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COMMON FIXED POINT THEOREMS FOR SIX MAPPINGS SATISFYING ψ-WEAKLY CONTRACTIVE CONDITIONS IN G-METRIC SPACE

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Abstract. In this paper, we introduce some common fixed point theorems for six mappings satisfying ψ- and $(\psi,\varphi)-$weakly contractive conditions in G-metric spaces. And we introduce an example to support the validity of our results.

Keywords: G-metric space; common fixed point; ψ-weakly contractive conditions; weakly compatible mappings

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1. Introduction

In 2006, Mustafa and Sims [1] introduced the generalized structure of metric spaces, called G-metric spaces. Afterwards, numerous fixed point theorems in this generalized structure relative to one, two or three mappings were proved by different authors(see[5-7]). 2015, Zeqing Liu and Xiaoping Zhang et al[8] introduced the existence and uniqueness of common fixed points for four mappings satisfying ψ- and $(\psi,\varphi)-$weakly contractive conditions in metric spaces which was motivated by the results in [9-12]. In this paper, we extended and generalize the

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results in [8] and introduce some common fixed point theorems for six mappings satisfying ψ- and $(\psi,\varphi)-$weakly contractive conditions in G-metric spaces.

2. Previous notations and results

We recall the definitions of G-metric space,the notion of convergence and other results that will be needed in the sequel.

Definition 2.1[1] Let X be a nonempty set. Suppose that G: $X \times X \times X \to [0,+\infty)$ is a function satisfying the following conditions:

(G1) $G(x,y,z) = 0$ if and only if $x = y = z$;

(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$;

(G3) $G(x,x,y) \leq\leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq Z$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$(symmetry in all three variables);

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$(rectangle inequality).

Then G is called a G-metric on X and (X,G) is called a G-metric space.

This notion of $G-metric$ was introduced by Mustafa and Sims [1] in 2006. It can be shown that if (X,d) is a metric space one can define G-metric on X by

$$G(x,y,z) = \max\{d(x,y),d(y,z),d(z,x)\} \text{ or } G(x,y,z) = d(x,y)+d(y,z)+d(z,x).$$

Definition 2.2[1] Let(X,G) be a G-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is $G-$*convergent* to a point $x \in X$ or $\{x_n\}$ $G-$*converges* to x if, for any $\varepsilon > 0$, there exists $k \in N$ such that $G(x,x_n,x_m) < \varepsilon$ for all $m,n \geq k$, that is, $\lim_{n,m\to+\infty} G(x,x_n,x_m)$. In this case, we write $x_n \to x(n \to \infty)$ or $\lim_{n\to+\infty} x_n = x$.

Proposition 2.1[1] Let (X,G) be a G-metric space. The following are equivalent:

(1) $\{x_n\}$ is G-convergent to x;

(2) $G(x_n,x_n,x) \to 0$ as $n \to +\infty$;

(3) $G(x_n,x,x) \to 0$ as $n \to +\infty$;

(4) $G(x_n,x_m,x) \to 0$ as $n,m \to +\infty$.

Definition 2.3[1] Let (X,G) be a G-metric space and $\{x_n\}$ be a sequence in X. We say that

$\{x_n\}$ is a $G-Cauchy\ sequence$ if, for any $\varepsilon > 0$, there exists $k \in N$ such that $G(x_n, x_m, x_l)$ for all $m, n, l \geq k$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 2.2[1] Let (X, G) be a G-metric space. Then the following are equivalent:

(1) The sequence $\{x_n\}$ is a G-Cauchy sequence.

(2) For any $\varepsilon > 0$, there exists $k \in N$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq k$.

Proposition 2.3[1] Let (X, G) be a G-metric space. Then, f: $X \to X$ is $G-continuous$ at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.

Definition 2.4[1] A G-metric space (X, G) is called $G-complete$ if every G-cauchy sequence is G-convergent in (X, G).

Definition 2.5[2] Let (X, G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequence $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively,$(F(x_n, y_n))$ is G-convergent to $F(x, y)$.

Definition 2.6[3] A pair of self mappings f and g in a metric space (X, d) are said to be weakly compatible if for all $t \in X$ the equality $ft = gt$ implies $fgt = gft$.

