1-\frac{y}{L})-\alpha_2 y-\frac{\lambda_2.z}{m+y^2}=0 \tag{19}$$

$$b_1\lambda_1 x+\frac{b_2\lambda_2 y}{m+y^2}-c \tag{20}$$

Eliminating x from (18) and (19) ,we get

$$f(x,y)=0 \tag{21}$$

Where

$$f(y,z)=rS(L-y)(m+y^2)-rKL(m+y^2)-rLz\lambda_2+\alpha_1\alpha_2KLy(m+y^2)+\lambda_1\alpha_2KLz(m+y^2) \tag{22}$$

Also eliminating x from (18) and (20) ,we get

$$g(y,z)=0 \tag{23}$$

Where

$$g(y,z)=r(m+y^2)[c-Kb_1\lambda_1]+Kb_1\lambda_1\alpha_1y(m+y^2)+\lambda_1{}^2b_1Kz(m+y^2)-rb_2\lambda_2y \tag{24}$$

From (22) as $z\to 0$, $y\to y_a$ is given by

$$y_a=\frac{rL(s-\alpha_2K)}{rs-\alpha_1\alpha_2KL}$$

We note that $y_a>0$ if the inequality $r>\alpha_1L$ and $s>\alpha_2K$ holds.

Also from the equation (21) and (22), we have

$$\frac{dy}{dz}=\frac{P_1}{Q_1}$$

Where

$$P_1=-rL\lambda_2+\lambda_1\alpha_2KL(m+y^2)$$

$$Q_1=rKL(\alpha-\frac{s}{K})+KL(m+y^2)[\frac{rs}{KL}-\alpha_1\alpha_2]+$$

It is clear that $\frac{dy}{dz}>0$,if

$P_1>0$ and $Q_1>0$ (or) $P_1<0$ and $Q_1<0$

The value of x^* calculated from (20)

$$x^*=\frac{c(m+y^{*2})-b_2\lambda_2y^*}{(m+y^{*2})b_1\lambda_1} \tag{25}$$

We can see that exists $E_6(x^*,y^*,z^*)$ if x^* to be positive, if $c(m+y^{*2})>b_2\lambda_2y^*$.

4. Dynamical behavior and Stability analysis

In order to check the stability of the model (1), the variational matrix corresponding to each equilibrium point is calculated.

$$E(x,y,z)=\begin{pmatrix} r-\frac{2rx}{K}-\alpha_1 y-\lambda_1 z & -\alpha_1 x & -\lambda_1 x \\ -\alpha_2 y & s-\frac{2sy}{L}-\alpha_2 x-\frac{\lambda_2 z(m-y^2)}{(m+y^2)^2} & \frac{-\lambda_2 y}{(m+y^2)} \\ b_1\lambda_1 z & \frac{b_2\lambda_2 z(m-y^2)}{(m+y^2)^2} & -c+b_1\lambda_1 x+\frac{b_2\lambda_2 y}{(m+y^2)} \end{pmatrix}$$

i) The variational matrix of equilibrium points at $E_0(0,0,0)$ is

$$E_0=\begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -c \end{pmatrix}$$

The eigen values of E_0 are r, s and $-c$ thus E_0 is always saddle point so that stable in z-direction and unstable manifold in the x-y plane .

ii) The variational matrix of equilibrium points at $E_1(K,0,0)$ is

$$E_1=\begin{pmatrix} -r & -\alpha_1 K & -\lambda_1 K \\ 0 & s-\alpha_2 K & 0 \\ 0 & 0 & b_1\lambda_1 K-c \end{pmatrix}$$

Thus E_1 is saddle point with locally stable manifold in x-direction and with locally unstable manifold in $y-z$ plane,if $s-\alpha_2 K>0$ and $b_1\lambda_1 K-c>0$ hold. But if $s-\alpha_2 K<0$ and $b_1\lambda_1 K-c<0$ then E_1 is locally asympotatically stable in $x-y-z$ plane.

iii) The variational matrix of equilibrium points at $E_2(0,L,0)$ is

$$E_2=\begin{pmatrix} r-\alpha_1 L & 0 & 0 \\ -\alpha_2 L & -s & \frac{-\lambda_2 L}{m+L^2} \\ 0 & 0 & \frac{b_2\lambda_2 L}{m+L^2}-c \end{pmatrix}$$

