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INTERVAL-VALUED INTUITIONISTIC $(\tilde{T},\tilde{S})$- FUZZY MEDIAL IDEALS OF BCI-ALGEBRAS

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Abstract. We consider the interval-valued intuitionistic $(\tilde{T},\tilde{S})$-fuzzification of the concept medial ideals. The image (preimage) of interval-valued intuitionistic $(\tilde{T},\tilde{S})$-fuzzy medial ideals under homomorphism of BCI-algebras are defined and investigated some of there properties. The notion of product of interval-valued intuitionistic $(\tilde{T},\tilde{S})$-fuzzy medial ideals in BCI-algebras is introduced, and investigated some related properties.

Keywords: medial BCI-algebras; fuzzy medial ideals.

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1. Introduction

The concept of fuzzy subset and various operations on it were first introduced by Zadeh in [13]. Since then, several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of “intuitionistic fuzzy set” was first published by Atanassov [1, 2] as a generalization of the notion of fuzzy set. Iseki [6, 7] introduced the notion of BCK-algebras. Iseki [5] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK / BCI-algebras and their relationship with other structures including lattices and Boolean algebras. There is a great deal of literature which has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. In [9] J.Meng and Y.B.Jun studied medial BCI-algebras.

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In [11] S.M.Mostafa, Y.B.Jun and A. El-menshawy introduced the notion of medial ideals in BCI-algebras, they stated the fuzzification of medial ideals and investigated its properties. On the other hand, triangular norm is a powerful tool in the theory research and application development of fuzzy sets. On the basis of the definition of the intuitionistic fuzzy groups, Li [12] generalized the operators " ∧" and " ∨ " to T-norm and S-norm and defined the intuitionistic fuzzy groups of (T,S) - norms. as a generalization of the notion of fuzzy set.

K. H. Kim [4] Using t-norm T and s-norm S, they introduced the notion of intuitionistic (T,S)-normed fuzzy subalgebra in BCK/BCI-algebra, and some related properties are investigated. Jun et al. [8] considered the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras, and investigated some of their properties. They introduced the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigated some related properties. A.M. Menshawy [10] introduced the notion of intuitionistic fuzzy medial ideals and investigated some simple but elegant results. Atanassove and Gargov [3] introduced the notion of interval-valued intuitionistic fuzzy sets which is a generalization of both intuitionistic fuzzy sets and interval-valued fuzzy sets.

In this paper, the notion of interval valued intuitionistic $(\tilde{T},\tilde{S})$ -fuzzy subalgebras and medial ideals of a BCI-algebra is introduced, and several properties are investigated. Relations between a interval valued intuitionistic $(\tilde{T},\tilde{S})$ fuzzy subalgebra and a interval valued intuitionistic $(\tilde{T},\tilde{S})$ fuzzy medial ideal are given. In connection with the notion of homomorphism, we study how the images and inverse images of interval valued intuitionistic fuzzy medial-ideal become interval valued intuitionistic $(\tilde{T},\tilde{S})$ -fussy medial-ideal. Furthermore, we give the concept of the Cartesian product of interval-valued intuitionistic $(\tilde{T},\tilde{S})$-fuzzy medial ideals in BCI-algebras, and investigate some related properties.

2. Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

Definition 2.1[5] An algebraic system $(X,*,0)$ of type (2, 0) is called a BCI-algebra if it satisfying the following conditions:

(BCI-1) $((x*y)*(x*z))*(z*y)=0,$

(BCI-2) $(x*(x*y))*y=0,$

(BCI-3) $x * x = 0$,

(BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all x, y and $z \in X$.

In a BCI-algebra *X*, we can define a partial ordering"$\leq$" by $x \leq y$ if and only if $x * y = 0$.

In what follows, *X* will denote a BCI-algebra unless otherwise specified.

A BCI-algebra *X* is called a medial BCI-algebra if it satisfying the following condition: $(x * y) * (z * u) = (x * z) * (y * u)$, for all x, y, z and $u \in X$.

In a medial BCI-algebra X, the following holds for all $x, y, z \in X$:

(1) $x * (y * z) = z * (y * x)$,

(2) $x * (x * y) = y$,

(3) $0 * (y * x) = x * y$.

Definition 2.2[9] A non empty subset *S* of a medial BCI-algebra *X* is said to be medial subalgebra of *X*, if $x * y \in S$, for all $x, y \in S$.

Definition 2.3 [9] A non-empty subset *I* of a BCI-algebra *X* is said to be a BCI-ideal of *X* if it satisfies:

(I_1) $0 \in I$,

(I_2) $x * y \in I$ and $y \in I$ implies $x \in I$ for all $x, y \in X$.

Definition 2.4[11] A non empty subset *M* of a BCI-algebra *X* is said to be a medial ideal of *X* if it satisfies:

(M_1) $0 \in M$,

(M_2) $z * (y * x) \in M$ and $y * z \in M$ imply $x \in M$ for all x, y and $z \in X$.

Proposition 2.5[11] Any medial ideal of a BCI-algebra must be a BCI- ideal but the converse is not true.

Lemma 2.6 Any BCI- ideal of a medial BCI-algebra is a medial ideal.