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{R}^+ = [0, +\infty)$, $M(x, y, z) = max\{G(Ax, By, Cz), G(Ax, Ax, Tx), G(By, By, Sy), G(Cz, Cz, Hz),$
$\frac{1}{2}[G(Ax, By, Cz) + G(Tx, Sy, Hz)]\}$ and

$\Phi_1 = \{\psi : \psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing, and $\psi(t) = 0$ if and only if $t = 0$ $\}$,

$\Phi_2 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semi-continuous, and $\varphi(t) = 0$ if and only if $t = 0\}$,

$\Phi_3 = \{\psi : \psi : \mathbb{R}^+ \to \mathbb{R}^+$ is upper semi-continuous, and $\lim_{n\to\infty} a_n = 0$ for each sequence $\{a_n\}_{n\in N} \subset \mathbb{R}^+$ with $a_{n+1} \leq \psi(a_n), \forall n \in \mathbb{N}\}$.

Lemma 2.1[4] Let $\psi \in \Phi_3$. Then $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$.

3. Main results

Our main results are as follows.

Lemma 3.1 Let A, B, C, S, T and H be self mappings in a G-metric space (X,G) satisfying

$$\psi(G(Tx,Sy,Hz)) \leq \psi(M(x,y,z)) - \varphi(M(x,y,z)), \tag{3.1}$$

where $(\psi,\varphi) \in \Phi_1 \times \Phi_2$. Assume that I: $\mathbb{R}^+ \to \mathbb{R}^+$ is the identity mapping and

$$\psi_1(t) = (\psi + I)^{-1}(\psi + I - \varphi)(t), \quad \forall t \in \mathbb{R}^+. \tag{3.2}$$

Then $\psi_1 \in \Phi_3$ and

$$G(Tx,Sy,Hz) \leq \psi_1(M(x,y,z)), \quad \forall x,y,z \in X. \tag{3.3}$$

Proof

It follows from $\psi \in \Phi_1$ that $\psi + I$: $\mathbb{R}^+ \to \mathbb{R}^+$ is continuous and increasing and $(\psi + I)(t) = 0$ if and only if $t = 0$. So does $(\psi + I)^{-1}$. Obviously, $(\psi,\varphi) \in \Phi_1 \times \Phi_2$ and (3.2) guarantee

$$\psi_1 \text{ is upper semi-continuous and } \psi_1(0) = 0. \tag{3.4}$$

Assume that $\{a_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence in $\mathbb{R}^+$ with

$$a_{n+1} \leq \psi_1(a_n), \quad \forall n \in \mathbb{N}. \tag{3.5}$$

Suppose that $a_{n_0} = 0$ for some $n_0 \in \mathbb{N}$. It follows from (3.2), (3.4) and (3.5) that

$$0 \leq a_{n_0+1} \leq \psi_1(a_{n_0}) = \psi_1(0) = 0,$$

that is, $a_{n_0+1} = 0$. Similarly we have $a_n = a_{n-1} = \ldots = a_{n_0} = 0$ for each $n > n_0$, that is, $\lim_{n\to\infty} a_n = 0$. Suppose that $a_n > 0$ for all $n \in \mathbb{N}$. If $a_{k+1} \geq a_k$ for some $k \in \mathbb{N}$, it follows from

(3.2), (3.5) and $(\psi,\varphi)\in\Phi_1\times\Phi_2$ that

$$\begin{aligned}\psi(a_k)+a_k &\leq \psi(a_{k+1})+a_{k+1}=(\psi+I)(a_{k+1})\leq(\psi+I)\psi_1(a_k)\\ &= (\psi+I-\varphi)(a_k)\\ &= \psi(a_k)+a_k-\varphi(a_k)<\psi(a_k)+a_k\end{aligned}$$

which is a contradiction. Consequently, $\{a_n\}_{n\in\mathbb{N}}$ is a positive and decreasing, which implies that $\{a_n\}_{n\in\mathbb{N}}$ converges to some $a\geq 0$. Suppose that $a>0$. By means of (3.4) and (3.5), we find

$$0<a=\limsup_{n\to\infty}a_{n+1}\leq\limsup_{n\to\infty}\psi_1(a_n)\leq\psi_1(a),$$

which together with (3.2) and $(\psi,\varphi)\in\Phi_1\times\Phi_2$ means

$$\psi(a)+a\leq\psi(a)+a-\varphi(a)<\psi(a)+a,$$

which is a contradiction. Hence $a=0$. Consequently, $\psi_1\in\Phi_3$.