Thus E_2 is saddle point with locally stable manifold in y-direction and with locally unstable manifold in $x-z$ plane, if $r-\alpha_1 L>0$ and $\frac{b_2\lambda_2 L}{m+L^2}-c>0$ hold. But if $r-\alpha_1 L<0$ and $\frac{b_2\lambda_2 L}{m+L^2}-c<0$ then E_2 is locally asymptomatically stable in $x-y-z$ plane.

iv) The variational matrix of equilibrium at $E_3(\tilde{x},\tilde{y},0)$

$$E_3=\begin{pmatrix} A_1^* & -\alpha_1\tilde{x} & -\lambda_1\tilde{x} \\ -\alpha_2\tilde{y} & B_1^* & \dfrac{-\lambda\tilde{y}}{m+\tilde{y}^2} \\ 0 & 0 & C_1^* \end{pmatrix}$$

Where $A_1^*=r-\dfrac{2r\tilde{x}}{K}-\alpha_1\tilde{y}, B_1^*=s-\dfrac{2s\tilde{y}}{L}-\alpha_2\tilde{x}, C_1^*=-c+b_1\lambda_1\tilde{x}+\dfrac{b_2\lambda_2\tilde{y}}{(m+\tilde{y}^2)}$

Here $\tilde{x}=\dfrac{sK(r-\alpha_1L)}{rs-\alpha_1\alpha_2KL}$, $\tilde{y}=\dfrac{rL(s-\alpha_2K)}{rs-\alpha_1\alpha_2KL}$

Then we get

$$E_3=\begin{pmatrix} r(1-\frac{2s(r-\alpha_1L)}{rs-\alpha_1\alpha_2KL})-\alpha_1\frac{rL(s-\alpha_2K)}{rs-\alpha_1\alpha_2KL} & -\alpha_1\frac{sK(r-\alpha_1L)}{rs-\alpha_1\alpha_2KL} & -\lambda_1\frac{sK(r-\alpha_1L)}{rs-\alpha_1\alpha_2KL} \\ -\alpha_2\frac{rL(s-\alpha_2K)}{rs-\alpha_1\alpha_2KL} & s(1-\frac{2r(s-\alpha_2K)}{rs-\alpha_1\alpha_2KL})-\alpha_2\frac{sK(r-\alpha_1L)}{rs-\alpha_1\alpha_2KL} & \frac{-\lambda_2(rL(s-\alpha_2K)(rs-\alpha_1\alpha_2KL)}{m(rs-\alpha_1\alpha_2KL)^2+(rL(s-\alpha_2K))^2} \\ 0 & 0 & -c+b_1\lambda_1\frac{sK(r-\alpha_1L)}{rs-\alpha_1\alpha_2KL}+\frac{b_2\lambda_2(rL(s-\alpha_2K)(rs-\alpha_1\alpha_2KL)}{m(rs-\alpha_1\alpha_2KL)^2+(rL(s-\alpha_2K))^2} \end{pmatrix}$$

Here sum of two eigen value is $\dfrac{sK(r-\alpha_1L)}{rs-\alpha_1\alpha_2KL}+\dfrac{rL(s-\alpha_2K)}{rs-\alpha_1\alpha_2KL}$

Product of the eigen value is $\dfrac{sK(r-\alpha_1L)}{rs-\alpha_1\alpha_2KL}.\dfrac{rL(s-\alpha_2K)}{rs-\alpha_1\alpha_2KL}$

If $r>\alpha_1L$ and $s>\alpha_2K$ holds,then the sum of two eigen value is negative and product is positive.in this case we say that $E(\tilde{x},\tilde{y},0)$ exists and is asymptotically stable in plane, but if $r<\alpha_1L$ and $s<\alpha_2K$ then the product of two eigenvalues in negative.then exists and in that case it will be unstable in $x-y$ plane. More over it will be stable in $x-y-z$ plane if other eigen value of the system is $b_1\lambda_1\dfrac{sK(r-\alpha_1L)}{rs-\alpha_1\alpha_2KL}+\dfrac{b_2\lambda_2(rL(s-\alpha_2K)(rs-\alpha_1\alpha_2KL)}{m(rs-\alpha_1\alpha_2KL)^2+(rL(s-\alpha_2K))^2}<c$

v) The variational matrix of equilibrium points at $E_4(\bar{x},0,\bar{z})$ is

$$E_4=\begin{pmatrix} A_2^* & \dfrac{-\alpha_1c}{b_1\lambda_1} & \dfrac{-c}{b_1} \\ 0 & B_2^* & 0 \\ rb_1(1-\dfrac{c}{Kb_1\lambda_1}) & \dfrac{b_2\lambda_2}{m\lambda_1}(1-\dfrac{c}{Kb_1\lambda_1}) & 0 \end{pmatrix}$$