Proof. Straightforward.

Example 2.7[11] Let X = {0,1,2,3,4,5} be a set with a binary operation $*$ defined by the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	1	0	4	4
2	2	2	0	0	4	4
3	3	2	1	0	4	4
4	4	4	4	4	0	0
5	5	4	5	4	1	0

Then $(X,*,0)$ is a BCI-algebra and A = {0, 1, 2, 3} is a medial-ideal of X.

In this section, we begin with the concepts of interval-valued fuzzy sets.

An interval number is $\tilde{a} = [a^L, a^U]$, where $0 \le a^L \le a^U \le 1$.Let D[0, 1] denote the family of all closed subintervals of [0, 1], i.e.,

$$D[0,1] = \{\tilde{a} = [a^L, a^U] : a^L \le a^U \quad for\ a^L, a^U \in I\}$$

We define the operations $\le$, $\ge$, $=$, rmin and rmax in case of two elements in D[0, 1]. We consider two elements $\tilde{a} = [a^L, a^U]$ and $\tilde{b} = [b^L, b^U]$ in D[0, 1].

Then

1- $\tilde{a} \le \tilde{b} \quad iff \ a^L \le b^L, a^U \le b^U$;

2- $\tilde{a} \ge \tilde{b} \quad iff \ a^L \ge b^L, a^U \ge b^U$;

3- $\tilde{a} = \tilde{b} \quad iff \ a^L = b^L, a^U = b^U$;

4- $r\min\{\tilde{a}, \tilde{b}\} = \left[\min\{a^L, b^L\}, \min\{a^U, b^U\}\right]$;

5- $r\max\{\tilde{a}, \tilde{b}\} = \left[\max\{a^L, b^L\}, \max\{a^U, b^U\}\right]$

Here we consider that $\tilde{0} = [0,0]$ as least element and $\tilde{1} = [1,1]$ as greatest element.

Let $\tilde{a}_i \in D[0,1]$, where $i \in \Lambda$ .We define

$$r\inf_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a^L{}_i, \inf_{i\in\Lambda} a^U{}_i\right] \text{ and } r\sup_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a^L{}_i, \sup_{i\in\Lambda} a^U{}_i\right]$$

An interval valued fuzzy set (briefly, i-v-f-set) $\tilde{\mu}$ on X is defined as

$\tilde{\mu} = \left\{\left\langle x\ ,\ [\mu^L(x), \mu^U(x)],\ x \in X \right\rangle\right\}$, where $\tilde{\mu} : X \to D[0,1]$ and $\mu^L(x) \le \mu^U(x)$,for all $x \in X$. Then the ordinary fuzzy sets $\mu^L : X \to [0,1]$ and $\mu^U : X \to [0,1]$ are called a lower fuzzy set and an upper fuzzy set of $\tilde{\mu}$ respectively.

Definition 2.8[3] An interval valued triangular norm (interval valued t-norm) is a function

$\tilde{T} : D[0,1] \times D[0,1] \to D[0,1]$

that satisfies following conditions:

(T_1) interval valued boundary condition : $\tilde{T}(\tilde{x}, \tilde{1}) = \tilde{x}$,

(T_2)) interval valued commutativity condition: $\tilde{T}(\tilde{x}, \tilde{y}) = \tilde{T}(\tilde{y}, \tilde{x})$,

(T_3) interval valued associativity condition : $\tilde{T}(\tilde{x}, \tilde{T}(\tilde{y}, \tilde{z})) = \tilde{T}(\tilde{T}(\tilde{x}, \tilde{y}), \tilde{z})$,

(T_4) interval valued monotonicity: $\tilde{T}(\tilde{x}, \tilde{y}) \le \tilde{T}(\tilde{y}, \tilde{z})$,whenever $\tilde{y} \le \tilde{z}$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in D[0,1]$.

A simple example of such defined interval valued *t*-norm is a function $\tilde{T}(\tilde{\alpha}, \tilde{\beta}) = r\min\{\tilde{\alpha}, \tilde{\beta}\}$

In the general case $\tilde{T}(\tilde{\alpha}, \tilde{\beta}) \le r\min\{\tilde{\alpha}, \tilde{\beta}\}$ and $\tilde{T}(\tilde{\alpha}, \tilde{0}) = \tilde{0}, \forall \tilde{\alpha}, \tilde{\beta} \in D[0,1]$.

Definition 2.9 Let X be a BCI-algebra. A fuzzy subset μ in X is called a fuzzy subalgebra of X with respect to interval valued t-norm $\tilde{T}$ (briefly, $\tilde{T}$ -fuzzy subalgbra of X) if

$\tilde{\mu}(x) \ge \tilde{T}\{\tilde{\mu}(x * y), \tilde{\mu}(y)\}$, for all $x, y \in X$.