In order to prove (3.3), we have to consider two possible cases as follows:

Case 1. $M(x_0,y_0,z_0)=0$ for some $x_0,y_0,z_0\in X$. It is easy to verify

$$\begin{aligned}&G(Ax_0,By_0,Cz_0)=G(Ax_0,Ax_0,Tx_0)=G(By_0,By_0,Sy_0)\\ &=G(Cz_0,Cz_0,Hz_0)=G(Tx_0,Sy_0,Hz_0),\end{aligned}$$

which yields

$$Ax_0=Tx_0=By_0=Sy_0=Cz_0=Hz_0$$

and

$$G(Tx_0,Sy_0,Hz_0)=\psi_1(M(x_0,y_0,z_0));$$

Case 2. $M(x,y,z)>0$ for all $x,y,z\in X$. It follows from (3.1), (3.2) and $(\psi,\varphi)\in\Phi_1\times\Phi_2$ that

$$\psi(G(Tx,Sy,Hz))\leq\psi(M(x,y,z))-\varphi(M(x,y,z))<\psi(M(x,y,z)),\quad\forall x,y,z\in X,$$

which yields

$$G(Tx,Sy,Hz)<M(x,y,z),\quad\forall x,y,z\in X,$$

and

$$\begin{aligned}(\psi+I)(G(Tx,Sy,Hz)) &= \psi(G(Tx,Sy,Hz))+G(Tx,Sy,Hz)\\ &< \psi(M(x,y,z))-\varphi(M(x,y,z))+M(x,y,z)\\ &= (\psi+I-\varphi)(M(x,y,z)), \quad \forall x,y,z\in X,\end{aligned}$$

which together with (3.2) gives (3.3). This completes the proof.

Remark 3.1 It follows from Lemma 3.1 that the (ψ,φ)-weakly contractive conditions (3.1) relative to six mappings A, B, C, S, T and H implies the ψ_1-weakly contractive conditions (3.3) relative to six mappings A, B, C, S, T and H.

Theorem 3.1 Let A, B, C, S, T and H be self mappings in a G-metric space (X,G) such that:

$\{A,T\}$, $\{B,S\}$ and $\{C,H\}$ are weakly compatible; (3.6)

$T(X)\subseteq B(X)$, $S(X)\subseteq C(X)$ and $H(X)\subseteq A(X)$; (3.7)

one of $A(X)$, $B(X)$, $C(X)$, $S(X)$, $T(X)$ and $H(X)$ is complete; (3.8)

$$G(Tx,Sy,Hz)\le\psi(M(x,y,z)),\forall x,y,z\in X, \tag{3.9}$$

Where ψ is in Φ_3.

Then A, B, C, S, T and H have a unique common fixed point in X.

Proof

Let $x_0\in X$. It follows from (3.7) that there exist two sequence $\{y_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ in X such that

$$y_{3n+1}:=Bx_{3n+1}=Tx_{3n},$$

$$y_{3n+2}:=Cx_{3n+2}=Sx_{3n+1},$$

$$y_{3n+3}:=Ax_{3n+3}=Hx_{3n+2}. \tag{3.10}$$

Put $G_n = G(y_n, y_{n+1}, y_{n+2})$ for all $n \in \mathbb{N}$. Now we prove

$$\lim_{n\to\infty} G_n = 0 \tag{3.11}$$

$$G_{3n} = G(Tx_{3n}, Sx_{3n+1}, Hx_{3n-1}) \leq \psi(M(x_{3n}, x_{3n+1}, x_{3n-1})), \forall n \in \mathbb{N} \tag{3.12}$$

and

$$\begin{aligned}
& M(x_{3n}, x_{3n+1}, x_{3n-1}) \\
= \ & \max\{G(Ax_{3n}, Bx_{3n+1}, Cx_{3n-1}), G(Ax_{3n}, Ax_{3n}, Tx_{3n}), \\
& G(Bx_{3n+1}, Bx_{3n+1}, Sx_{3n+1}), G(Cx_{3n-1}, Cx_{3n-1}, Hx_{3n-1}), \\
& \frac{1}{2}[G(Ax_{3n}, Bx_{3n+1}, Cx_{3n-1}) + G(Tx_{3n}, Sx_{3n+1}, Hx_{3n-1})]\} \\
= \ & \max\{G(y_{3n}, y_{3n+1}, y_{3n-1}), G(y_{3n}, y_{3n}, y_{3n+1}), \\
& G(y_{3n+1}, y_{3n+1}, y_{3n+2}), G(y_{3n-1}, y_{3n-1}, y_{3n}), \\
& \frac{1}{2}(G(y_{3n}, y_{3n+1}, y_{3n-1}) + G(y_{3n+1}, y_{3n+2}, y_{3n}))\} \\
= \ & \max\{G_{3n-1}, G(y_{3n}, y_{3n}, y_{3n+1}), G(y_{3n+1}, y_{3n+1}, y_{3n+2}), \\
& G(y_{3n-1}, y_{3n-1}, y_{3n}), \frac{1}{2}(G_{3n-1} + G_{3n})\} \\
\leq \ & \max\{G_{3n-1}, G_{3n}, \frac{1}{2}(G_{3n-1} + G_{3n})\} \\
= \ & \max\{G_{3n-1}, G_{3n}\}, \quad \forall n \in \mathbb{N}
\end{aligned} \tag{3.13}$$