Where

$$A_2^* = r - \frac{2r\overline{x}}{K} - \lambda_1 \overline{z}\ , B_2^* = s - \alpha_2 \overline{x} - \frac{\lambda_2 \overline{z}}{m}$$

Here $\overline{x} = \dfrac{c}{b_1\lambda_1}$, $\overline{z} = \dfrac{r}{\lambda_1}(1 - \dfrac{c}{Kb_1\lambda_1})$

Then

$$E_4 = \begin{pmatrix} \dfrac{-rc}{Kb_1\lambda_1} & \dfrac{-\alpha_1 c}{b_1\lambda_1} & \dfrac{-c}{b_1} \\ 0 & s - \dfrac{\alpha_2 c}{b_1\lambda_1} - \dfrac{r\lambda_2}{m\lambda_1}(1 - \dfrac{c}{Kb_1\lambda_1}) & 0 \\ rb_1(1 - \dfrac{c}{Kb_1\lambda_1}) & \dfrac{b_2\lambda_2}{m\lambda_1}(1 - \dfrac{c}{Kb_1\lambda_1}) & 0 \end{pmatrix}$$

Thus $E_4(\overline{x}, 0, \overline{z})$ exists and is asymptotically stable in $x - y - z$ plane if the inequality $Kb_1\lambda_1 > c$

and $\dfrac{\alpha_2 c}{b_1\lambda_1} - \dfrac{r\lambda_2}{m\lambda_1}(1 - \dfrac{c}{Kb_1\lambda_1}) > s$ holds.

vi) The variational matrix of equilibrium point at $E_5(0, \hat{y}, \hat{z})$

$$E_5 = \begin{pmatrix} r - \alpha_1\hat{y} - \lambda_1\hat{z} & 0 & 0 \\ -\alpha_2\hat{y} & s - \dfrac{2s\hat{y}}{L} - \dfrac{\lambda_2 z(m - \hat{y})}{(m + \hat{y})^2} & \dfrac{-\lambda_2 y}{(m + y^2)} \\ b_1\lambda_1\hat{z} & \dfrac{b_2\lambda_2\hat{z}(m - \hat{y})}{(m + \hat{y})^2} & -c + \dfrac{b_2\lambda_2 y}{(m + y^2)} \end{pmatrix}$$

Where

$$\hat{y} = \frac{b_2\lambda_2 \pm \sqrt{(b_2\lambda_2)^2 - 4c^2 m}}{2c} \quad , \quad \hat{z} = \frac{s}{\lambda_2}(m + \hat{y}^2)(1 - \frac{\hat{y}}{L})$$

Thus $E_5(0, \hat{y}, \hat{z})$ exists and is asymptotically stable in $x - y - z$ plane if $r - \alpha_1\hat{y} - \lambda_1\hat{z}$ also

if $b_2\lambda_2 > c^2 m$

vii) The variational matrix at the equilibrium points E_6

$$E_6 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Where

$$a_{11} = r - \frac{2rx^*}{K} - \alpha_1 y^* - \lambda_1 z^*, a_{12} = -\alpha_1 x^*, a_{13} = -\lambda_1 x^*, a_{21} = -\alpha_1 y^*, a_{22} = s - \frac{2sy^*}{L} - \alpha_2 x^* - \frac{\lambda_2 z^*(m - y^{*2})}{(m + y^{*2})^2}$$

$$a_{23} = \frac{-\lambda_2 y^*}{(m + y^{*2})}, a_{31} = b_1 \lambda_1 z^*, a_{32} = \frac{b_2 \lambda_2 z^*(m - y^{*2})}{(m + y^{*2})^2}, a_{33} = -c + b_1 \lambda_1 x^* + \frac{b_2 \lambda_2 y^*}{(m + y^{*2})}$$