Definition 2.10[3] An interval valued triangular conorm (interval valued t-conorm $\tilde{S}$) is a mapping

$\tilde{S} : D[0,1] \times D[0,1] \to D[0,1]$ that satisfies following conditions:

(S1) $\tilde{S}(\tilde{x}, \tilde{0}) = \tilde{x}$,

(S2) $\tilde{S}(\tilde{x}, \tilde{y}) = \tilde{S}(\tilde{y}, \tilde{x})$,

(S3) $\tilde{S}(\tilde{x}, \tilde{S}(\tilde{y}, \tilde{z})) = \tilde{S}(\tilde{S}(\tilde{x}, \tilde{y}), \tilde{z})$

(S4) interval valued monotonicity: $\tilde{S}(\tilde{x}, \tilde{y}) \le \tilde{S}(\tilde{y}, \tilde{z})$,whenever $\tilde{y} \le \tilde{z}$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in D[0,1]$.A simple example of such definition interval valued s-norm S is a function

$\tilde{S}(\tilde{\alpha}, \tilde{\beta}) = r\max\{\tilde{\alpha}, \tilde{\beta}\}$.In the general case $\tilde{S}(\tilde{\alpha}, \tilde{\beta}) \le r\max\{\tilde{\alpha}, \tilde{\beta}\}$ and

$\tilde{S}(\tilde{\alpha}, \tilde{1}) = \tilde{1}, \forall \tilde{\alpha}, \tilde{\beta} \in D[0,1]$

Definition 2.11[3] Let X be a BCI-algebra. A interval valued $\tilde{\mu}$ in X is called interval valued subalgebra of X with respect interval valued to s-conorm $\tilde{S}$ (briefly, i.v $\tilde{S}$ - sub-algebra of X) if $\tilde{\mu}(x) \le \tilde{S}\{\mu(x * y), \tilde{\mu}(y)\}$, for all $x, y \in X$.

3. interval-valued $\tilde{T}, \tilde{S}$ -medial ideals

Definition 3.1 [3] Let X be a BCI-algebra. An interval valued $\tilde{\mu}$ in X is called interval-valued $\tilde{T}$ - BCI- ideal of X if it satisfies:

(TI$_1$) $\tilde{\mu}(0) \ge \tilde{\mu}(x)$,

(TI$_2$) $\tilde{\mu}(x) \ge \tilde{T}\{\tilde{\mu}(x * y), \tilde{\mu}(y)\}$, for all x, y and $z \in X$.

Definition 3.2[3] Let X be a BCI-algebra. An interval valued $\tilde{\mu}$ in X is called $\tilde{S}$ - BCI- ideal of X if it satisfies:

(SI$_1$) $\tilde{\mu}(0) \le \tilde{\mu}(x)$,

(SI$_2$) $\tilde{\mu}(x) \le \tilde{S}\{\tilde{\mu}(x * y), \tilde{\mu}(y)\}$, for all x, y and $z \in X$.

Definition 3.3 Let X be a BCI-algebra. An interval valued $\tilde{\mu}$ in X is called $\tilde{T}$ - medial -ideal of X if it satisfies:

(FM$_1$) $\tilde{\mu}(0) \ge \tilde{\mu}(x)$,

(FM$_2$) $\tilde{\mu}(x) \ge \tilde{T}\{\tilde{\mu}(z * (y * x)), \tilde{\mu}(y * z)\}$, for all x, y and $z \in X$.

Example.3.4 Let X = {0, 1, 2, 3, 4, 5} be a set with a binary operation $*$ as example 2.6

The function $\tilde{T}_m$ defined by $\tilde{T}_m(\tilde{\alpha}, \tilde{\beta}) = r\max\{\tilde{\alpha} + \tilde{\beta} - \tilde{1}, \tilde{0}\}, \forall \tilde{\alpha}, \tilde{\beta} \in D[0,1]$ is interval valued t-norm . By routine calculations, we known that a fuzzy set μ in X defined by

$\tilde{\mu}$ (1) =[0.3,0.5] and $\tilde{\mu}$ (0) = $\tilde{\mu}$ (2) = $\tilde{\mu}$ (3) = $\tilde{\mu}$ (4) = $\tilde{\mu}$ (5) = [0.3 0.9] is interval valued T_m-fuzzy BCI-ideal of X, which is interval valued T_m-fuzzy medial-ideal because

$\tilde{\mu}_A(x) \ge \tilde{T}\{\tilde{\mu}_A(z * (y * x), \tilde{\mu}_A(y * z)\}$.

Lemma 3.5 Any $\tilde{T}$ -fuzzy medial- ideal of a BCI-algebra is $\tilde{T}$ - fuzzy BCI- ideal of X.

Proof. Straightforward.

Lemma 3.6 Any $\tilde{S}$ -fuzzy medial- ideal of a BCI-algebra is $\tilde{S}$ - fuzzy BCI- ideal of X.

Proof. Straightforward.