Suppose that $G_{3n_0-1} < G_{3n_0}$ for some $n_0 \in \mathbb{N}$. It follows (3.9), (3.13) and Lemma 2.1 that

$$\begin{aligned}
& G_{3n_0} \leq \psi(M(x_{3n_0}, x_{3n_0+1}, x_{3n_0-1})) \\
& \leq \psi(\max\{G_{3n_0-1}, G_{3n_0}\}) = \psi(G_{3n_0}) < G_{3n_0},
\end{aligned}$$

which is a contradiction. Hence

$$G_{3n} \leq G_{3n-1}, \quad \forall n \in \mathbb{N}. \tag{3.14}$$

Similarly we infer

$$G_{3n+1} \leq G_{3n}, \quad \forall n \in \mathbb{N}. \tag{3.15}$$

and

$$G_{3n+2} \leq G_{3n+1}, \ \forall n \in \mathbb{N}. \tag{3.16}$$

From (3.14), (3.15) and (3.16) we have

$$G_{n+1} \leq G_n, \quad \forall n \in \mathbb{N},$$

which means that the sequence $\{G_n\}_{n\in\mathbb{N}}$ is nonincreasing and bounded. consequently there exists $r \geq 0$ with $\lim_{n\to\infty} G_n = r$. Suppose that $r > 0$. It follows from (3.9), (3.14), $\psi \in \Phi_3$, and Lemma 2.1 that

$$\begin{aligned} r &= \limsup_{n\to\infty} G_{3n} \leq \limsup_{n\to\infty} \psi(M(x_{3n}, x_{3n+1}, x_{3n-1})) \\ &\leq \limsup_{n\to\infty} \psi(G_{3n-1}) \leq \psi(r) < r, \end{aligned}$$

which is a contradiction. Hence $r = 0$, that is, (3.11) holds.

Next we prove that $\{y_n\}_{n\in\mathbb{N}}$ is a cauchy sequence. Because of (3.11) it is sufficient to verify that $\{y_{3n}\}_{n\in\mathbb{N}}$ is a cauchy sequence. Suppose to the contrary: that is, $\{y_{3n}\}$ is not a cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequence $\{y_{3m_k}\}$ and $\{y_{3n_k}\}$ of $\{y_{3n}\}$ such that m_k is the smallest index for which $3m_k > 3n_k > k$, and

$$G(y_{3n_k}, y_{3m_k}, y_{3m_k}) \geq \varepsilon \tag{3.17}$$

This means that

$$G(y_{3n_k}, y_{3m_k-3}, y_{3m_k-3}) < \varepsilon \tag{3.18}$$

Taking advantage of (3.17), (3.18), and (G3)-(G5), we get

$$\begin{aligned}
\varepsilon &\leq G(y_{3n_k}, y_{3m_k}, y_{3m_k}) \\
&\leq G(y_{3n_k}, y_{3m_k-3}, y_{3m_k-3}) + G(y_{3m_k-3}, y_{3m_k}, y_{3m_k}) \\
&\leq G(y_{3n_k}, y_{3m_k-3}, y_{3m_k-3}) + G(y_{3m_k-3}, y_{3m_k-2}, y_{3m_k-2}) \\
&\quad +G(y_{3m_k-2}, y_{3m_k}, y_{3m_k}) \\
&\leq G(y_{3n_k}, y_{3m_k-3}, y_{3m_k-3}) + G(y_{3m_k-3}, y_{3m_k-1}, y_{3m_k-2}) \\
&\quad +G(y_{3m_k-2}, y_{3m_k-1}, y_{3m_k}) \\
&< \varepsilon + G_{3m_k-3} + G_{3m_k-2}
\end{aligned} \tag{3.19}$$

and

$$|G(y_{3m_k+1}, y_{3m_k+2}, y_{3n_k}) - G(y_{3m_k}, y_{3m_k}, y_{3n_k})| \leq 2G_{3m_k},$$

$$|G(y_{3m_k}, y_{3m_k+1}, y_{3n_k-1}) - G(y_{3m_k+1}, y_{3m_k+2}, y_{3n_k})| \leq G_{3m_k} + G_{3n_k-1}; \tag{3.20}$$