Then corresponding characteristic equation becomes

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0$$

Where

$$A_1 = -(a_{11} + a_{22} + a_{33})$$

$$= [c - r - s + \frac{2rx^*}{K} + \alpha_1 y^* + \lambda_1 z^* + \frac{2sy^*}{L} + \alpha_2 x^* + \frac{\lambda_2 z^*(m - y^{*2})}{(m + y^{*2})^2} - b_1 \lambda_1 x^* - \frac{b_2 \lambda_2 y^*}{(m + y^{*2})}]$$

$$A_2 = a_{22}a_{33} - a_{23}a_{32} + a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31}$$

$$= [(r - \frac{2rx^*}{K} - \alpha_1 y^* - \lambda_1 z^*).(s - \frac{2sy^*}{L} - \alpha_2 x^* - \frac{\lambda_2 z^*(m - y^{*2})}{(m + y^{*2})^2}) - (\alpha_1^{\ 2} x^* y^*)] +$$

$$[(s - \frac{2sy^*}{L} - \alpha_2 x^* - \frac{\lambda_2 z^*(m - y^{*2})}{(m + y^{*2})^2}).(-c + b_1 \lambda_1 x^* + \frac{b_2 \lambda_2 y^*}{(m + y^{*2})}) + (\frac{b_2 \lambda_2^{\ 2} y^* z^*(m - y^{*2})}{(m + y^{*2})^4})]$$

$$+[(r - \frac{2rx^*}{K} - \alpha_1 y^* - \lambda_1 z^*).(-c + b_1 \lambda_1 x^* + \frac{b_2 \lambda_2 y^*}{(m + y^{*2})}) + (\alpha_1 x^*.b_1 \lambda_1 z^*)]$$

$$A_3 = \det(E^*)$$

$$= a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31} + a_{13}a_{22}a_{31} - a_{13}a_{21}a_{32}$$

$$= (r - \frac{2rx^*}{K} - \alpha_1 y^* - \lambda_1 z^*).[(s - \frac{2sy^*}{L} - \alpha_2 x^* - \frac{\lambda_2 z^*(m - y^{*2})}{(m + y^{*2})^2}).(-c + b_1 \lambda_1 x^* + \frac{b_2 \lambda_2 y^*}{(m + y^{*2})}) + (\frac{b_2 \lambda_2^{\ 2} y^* z^*(m - y^{*2})}{(m + y^{*2})^4})]$$

$$+\alpha_1 x^*[\left(-\alpha_1 y^*\right).(-c + b_1 \lambda_1 x^* + \frac{b_2 \lambda_2 y^*}{(m + y^{*2})}) + \frac{\lambda_2 y^*}{(m + y^{*2})}.b_1 \lambda_1 z^*]$$

$$-\lambda_1 x^*[\left(-\lambda_1 x^*\right).(-c + b_1 \lambda_1 x^* + \frac{b_2 \lambda_2 y^*}{(m + y^{*2})}) + b_1 \lambda_1 z^*.\frac{\lambda_2 y^*}{(m + y^{*2})}]$$

Therefore an application of Routh-Hurwitz criterion shows that

$$a_{11} < 0, a_{22} < 0$$

Then the following conditions are satisfied

$$A_1 > 0, A_3 > 0 \text{ and } A_1 A_2 - A_3 > 0$$

Hence the positive equilibrium point $E_6(x^*, y^*, z^*)$ is asymptotically stable.

5. Global stability and persistence

Theorem 2:

The interior equilibrium E_3 is globally asymptotically stable in the interior of the quadrant of the $x - y$ plane.

Proof:

Let $H_1(x, y) = \dfrac{1}{xy}$

Clearly $H_1(x, y)$ is positive in the interior of the positive quadrant of $x - y$ plane.

$$h_1(x, y) = rx(1 - \frac{x}{K}) - \alpha_1 xy$$

$$h_2(x, y) = sy(1 - \frac{y}{L}) - \alpha_2 xy$$

Then $\Delta(x, y) = \dfrac{\partial}{\partial x}(h_1 H_1) + \dfrac{\partial}{\partial y}(h_2 H_1)$

$$= \frac{-r}{yK} - \frac{s}{xL}$$

$$<0$$

From the above equation we note that $\Delta(x, y)$ does not change sign and is not identically zero in the interior of the positive quadrant of the $x - y$ plane. In the following theorem, we show that E_3 is globally asymptotically stable.