4. Interval valued Intuitionistic $(\tilde{T},\tilde{S})$-fuzzy medial ideals

An interval valued intuitionistic fuzzy set (briefly IVIFS) is an object having the form $A=\{(x,\tilde{\mu}_A(x),\tilde{\lambda}_A(x))\mid x\in X\}$, where the function $\tilde{\mu}_A : X\to D[0,1]$, $\tilde{\lambda}_A : X\to D[0,1]$ and $0\le \mu^L{}_A(x)+\lambda^L{}_A(x)\le 1$, $0\le \mu^U{}_A(x)+\lambda^U{}_A(x)\le 1$ for all $x\in X$. An an interval valued intuitionistic fuzzy set $A=\{(x,\tilde{\mu}_A(x),\tilde{\lambda}_A(x))\mid x\in X\}$ in X can be identified to an order pair $(\tilde{\mu}_A,\tilde{\lambda}_A)$ in $I^X\times I^X$, *where* $I=D[0,1]$. We shall use the symbol $A=(\tilde{\mu}_A,\tilde{\lambda}_A)$ for an interval valued intuitionistic fuzzy set $A=\{(x,\tilde{\mu}_A(x),\tilde{\lambda}_A(x))\mid x\in X\}$.

Definition 4.1 An IVIFS $A=(\tilde{\mu}_A,\tilde{\lambda}_A)$ in a BCI-algebra *X* is called an interval valued intuitionistic $(\tilde{T},\tilde{S})$-fuzzy subalgebra of *X* if it satisfies the following

(IFMS1) $\tilde{\mu}_A(x*y)\ge \tilde{T}\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$,

(IFMS2) $\tilde{\lambda}_A(x*y)\le \tilde{S}\{\tilde{\lambda}_A(x),\tilde{\lambda}_A(y)\}$, for all $x,y\in X$.

Example 4.2 Let $X=\{0,1,2,3,4,5\}$ as in example 2.6, and $A=(\tilde{\mu}_A,\tilde{\lambda}_A)$ be an IVIFS in *X* defined by $\tilde{\mu}_A(0)$ = $\tilde{\mu}_A$ (2) = $\tilde{\mu}_A(3)$ = $\tilde{\mu}_A$ (4) = $\tilde{\mu}_A$ (5) = [0.1,0.7]>[0.1, 0.3] = $\tilde{\mu}_A$ (1), and $\tilde{\lambda}_A(0)$ = $\tilde{\lambda}_A$ (2) = $\tilde{\lambda}_A$ (3) = $\tilde{\lambda}_A$ (4) = $\tilde{\lambda}_A$ (5) = [0.1,0.2] <[0.1 ,0.5] = $\tilde{\lambda}_A$ (1). Let $\tilde{T}_m:[0,1]\times[0,1]\to[0,1]$ be a function defined by $\tilde{T}_m(\tilde{\alpha},\tilde{\beta})=r\max\{\tilde{\alpha}+\tilde{\beta}-\tilde{1},\tilde{0}\},\forall\tilde{\alpha},\tilde{\beta}\in D[0,1]$, and $\tilde{S}_m:[0,1]\times[0,1]\to[0,1]$ be a function defined by $\tilde{S}_m(\tilde{\alpha},\tilde{\beta})=r\min\{\tilde{\alpha}+\tilde{\beta},\tilde{1}\}$. Then by routine calculations, we can prove that $A=(\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval valued intuitionistic $(\tilde{T}_m,\tilde{S}_m)$-fuzzy sub-algebra of *X*.

Lemma 4.3 Every interval valued intuitionistic $(\tilde{T},\tilde{S})$-fuzzy subalgebra $A=(\tilde{\mu}_A,\tilde{\lambda}_A)$ of *X* satisfies the inequalities $\tilde{\mu}_A(0)\ge\tilde{\mu}_A(x)$, and $\tilde{\lambda}_A(0)\le\tilde{\lambda}_A(x)$ for all $x\in X$.

Proof. Straightforward.

Definition 4.4 An IVIFS $A=(\tilde{\mu}_A,\tilde{\lambda}_A)$ in *X* is called an interval valued intuitionistic $(\tilde{T},\tilde{S})$-fuzzy ideal of *X* if it satisfies the following inequalities:

(IFI1) $\tilde{\mu}_A(0)\ge\tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0)\le\tilde{\lambda}_A(x)$

(IFI2) $\tilde{\mu}_A(x)\ge\tilde{T}\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(y)\}$, (IFI3) $\tilde{\lambda}_A(x) \le \tilde{S}\{\tilde{\lambda}_A(x * y), \tilde{\lambda}_A(y)\}$, for all $x, y \in X$.

Definition 4.5. AnIVIFS $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in X is called an interval valued intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy medial ideal of X if it satisfies the following inequalities:

(IFM1) $\tilde{\mu}_A(0) \ge \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) \le \tilde{\lambda}_A(x)$

(IFM2) $\tilde{\mu}_A(x) \ge \tilde{T}\{\tilde{\mu}_A(z * (y * x)), \tilde{\mu}_A(y * z)\}$,

(IFM3) $\tilde{\lambda}_A(x) \le \tilde{S}\{\tilde{\lambda}_A(z * (y * x)), \tilde{\lambda}_A(y * z)\}$, for all $x, y, z \in X$.