Letting $k \to \infty$ in (3.19) and (3.20) and using (3.11), we have

$$\begin{aligned}
&\lim_{k\to\infty} G(y_{3m_k}, y_{3m_k}, y_{3n_k}) = \lim_{k\to\infty} G(y_{3m_k+1}, y_{3m_k+2}, y_{3n_k}) \\
&= \lim_{k\to\infty} G(y_{3m_k}, y_{3m_k+1}, y_{3n_k-1}) = \varepsilon
\end{aligned}$$

And also, from (3.9) and (3.10) we have

$$\begin{aligned}
&G(y_{3m_k+1}, y_{3m_k+2}, y_{3n_k}) \\
&= G(Tx_{3m_k}, Sx_{3m_k+1}, Hx_{3n_k-1}) \\
&\leq \psi(M(x_{3m_k}, x_{3m_k+1}, x_{3n_k-1})),
\end{aligned}$$

where

$$
\begin{aligned}
& M(x_{3m_k}, x_{3m_k+1}, x_{3n_k-1}) \\
= \ & \max\{G(Ax_{3m_k}, Bx_{3m_k+1}, Cx_{3n_k-1}), G(Ax_{3m_k}, Ax_{3m_k}, Tx_{3m_k}), \\
& G(Bx_{3m_k+1}, Bx_{3m_k+1}, Sx_{3m_k+1}), G(Cx_{3n_k-1}, Cx_{3n_k-1}, Hx_{3n_k-1}), \\
& \frac{1}{2}[G(Ax_{3m_k}, Bx_{3m_k+1}, Cx_{3n_k-1}) + G(Tx_{3m_k}, Sx_{3m_k+1}, Hx_{3n_k-1})]\} \\
= \ & \max\{G(y_{3m_k}, y_{3m_k+1}, y_{3n_k-1}), G(y_{3m_k}, y_{3m_k}, y_{3m_k+1}), \\
& G(y_{3m_k+1}, y_{3m_k+1}, y_{3m_k+2}), G(y_{3n_k-1}, y_{3n_k-1}, y_{3n_k}), \\
& \frac{1}{2}[G(y_{3m_k}, y_{3m_k+1}, y_{3n_k-1}) + G(y_{3m_k+1}, y_{3m_k+2}, y_{3n_k})]\} \\
& \to \max\{\varepsilon, 0, 0, 0, \varepsilon\} \\
= \ & \varepsilon \quad as \ \ k \to \infty. \qquad (3.21)
\end{aligned}
$$

In view of (3.9), (3.10), (3.21), $\psi \in \Phi_3$ and Lemma 2.1, we gain

$$
\begin{aligned}
\varepsilon &= \limsup_{k\to\infty} G(y_{3m_k+1}, y_{3m_k+2}, y_{3n_k}) = \limsup_{k\to\infty} G(Tx_{3m_k}, Sx_{3m_k+1}, Hx_{3n_k-1}) \\
&\le \limsup_{k\to\infty} \psi(M(x_{3m_k}, x_{3m_k+1}, x_{3n_k-1})) \le \psi(\varepsilon) < \varepsilon,
\end{aligned}
$$

which is a contradiction. Hence $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Assume that $A(X)$ is complete. Observe that $\{y_{3n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A(X)$. Consequently there exists $(z, v) \in A(X) \times X$ with $\lim_{n\to\infty} y_{3n+3} = z = Av$. It is easy to see

$$
z = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Bx_{3n+1} = \lim_{n\to\infty} Tx_{3n} = \lim_{n\to\infty} Cx_{3n+2}
$$

$$
= \lim_{n\to\infty} Sx_{3n+1} = \lim_{n\to\infty} Hx_{3n+2} = \lim_{n\to\infty} Ax_{3n+3} = Av. \qquad (3.22)
$$