Theorem 3:

The interior equilibrium E_4 is globally asymptotically stable in the interior of the quadrant of the $x - z$ plane.

Proof:

Let $H_2(x,z)=\dfrac{1}{xz}$

Clearly $H_2(x,z)$ is positive in the interior of the positive quadrant of $x-z$ plane.

$$h_1(x,z)=rx(1-\frac{x}{K})-\lambda_1 xz$$

$$h_2(x,z)=b_1\lambda_1 xz-cz$$

Then $\Delta(x,z)=\dfrac{\partial}{\partial x}(h_1H_2)+\dfrac{\partial}{\partial z}(h_2H_2)$

$$=\frac{-r}{zK}$$

$$<0$$

From the above equation we note that $\Delta(x,z)$ does not change sign and is not identically zero in the interior of the positive quadrant of the $x-z$ plane. In the following theorem, we show that E_4 is globally asymptotically stable.

Theorem 4:

The interior equilibrium E_5 is globally asymptotically stable in the interior of the quadrant of the $y-z$ plane.

Proof:

Let $H_3(y,z)=\dfrac{1}{yz}$

Clearly $H_3(y,z)$ is positive in the interior of the positive quadrant of $y-z$ plane.

$$h_1(y,z)=sy(1-\frac{y}{L})-\frac{\lambda_1 yz}{m+y^2}$$

$$h_2(y,z)=z[-c+\frac{b_2\lambda_2 y}{m+y^2}]$$

Then $\Delta(y,z)=\dfrac{\partial}{\partial y}(h_1H_3)+\dfrac{\partial}{\partial z}(h_2H_3)$

$$=-(\frac{s}{zL}-\frac{2\lambda y}{(m+y^2)^2})$$

$$<0$$

From the above equation we note that $\Delta(y,z)$ does not change sign and is not identically zero in the interior of the positive quadrant of the $y-z$ plane. In the following theorem, we show that E_5 is globally asymptotically stable.

Theorem 5: The co-existence equilibrium point $E_6(x^*, y^*, z^*)$ is globally asymptotically stable with respect to all solutions initiating in the interior of B satisfy the following conditions

$$a_{12}{}^2 < 4a_{11}a_{22} \tag{26}$$

Proof: The proof can be reached by using Lypunov stability theorem which gives sufficient condition. Now let us consider a positive definite function $V(x,y,z)$

$$V(x,y,z) = (x-x^*) - x^* \ln(\frac{x}{x^*}) + c_1(y-y^*) - y^* \ln(\frac{y}{y^*}) + c_2(z-z^*) - z^* \ln(\frac{z}{z^*}) \tag{27}$$

in the interior of the positive octant,

Differentiating (27) with respect to time t ,we get

$$\dot{V} = (x-x^*)\frac{\dot{x}}{x} + (y-y^*)\frac{c_1\dot{y}}{y} + (z-z^*)\frac{c_2\dot{z}}{z} \tag{28}$$

Using system of equation (1) in (28) which simplifies

$$\dot{V} = -(x-x^*)^2\frac{r}{K} - (x-x^*)(y-y^*)(\alpha_1 + c_1\alpha_2) - (x-x^*)(z-z^*)(\lambda_1 - c_2b_1\lambda_1) - c_1(y-y^*)^2\frac{s}{L}$$
$$-(y-y^*)(z-z^*)\frac{m\lambda_2(c_1-c_1b_2)}{(m+y^2)(m+y^{*2})} - \frac{\lambda_2 y^* c_1}{(m+y^2)(m+y^{*2})}(y-y^*)(z-z^*) + \frac{\lambda_2 c_1 z^*}{(m+y^2)(m+y^{*2})}(y-y^*)^2$$

The above equation can be written as

$$\dot{V} = -[a_{11}(x-x^*)^2 + a_{12}(x-x^*)(y-y^*) + a_{22}(y-y^*)^2 + a_{13}(x-x^*)(z-z^*) + a_{23}(y-y^*)(z-z^*)]$$

Where

$$a_{11} = \frac{r}{K}, a_{12} = (\alpha_1 + c_1\alpha_2), a_{13} = (\lambda_1 - c_2b_1\lambda_1),$$
$$a_{22} = c_1\frac{s}{L} - \frac{\lambda_2 c_1 z^*}{(m+y^2)(m+y^{*2})}, a_{23} = \frac{m\lambda_2(c_1-c_1b_2) - \lambda_2 y^* c_1}{(m+y^2)(m+y^{*2})}$$