Example 4.6. Let $X = \{0,1,2,3\}$ be a set with a binary operation $*$ define by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define IVIFS $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in X as follows $\tilde{\mu}_A(1) = \tilde{0}$, $\tilde{\mu}_A(0) = \tilde{\mu}_A(3) = \tilde{\mu}_A(3) = \tilde{1}$.
$\lambda_A(1) = \tilde{0}$, $\tilde{\lambda}_A(0) = \tilde{\lambda}_A(2) = \tilde{\lambda}_A(3) = \tilde{1}$. Let $\tilde{T}_m : D[0,1] \times D[0,1] \to D[0,1]$ be a function defined by $\tilde{T}_m(\tilde{\alpha}, \tilde{\beta}) = r\max\{\tilde{\alpha} + \tilde{\beta} - \tilde{1}, \tilde{0}\}$, and $\tilde{S}_m : D[0,1] \times D[0,1] \to D[0,1]$ be a function defined by $\tilde{S}_m(\tilde{\alpha}, \tilde{\beta}) = r\min\{\tilde{\alpha} + \tilde{\beta}, \tilde{1}\}$ .By routine calculations ,we can prove that $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval valued intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy medial ideal of X.

Lemma 4.7. Let $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval valued intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy medial ideal of X. If $x \le y$ in X, then $\tilde{\mu}_A(x) \ge \tilde{\mu}_A(y)$, $\tilde{\lambda}_A(x) \le \tilde{\lambda}_A(y)$, for all $x, y \in X$. That is $\tilde{\mu}_A$ is order reserving and $\tilde{\lambda}_A$ is order preserving.

Proof. Straightforward

Lemma 4.8. Let $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval valued intuitionistic (($\tilde{T}, \tilde{S}$)-fuzzy medial ideal of X. If the inequality $x * y \le z$ holds in X, then

$\tilde{\mu}_A(x) \ge \tilde{T}\{\tilde{\mu}_A(y), \tilde{\mu}_A(z)\}$, $\tilde{\lambda}_A(x) \le \tilde{S}\{\tilde{\lambda}_A(y), \tilde{\lambda}_A(z)\}$, for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$ be such that $x * y \le z$. Thus, put $z = 0$ in (IFM$_2$) and using (lemma 4.7), we get $\tilde{\mu}_A(x) \ge \tilde{T}\{\tilde{\mu}_A(0 * (y * x), \tilde{\mu}_A(y * 0)\}$

$$= \tilde{T}\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\} \ge \tilde{T}\{\tilde{\mu}_A(z), \tilde{\mu}_A(y)\}.$$

Similarly for $\tilde{\lambda}_A(x)$.

Theorem 4.9. Every interval valued intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy medial ideal of X is an interval valued intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy medial sub-algebra of X.

Proof. Let $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in X be an interval valued intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy medial ideal of X. Since $x * y \le x$, for all $x, y \in X$, then $\tilde{\mu}_A(x * y) \ge \tilde{\mu}_A(x)$, $\tilde{\lambda}_A(x * y) \le \tilde{\lambda}_A(x)$. Put z = 0 in (IFM2), (IFM3), we have $\tilde{\mu}_A(x * y) \ge \tilde{\mu}_A(x) \ge \tilde{T}\{\tilde{\mu}_A(0 * (y * x)), \tilde{\mu}_A(y * 0)\} = \tilde{T}\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$

$$\ge \tilde{T}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}.$$

Similarly for $\tilde{\lambda}_A(x)$. Then $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval valued intuitionistic $(\tilde{T}, \tilde{S})$ fuzzy subalgebra of X.

The converse of theorem 4.9 may not be true. For example, the interval valued intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy subalgebra $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in example 4.2 is not an interval valued intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy medial ideal of X since $\tilde{\mu}_A(1) = \tilde{0} < \tilde{1} = \tilde{T}_m\{\tilde{\mu}_A(4 * (4 * 1)), \tilde{\mu}_A(4 * 4)\}$.

Theorem 4.10 Let $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be interval valued an intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy medial subalgebra of X such that $x * y \le z$ for all $x, y, z \in X$. Then $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval valued intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy medial ideal of X.

Proof. Let $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval valued intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy medial subalgebra of X. Recall that $\tilde{\mu}_A(0) \ge \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) \le \tilde{\lambda}_A(x)$, for all $x \in X$. Since, for all $x, y, z \in X$, we have $x * (z * (y * x)) = (y * x) * (z * x) \le y * z$, it follows from lemma 4.8 that

$\tilde{\mu}_A(x) \ge \tilde{T}\{\tilde{\mu}_A(z * (y * x), \tilde{\mu}_A(y * z)\}$, $\tilde{\lambda}_A(x) \le \tilde{S}\{\tilde{\lambda}_A(z * (y * x)), \tilde{\lambda}_A(y * z)\}$.

Hence $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval valued intuitionistic $(\tilde{T}, \tilde{S})$-fuzzy medial ideal of X.

Definition 4.11: For any $\tilde{t} \in D[0,1]$ and an interval valued fuzzy set $\tilde{\mu}$ in a nonempty set of *D[0,1]*, the set $U(\tilde{\mu},\tilde{t}) := \{x \in X \mid \tilde{\mu}(x) \geq \tilde{t}\}$ is called an upper $\tilde{t}$ -level cut of $\tilde{\mu}$, and the set $L(\tilde{\mu},\tilde{t}) := \{x \in X \mid \tilde{\mu}(x) \leq \tilde{t}\}$ is called a lower $\tilde{t}$ -level cut of $\tilde{\mu}$.