Suppose that $Tv \neq z$. From (3.22) we have

$$\begin{aligned}
& M(v, x_{3n+1}, x_{3n+2}) \\
= \;& \max\{G(Av, Bx_{3n+1}, Cx_{3n+2}), G(Av, Av, Tv), \\
& G(Bx_{3n+1}, Bx_{3n+1}, Sx_{3n+1}), G(Cx_{3n+2}, Cx_{3n+2}, Hx_{3n+2}), \\
& \frac{1}{2}[G(Av, Bx_{3n+1}, Cx_{3n+2}) + G(Tv, Sx_{3n+1}, Hx_{3n+2})]\} \\
= \;& \max\{G(z, y_{3n+1}, y_{3n+2}), G(z, z, Tv), G(y_{3n+1}, y_{3n+1}, y_{3n+2}), \\
& G(y_{3n+2}, y_{3n+2}, y_{3n+3}), \frac{1}{2}[G(z, y_{3n+1}, y_{3n+2}) + G(Tv, y_{3n+2}, y_{3n+3})]\} \\
\to \;& \max\{G(z,z,z), G(z,z,Tv), G(z,z,z), \\
& G(z,z,z), \frac{1}{2}[G(z,z,z) + G(Tv,z,z)]\} \\
= \;& \max\{0, G(z,z,Tv), 0, 0, \frac{1}{2}G(z,z,Tv)\} \\
= \;& G(z,z,Tv) \quad \text{as } n \to \infty,
\end{aligned}$$

which together with (3.9), $\psi \in \Phi_3$, and Lemma 2.1 yields

$$\begin{aligned}
G(Tv,z,z) &= \limsup_{n\to\infty} G(Tv, y_{3n+2}, y_{3n+3}) = \limsup_{n\to\infty} G(Tv, Sx_{3n+1}, Hx_{3n+2}) \\
&\leq \limsup_{n\to\infty} \psi(M(v, x_{3n+1}, x_{3n+2})) \leq \psi(G(Tv,z,z)) < G(Tv,z,z),
\end{aligned}$$

which is a contradiction. Hence $Tv = z$. It follows from (3.7) that there exists a point $w \in X$ with $z = Bw = Tv$. Suppose that $Sw \neq z$. In light of (3.22), we deduce

$$\begin{aligned}
& M(x_{3n}, w, x_{3n+2}) \\
= \;& \max\{G(Ax_{3n}, Bw, Cx_{3n+2}), G(Ax_{3n}, Ax_{3n}, Tx_{3n}), \\
& G(Bw, Bw, Sw), G(Cx_{3n+2}, Cx_{3n+2}, Hx_{3n+2}) \\
& \frac{1}{2}[G(Ax_{3n}, Bw, Cx_{3n+2}) + G(Tx_{3n}, Sw, Hx_{3n+2})]\} \\
\to \;& \max\{G(z,z,z), G(z,z,z), G(z,z,Sw), G(z,z,z) \\
& \frac{1}{2}[G(z,z,z) + G(z,Sw,z)]\}
\end{aligned}$$

(1)

$$
\begin{aligned}
&= \max\{0,0,G(z,z,Sw),0,\frac{1}{2}G(z,z,Sw)\} \\
&= G(z,z,Sw) \quad as \;\; n\to\infty,
\end{aligned}
$$

which together with (3.9), (3.10), (3.22), $\psi \in \Phi_3$, and Lemma 2.1 yields

$$
\begin{aligned}
G(z,Sw,z) &= \limsup_{n\to\infty} G(y_{3n+1},Sw,y_{3n+3}) = \limsup_{n\to\infty} G(Tx_{3n},Sw,Hx_{3n+2}) \\
&\leq \limsup_{n\to\infty} \psi(M(x_{3n},w,x_{3n+2})) \leq \psi(G(z,z,Sw)) < G(z,z,Sw),
\end{aligned}
$$

which is a contradiction, and hence $Sw = z$. It follows from (3.7) that there exists a point $u \in z$ with $z = Cu = Sw$. Suppose that $Hu \neq z$. In light of (3.22), we deduce

$$
\begin{aligned}
& M(x_{3n},x_{3n+1},u) \\
= \;& \max\{G(Ax_{3n},Bx_{3n+1},Cu),G(Ax_{3n},Ax_{3n},Tx_{3n}), \\
& G(Bx_{3n+1},Bx_{3n+1},Sx_{3n+1}),G(Cu,Cu,Hu) \\
& \frac{1}{2}[G(Ax_{3n},Bx_{3n+1},Cu)+G(Tx_{3n},Sx_{3n+1},Hu)]\} \\
\to \;& \max\{G(z,z,z),G(z,z,z),G(z,z,z),G(z,z,Hu) \\
& \frac{1}{2}[G(z,z,z)+G(z,z,Hu)]\} \\
= \;& \max\{0,0,0,G(z,z,Hu),\frac{1}{2}G(z,z,Hu)\} \\
= \;& G(z,z,Hu) \quad as \;\; n\to\infty,
\end{aligned}
$$

which together with (3.9), (3.10), (3.22), $\psi \in \Phi_3$, and Lemma 2.1 yields