Let us choose $c_1 = \frac{mb_2}{b_1(m+y^{2*})}, c_2 = \frac{1}{b_1}$,then the sufficient condition that $\dot{V}$ to be negative definite is $a_{11} > 0, a_{12}{}^2 < 4a_{11}a_{22}$ .Hence V is a Lipunov function with respect to $E_6(x^*, y*, z*)$

Numerical discussion

Analytical studies become complete only with the numerical justification of the results. A qualitative analysis of the main features in the system is described by numerical simulations. Therefore, we assign some hypothetical data in order to verify the analytical result that has been obtained. The numerical experiments are conducted to examine the dynamical behavior of the system in three different parameter sets. It is obvious that changing the parameter value changes the numerical outcomes. So every different set of parameter gives unique results.

Let R_1 be the parameter set taken as

$$r = 3.5, s = 5.5, K = 150, L = 130, \alpha_1 = 0.001, \alpha_2 = 0.1, b_1 = 0.5, b_2 = 0.6, c = 3.9, \lambda_1 = 0.24, \lambda_2 = 0.21, m = 15$$

With the above parameter set, the system (1) has positive equilibrium(*******) which is globally asymptotically stable. (See Fig.1, 2). By using Liu's criteria, it is noteworthy that, when the inter-species interference co-efficient α_1 increases ,the positive equilibrium losses its stability and a Hopf bifurcation occurs.

Let R_2 be the parameter set taken as

$$r = 3.5, s = 5.5, K = 150, L = 130, \alpha_2 = 0.271, b_1 = 0.5, b_2 = 0.6, c = 1.9, \lambda_1 = 0.24, \lambda_2 = 0.21, m = 15$$

With the above values of parameter,if we gradually increase the value of α_1 and keep other parameters fixed,we observe that the system(1) losses stability (see Fig.3,5) and super critical Hopf bifurcation occurs(see Fig.2). Also the phase portrait of the system is plotted.(see Fig.4,6)

Let R_3 be the parameter set taken as

$$r = 3.5, s = 5.5, K = 150, L = 130, \alpha_1 = 0.015, \alpha_2 = 0.271, b_1 = 0.5, b_2 = 0.6, \lambda_1 = 0.24, \lambda_2 = 0.21, m = 10$$

With the above values of parameter, if we gradually increase the value of c and keep other parameters fixed,we observe that the system(1) losses stability (see Fig.7,8) and super critical Hopf bifurcation occurs(see Fig.9).The numerical study presented here shows that using α_1 and c it is possible to break unstable behavior of system (1) and drive it to a stable state. Also it is possible to keep the population level at the required state using the above control parameter.