Theorem 4.12: An IVIFS $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval valued intuitionistic fuzzy medial ideal of X if and only if for all $\tilde{s},\tilde{t} \in D[0,1]$, the set $U(\tilde{\mu}_A,\tilde{t})$ and $L(\tilde{\lambda}_A,\tilde{s})$ are either empty or medial ideals of *X*.

Proof. Straightforward.

5. The image (preimage) interval valued intuitionistic of $(\tilde{T},\tilde{S})$-fuzzy medial ideals

Let $(X,*,0)$ and $(Y,*',0')$ be BCI-algebras. A mapping $f: X \to Y$ is said to be a homomorphism if $f(x*y) = f(x)*' f(y)$ for all $x, y \in X$. Note that if $f: X \to Y$ is a homomorphism of BCI-algebras, then $f(0) = 0'$. Let $f: X \to Y$ be a homomorphism of BCI-algebras. For any IFS $A = (\mu_A, \lambda_A)$ in *Y*, define an IFS $A^f = (\mu_A^f, \lambda_A^f)$ in X by $\mu_A^f(x) := \mu_A(f(x))$, and $\lambda_A^f(x) := \lambda_A(f(x))$ for all $x \in X$.

Theorem 5.1. Let $f: X \to Y$ be a homomorphism of BCI-algebras. If an IVIFS $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in Y is an interval valued intuitionistic $(\tilde{T},\tilde{S})$ -fuzzy medial ideal, then $A^f = (\tilde{\mu}_A^f, \tilde{\lambda}_A^f)$ is an interval valued intuitionistic $(\tilde{T},\tilde{S})$ -fuzzy medial ideal of *X*.

Proof. For all $x, y, z \in X$, we have $\tilde{\mu}_A^f(x) := \tilde{\mu}_A(f(x)) \leq \tilde{\mu}_A(0) = \tilde{\mu}_A(f(0)) = \tilde{\mu}_A^f(0)$, and $\tilde{\lambda}_A^f(x) := \tilde{\lambda}_A(f(x)) \geq \tilde{\lambda}_A(0) = \tilde{\lambda}_A(f(0)) = \tilde{\lambda}_A^f(0)$. Now

$\tilde{\mu}_A^f(x) := \tilde{\mu}_A(f(x)) \geq \tilde{T}\{\tilde{\mu}_A(f(z)*(f(y)*f(x))), \tilde{\mu}_A(f(y)*f(z))\}$

$= \tilde{T}\{\tilde{\mu}_A(f(z)*f(y*x)), \tilde{\mu}_A(f(y*z))\} = \tilde{T}\{\tilde{\mu}_A(f(z*(y*x)), \tilde{\mu}_A(f(y*z))\}$

$= \tilde{T}\{\tilde{\mu}_A^f(z*(y*x)), \tilde{\mu}_A^f(y*z)\}$. Similarly, $\tilde{\lambda}_A^f(x) \leq \tilde{S}\{\tilde{\lambda}_A^f(z*(y*x)), \tilde{\lambda}_A^f(y*z)\}$.

Hence $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval valued intuitionistic $(\tilde{T},\tilde{S})$ -fuzzy medial ideal in *X*.

This completes the proof.

Theorem 5.2: Let $f: X \to Y$ be an epimorphism of BCI-algebras and let $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an IVIFS in Y. If $A^f = (\tilde{\mu}_A^f, \tilde{\lambda}_A^f)$ is an interval valued intuitionistic $(\tilde{T}, \tilde{S})$- fuzzy medial ideal of *X*, then $A = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval valued intuitionistic $(\tilde{T}, \tilde{S})$ -fuzzy medial ideal in *Y*.

Proof. For any $a \in Y$, there exists $x \in X$ such that $f(x) = a$ .Then

$$\tilde{\mu}_A(a) = \tilde{\mu}_A(f(x)) = \tilde{\mu}_A^f(x) \le \tilde{\mu}_A^f(0) = \tilde{\mu}_A(f(0)) = \tilde{\mu}_A(0),$$

$$\tilde{\lambda}_A(a) = \tilde{\lambda}_A(f(x)) = \tilde{\lambda}_A^f(x) \ge \tilde{\lambda}_A^f(0) = \tilde{\lambda}_A(f(0)) = \tilde{\lambda}_A(0).$$

Let $a, b, c \in Y$, there exists $x, y, z \in X$ such that $f(x) = a,\ f(y) = b,\ f(z) = c$. It follows that

$$\tilde{\mu}_A(a) = \tilde{\mu}_A(f(x)) = \tilde{\mu}_A^f(x) \ge \tilde{T}\{\tilde{\mu}_A^f(z * (y * x)), \tilde{\mu}_A^f(y * z)\} =$$

$$= \tilde{T}\{\tilde{\mu}_A(f(z * (y * x)), \tilde{\mu}_A(f(y * z))\} = \tilde{T}\{\tilde{\mu}_A(f(z) * f(y * x)), \tilde{\mu}_A(f(y) * f(z))\}$$

$$= \tilde{T}\{\tilde{\mu}_A(f(z) * (f(y) * f(x))), \tilde{\mu}_A(f(y) * f(z))\} = \tilde{T}\{\tilde{\mu}_A(c * (b * a)), \tilde{\mu}_A(b * c)\}.$$

Similarly, $\tilde{\lambda}_A(a) \le \tilde{S}\{\tilde{\lambda}_A(c * (b * a)), \tilde{\lambda}_A(b * c)\}$. This completes the proof.