$$
\begin{aligned}
G(z,z,Hu) &= \limsup_{n\to\infty} G(y_{3n+1},y_{3n+2},Hu) = \limsup_{n\to\infty} G(Tx_{3n},Sx_{3n+1},Hu) \\
&\leq \limsup_{n\to\infty} \psi(M(x_{3n},x_{3n+1},u)) \leq \psi(G(z,z,Hu)) < G(z,z,Hu),
\end{aligned}
$$

which is impossible, and hence $Hu = z$. Thus (3.6) means $Az = ATv = TAv = Tz$, $Bz = BSw = Sz$ and $Cz = CHu = HCu = Hz$. Suppose that $G(Tz, Sz, Hz) \neq 0$. Then we have

$$\begin{aligned} & M(z,z,z) \\ = \ & \max\{G(Az,Bz,Cz), G(Az,Az,Tz), G(Bz,Bz,Sz), \\ & G(Cz,Cz,Hz), \frac{1}{2}[G(Az,Bz,Cz)+G(Tz,Sz,Hz)]\} \\ = \ & \max\{G(Tz,Sz,Hz),0,0,0,G(Tz,Sz,Hz)\} \\ = \ & G(Tz,Sz,Hz), \end{aligned}$$

which together with (3.9), $\psi \in \Phi_3$, and Lemma 2.1 yields

$$G(Tz,Sz,Hz) \leq \psi(M(z,z,z)) = \psi(G(Tz,Sz,Hz)) < G(Tz,Sz,Hz),$$

which is impossible, and hence $G(Tz,Sz,Hz) = 0$. So $Tz = Sz = Hz$. Suppose that $Tz \neq z$. Then we have

$$\begin{aligned} & M(z,w,u) \\ = \ & \max\{G(Az,Bw,Cu), G(Az,Az,Tz), G(Bw,Bw,Sw), \\ & G(Cu,Cu,Hu), \frac{1}{2}[G(Az,Bw,Cu)+G(Tz,Sw,Hu)]\} \\ = \ & \max\{G(Tz,Sw,Hu),0,0,0,G(Tz,Sw,Hu)\} \\ = \ & G(Tz,Sw,Hu) \\ = \ & G(Tz,z,z), \end{aligned}$$

which together with (3.9), $\psi \in \Phi_3$, and Lemma 2.1 implies

$$G(Tz,z,z) = G(Tz,Sw,Hu) \leq \psi(M(z,w,u)) = \psi(G(Tz,z,z)) < G(Tz,z,z),$$

which is impossible and hence $Tz = z$, that is, z is a common fixed point of A, B, C, S, T and H. Suppose that A, B, C, S, T and H have another common fixed point $u \in X \setminus \{z\}$. Then we have

$$\begin{aligned} & M(z,z,u) \\ = & \max\{G(Az,Bz,Cu), G(Az,Az,Tz), G(Bz,Bz,Sz), \\ & G(Cu,Cu,Hu), \frac{1}{2}[G(Az,Bz,Cu)+G(Tz,Sz,Hu)]\} \\ = & \max\{G(z,z,u),0,0,0,G(z,z,u)\} \\ = & G(z,z,u), \end{aligned}$$

and

$$G(z,z,u) = G(Tz,Sz,Hu) \le \psi(M(z,z,u)) = \psi(G(z,z,u)) < G(z,z,u),$$

which is a contradiction and hence z is a unique common fixed point of A, B, C, S, T and H in X.

Similarly we conclude that A, B, C, S, T and H have a unique common fixed point in X if one of $B(X)$, $C(X)$, $S(X)$, $T(X)$ and $H(X)$ is complete. Then the proof is complete.

Utilizing Theorems 3.1 and Remark 3.1, we get the following results.

Theorem 3.2 Let A, B, C, S, T and H be self mappings in a G-metric space (X,G) satisfying (3.6)-(3.8) and

$$\psi(G(Tx,Sy,Hz)) \le \psi(M(x,y,z)) - \varphi(M(x,y,z)), \quad \forall x,y,z \in X,$$

where (ψ,φ) is in $\Phi_1 \times \Phi_2$. Then A, B, C, S, T and H have a unique common fixed point in X.