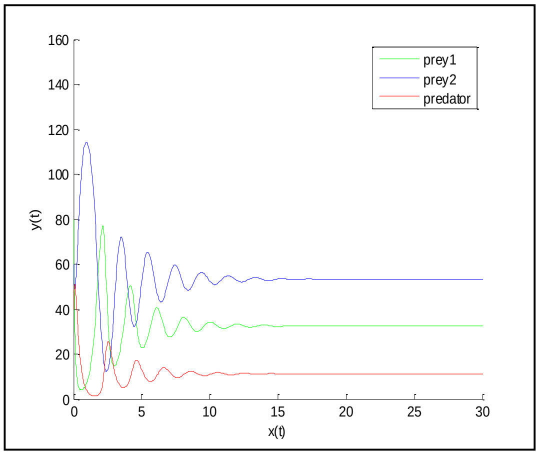

Fig 1.Numerical solution for the system(1) with parameter set R_1

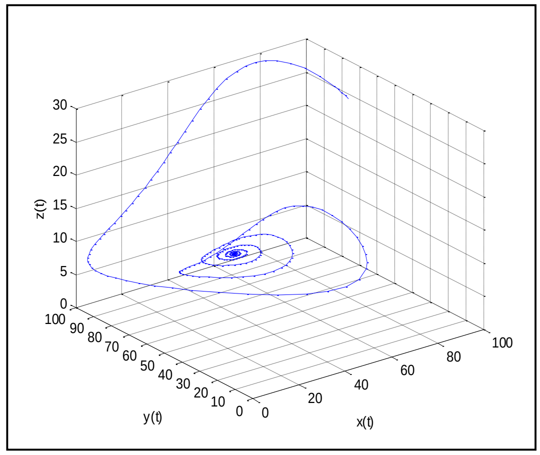

Fig 2. Phase portrait of system(1) with parameter set R_1

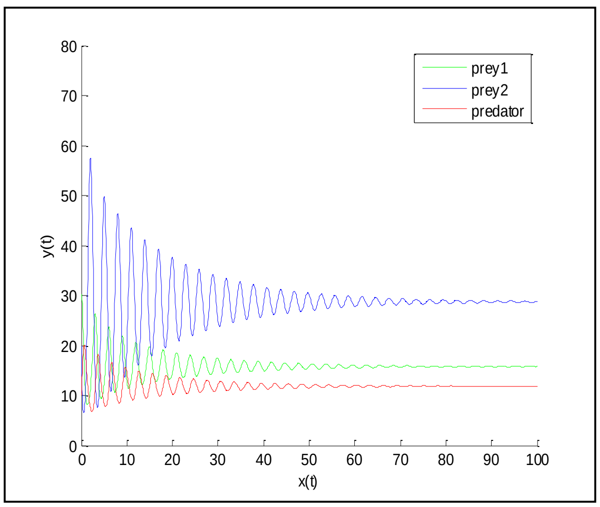

Fig 3. Numerical solution of the system(1) for $\alpha_1 = 0.01$ with parameter set R_2

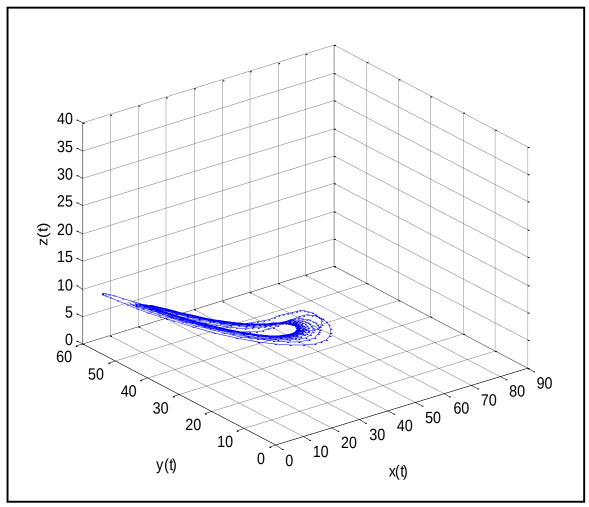

Fig 4. Phase portrait of system(1) for $\alpha_1 = 0.01$ with parameter set R_2

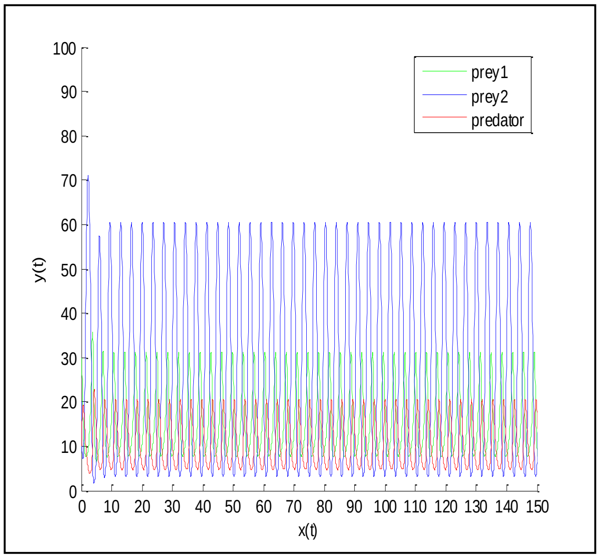

Fig 5. Numerical solution of the system(1) for $\alpha_1 = 0.021$ with parameter set R_2