6. Cartesian product of interval valued intuitionistic $(\tilde{T}, \tilde{S})$- fuzzy medial ideals

Let $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ and $B = (X, \tilde{\mu}_B, \tilde{\lambda}_B)$ be two interval valued intuitionistic $(\tilde{T}, \tilde{S})$ - of *X*, the Cartesian product $A \times B = (X \times X, \tilde{\mu}_A \times \tilde{\mu}_B, \tilde{\lambda}_A \times \tilde{\lambda}_B)$ is defined by $(\tilde{\mu}_A \times \tilde{\mu}_B)(x, y) = \tilde{T}\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\}$ and $(\tilde{\lambda}_A \times \tilde{\lambda}_B)(x, y) = \tilde{S}\{\tilde{\lambda}_A(x), \tilde{\lambda}_B(y)\}$, where $\tilde{\mu}_A \times \tilde{\mu}_B : X \times X \to D[0,1]$, $\tilde{\lambda}_A \times \tilde{\lambda}_B : X \times X \to D[0,1]$ for all $x, y \in X$.

Remark 6.1. Let X and Y be medial BCI-algebras, we define* on $X \times Y$ by, for every $(x, y), (u, v) \in X \times Y$, $(x, y) * (u, v) = (x * u, y * v)$. Clearly $(X \times Y; *, (0,0))$ is a medial BCI-algebra.

Proposition 6.2. Let $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$, $B = (X, \tilde{\mu}_B, \tilde{\lambda}_B)$ be interval valued intuitionistic $(\tilde{T}, \tilde{S})$ - fuzzy medial ideals of *X*, then $A \times B$ is interval valued intuitionistic $(\tilde{T}, \tilde{S})$ -fuzzy medial ideal of $X \times X$.

Proof. $(\tilde{\mu}_A \times \tilde{\mu}_B)(0,0) = \tilde{T}\{\tilde{\mu}_A(0), \tilde{\mu}_B(0)\} \ge \tilde{T}\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\} = (\tilde{\mu}_A \times \tilde{\mu}_B)(x, y)$,for all $x, y \in X$.

And

$(\tilde{\lambda}_A \times \tilde{\lambda}_B)(0,0) = \tilde{S}\{\tilde{\lambda}_A(0), \tilde{\lambda}_B(0)\} \le \tilde{S}\{\tilde{\lambda}_A(x), \tilde{\lambda}_B(y)\} = (\tilde{\lambda}_A \times \tilde{\lambda}_B)(x,y)$, for all $x, y \in X$. Now , let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, then

$$\tilde{T}\{(\tilde{\mu}_A \times \tilde{\mu}_B)((z_1, z_2) * ((y_1, y_2) * (x_1, x_2))), (\tilde{\mu}_A \times \tilde{\mu}_B)((y_1, y_2) * (z_1, z_2))\}$$
$$= \tilde{T}\{(\tilde{\mu}_A \times \tilde{\mu}_B)((z_1, z_2) * (y_1 * x_1, y_2 * x_2)), (\tilde{\mu}_A \times \tilde{\mu}_B)(y_1 * z_1, y_2 * z_2)\}$$
$$= \tilde{T}\{(\tilde{\mu}_A \times \tilde{\mu}_B)(z_1 * (y_1 * x_1), z_2 * (y_2 * x_2)), (\tilde{\mu}_A \times \tilde{\mu}_B)(y_1 * z_1, y_2 * z_2)\}$$
$$= \tilde{T}\{\tilde{T}\{\tilde{\mu}_A(z_1 * (y_1 * x_1)), \tilde{\mu}_B(z_2 * (y_2 * x_2))\}, \tilde{T}\{\tilde{\mu}_A(y_1 * z_1), \tilde{\mu}_B(y_2 * z_2)\}\}$$
$$= \tilde{T}\{\tilde{T}\{\tilde{\mu}_A(z_1 * (y_1 * x_1)), \tilde{\mu}_A(y_1 * z_1)\}, \tilde{T}\{\tilde{\mu}_B(z_2 * (y_2 * x_2)), \tilde{\mu}_B(y_2 * z_2)\}$$

$\le \tilde{T}\{\tilde{\mu}_A(x_1), \tilde{\mu}_B(x_2) = (\tilde{\mu}_A \times \tilde{\mu}_B)(x_1, x_2)$. Similarly we can prove that,

$$\tilde{S}\{(\tilde{\lambda}_A \times \tilde{\lambda}_B)((z_1, z_2) * ((y_1, y_2) * (x_1, x_2))), (\tilde{\lambda}_A \times \tilde{\lambda}_B)((y_1, y_2) * (z_1, z_2))\} \ge (\tilde{\lambda}_A \times \tilde{\lambda}_B)(x_1, x_2) .$$