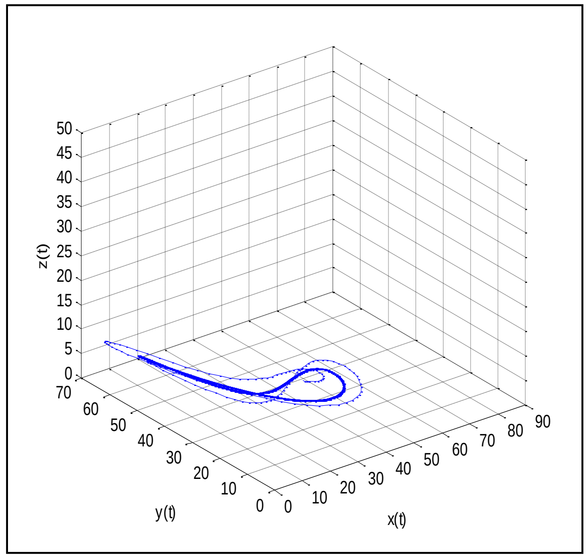

Fig 6. Phase portrait of system(1) for $\alpha_1 = 0.021$ with parameter set R_2

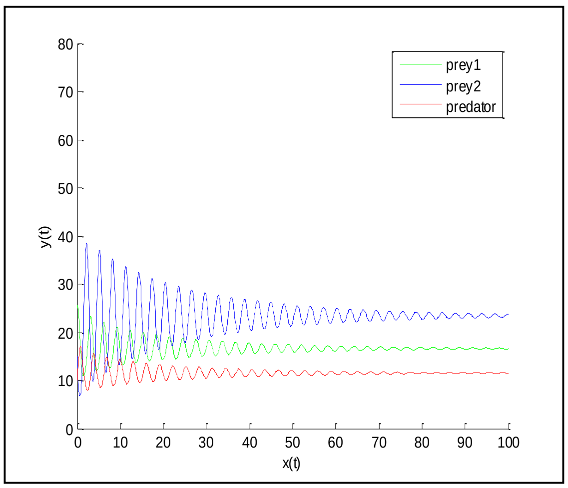

Fig 7 . Numerical solution of the system(1) for $c = 2$ with parameter set R_3

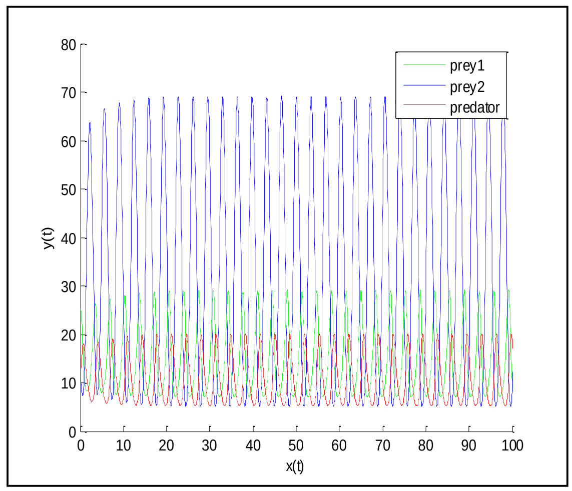

Fig 8. Numerical solution of system(1) for c=1.8 with parameter set R_3

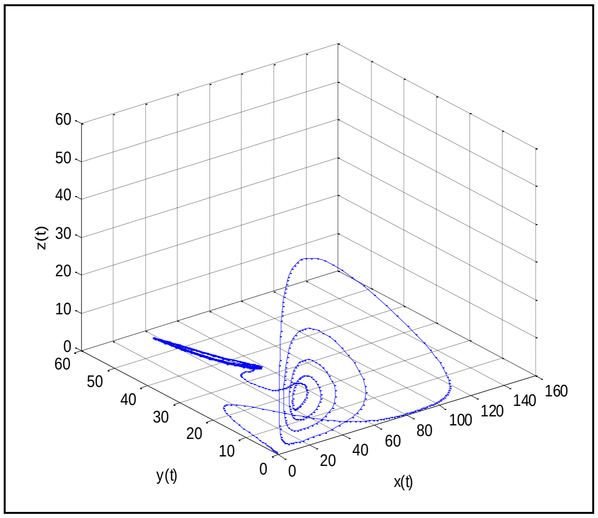

Fig 9.phase portrait of system(1) for $c = 1.9$

with parameter set R_3

7. Conclusion

In this paper, we studied the quality analysis of a two prey and one predator system. The local and global stability at various equilibrium points are analyzed and discussed. The system is driven from its unstable behavior to stable state by the control parameters α_1 and c using which the population level is maintained at the required state. The persistence of the system is evaluated from the occurrence of limit cycle.

Conflict of Interests

The authors declare that there is no conflict of interests.

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