Definition 6.3. Let $A = (X, \tilde{\lambda}_A, \tilde{\mu}_A)$ and $B = (X, \tilde{\mu}_B, \tilde{\lambda}_B)$ be interval valued intuitionistic $(\tilde{T}, \tilde{S})$ of a BCI-algebra *X*. for $\tilde{s}, \tilde{t} \in D[0,1]$, the set

$U(\tilde{\mu}_A \times \tilde{\mu}_B, \tilde{s}) := \{(x, y) \in X \times X \mid (\tilde{\mu}_A \times \tilde{\mu}_B)(x, y) \ge \tilde{s}\}$

is called upper $\tilde{s}$ -level of $(\tilde{\mu}_A \times \tilde{\mu}_B)(x, y)$ and the

set $L(\tilde{\lambda}_A \times \tilde{\lambda}_B, \tilde{t}) := \{(x, y) \in X \times X \mid (\tilde{\lambda}_A \times \tilde{\lambda}_B)(x, y) \le \tilde{t}\}$

is called lower $\tilde{t}$ -level of $(\tilde{\lambda}_A \times \tilde{\lambda}_B)(x, y)$.

Theorem 6.4. The interval valued intuitionistic fuzzy sets $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ and $B = (X, \tilde{\mu}_B, \tilde{\lambda}_B)$ are interval valued intuitionistic $(\tilde{T}, \tilde{S})$ -fuzzy medial ideals of *X* if and only if the non-empty set upper $\tilde{s}$ -level cut $U(\tilde{\mu}_A \times \tilde{\mu}_B, \tilde{s})$ and the non-empty lower $\tilde{t}$ -level cut $L(\tilde{\lambda}_A \times \tilde{\lambda}_B, \tilde{t})$ are medial ideals of $X \times X$ for all $\tilde{s}, \tilde{t} \in D[0,1]$.

Proof. Let $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ and $B = (X, \tilde{\mu}_B, \tilde{\lambda}_B)$ be interval valued intuitionistic $(\tilde{T}, \tilde{S})$ -fuzzy medial ideals of *X*, therefore for any $(x, y) \in X \times X$, we have

$$(\tilde{\mu}_A \times \tilde{\mu}_B)(0,0) = \tilde{T}\{\tilde{\mu}_A(0), \tilde{\mu}_B(0)\} \ge \tilde{T}\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\} = (\tilde{\mu}_A \times \tilde{\mu}_B)(x, y) .$$

Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$ and $\tilde{s} \in D[0,1]$, such that

$((z_1, z_2) * ((y_1, y_2) * (x_1, x_2))) \in U(\tilde{\mu}_A \times \tilde{\mu}_B, \tilde{s})$ and $(y_1, y_2) * (z_1, z_2) \in U(\tilde{\mu}_A \times \tilde{\mu}_B, \tilde{s})$.

Now $(\tilde{\mu}_A \times \tilde{\mu}_B)(x_1, x_2) \ge \tilde{T}\{(\tilde{\mu}_A \times \tilde{\mu}_B)((z_1, z_2) * ((y_1, y_2) * (x_1, x_2))), (\tilde{\mu}_A \times \tilde{\mu}_B)((y_1, y_2) * (z_1, z_2))\}$

$=\tilde{T}\{(\tilde{\mu}_A\times\tilde{\mu}_B)((z_1,z_2)*(y_1*x_1,y_2*x_2)),(\tilde{\mu}_A\times\tilde{\mu}_B)(y_1*z_1,y_2*z_2)\}$

$=\tilde{T}\{(\tilde{\mu}_A\times\tilde{\mu}_B)(z_1*(y_1*x_1),z_2*(y_2*x_2)),(\tilde{\mu}_A\times\tilde{\mu}_B)(y_1*z_1,y_2*z_2)\}\geq\tilde{T}\{\tilde{s},\tilde{s}\}=\tilde{s}$,

Therefore $(x_1,x_2)\in U(\tilde{\mu}_A\times\tilde{\mu}_B,\tilde{s})$ is a medial ideal of $X\times X$. Similarly we can prove that $L((\tilde{\lambda}_A\times\tilde{\lambda}_B)(x,y),\tilde{t})$ is a medial ideal of $X\times X$ .This completes the proof.

7. CONCLUSIONS

In the present paper, we have introduced the concept of interval valued intuitionistic $(\tilde{T},\tilde{S})$ fuzzy medial subalgebras and interval valued intuitionistic $(\tilde{T},\tilde{S})$ fuzzy medial ideals in BCI – algebras and investigated some of their useful properties. We believe that these results are very useful in developing algebraic structures also these definitions and main results can be similarly extended to some other algebraic structure such as PS-algebras, Q-algebras, SU-algebras , IS-algebras, β algebras and semi-rings . It is our hope that this work would other foundations for further study of the theory of BCI-algebras. In our future study of fuzzy structure of BCI-algebras, may be the following topics should be considered:

(1) To consider the structure of $(\tilde{\tau})$- interval–valued fuzzy medial subalgebras (ideals) of BCI-algebras.

(2) soft sets with applications of interval valued fuzzy medial subalgebras (ideals) of BCI-algebras.

Conflict of Interests

The authors declare that there is no conflict of interests.

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