Example 3.1 Let $X = [0,1]$ be endowed with the Euclidean G-metric

$$G(x,y,z) = \begin{cases} 0 & x = y = z; \\ \max\{x,y,z\} & else. \end{cases}$$

Let A, B, C, S, T, H: $X \to X$ be defined by $Ax = 2x$, $Bx = x$, $Cx = x^2$, $Sx = 0$,

$$Tx = \begin{cases} 0 & \forall x \in X \backslash \{\frac{1}{2}\}; \\ \frac{1}{2} & x = \frac{1}{2}. \end{cases}$$

$$Hx = \begin{cases} 0 & \forall x \in X \backslash \{\frac{1}{2}\}; \\ \frac{1}{6} & x = \frac{1}{2}. \end{cases}$$

And define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by:

$$\psi(t) = \frac{2}{3}t.$$

It is easy to verify that (3.6)-(3.8) holds and $\psi \in \Phi_3$. Put $x, y, z \in X$, in order to verify (3.9), we consider four cases as follows:

Case 1. $x \in X \backslash \{\frac{1}{2}\}$, $z \in X \backslash \{\frac{1}{2}\}$. It is clear that

$$G(Tx, Sy, Hz) = 0 \leq \psi(M(x, y, z));$$

Case 2. $x \in X \backslash \{\frac{1}{2}\}$, $z = \frac{1}{2}$. Clearly we have

$$\begin{aligned} & M(x,y,z) \\ = \; & max\{G(Ax,By,Cz), G(Ax,Ax,Tx), G(By,By,Sy), G(Cz,Cz,Hz), \\ & \frac{1}{2}[G(Ax,By,Cz) + G(Tx,Sy,Hz)]\} \\ \geq \; & G(Cz,Cz,Hz) = \frac{1}{4} \end{aligned}$$

It follows that

$$\psi(M(x,y,z)) \geq \frac{2}{3} \times \frac{1}{4} = \frac{1}{6},$$

$$G(Tx, Sy, Hz) = \frac{1}{6} \leq \psi(M(x,y,z)).$$

Case 3. $x = \frac{1}{2}$, $z \in X \setminus \{\frac{1}{2}\}$. It is clear that

$$\begin{aligned} & M(x,y,z) \\ = \; & max\{G(Ax,By,Cz), G(Ax,Ax,Tx), G(By,By,Sy), G(Cz,Cz,Hz), \\ & \frac{1}{2}[G(Ax,By,Cz)+G(Tx,Sy,Hz)]\} \\ \geq \; & G(Ax,Ax,Tx) = 1 \end{aligned}$$

It follows that

$$\psi(M(x,y,z)) = \frac{2}{3} \times 1 \geq \frac{2}{3}$$

$$G(Tx,Sy,Hz) = \frac{1}{2} < \frac{2}{3} \leq \psi(M(x,y,z))$$

Case 4. $x = \frac{1}{2}$, $z = \frac{1}{2}$. Clearly we have

$$\begin{aligned} & M(x,y,z) \\ = \; & max\{G(Ax,By,Cz), G(Ax,Ax,Tx), G(By,By,Sy), G(Cz,Cz,Hz), \\ & \frac{1}{2}[G(Ax,By,Cz)+G(Tx,Sy,Hz)]\} \\ \geq \; & G(Ax,By,Cz) = 1 \end{aligned}$$

It follows that

$$\psi(M(x,y,z)) = \frac{2}{3} \times 1 = \frac{2}{3}$$

$$G(Tx,Sy,Hz) = \frac{1}{2} < \frac{2}{3} \leq \psi(M(x,y,z))$$

Note that A, B, C, S, T and H satisfy all the hypotheses of Theorem 3.1. Hence A, B, C, S, T and H have a unique common fixed point. Here 0 is the fixed point of A, B, C, S, T and H.

REFERENCES

[1] Zead.M, Brailey.S. A new approach to generalized metric spaces[J]. Nonlinear Convex Anal. 7, 2006: 289-297.

[2] B.s.Choudhury, P.maity. Coupled fixed point results in generalized metric spaces[J]. Math.Comput.Modelling 54, 2011: 73-79.

[3] Jungck, G, Rhoades, BE. Fixed point for set valued functions without continuity[J]. Indian J.Pure Appl.Math. 29, 1998:227-238.

[4] Pasicki, L. Fixed point theorems for contracting mappings in partial metric spaces[J]. Fixed Point Theory Appl.2014: 185

[5] Abbas M, Nazir T, Vetro P. Common fixed point results for three maps in G-metric spaces[J]. Filomat. 25, 2011: 1-17.

[6] Aydi H, Shatanawi W, Vetro C. On generalized weakly G-contraction mapping in G-metric spaces[J]. Comput Math Appl. 62, 2011: 4222-4229.

[7] Rao K P R, BhanuLakshmi K, Mustafa Z, Raju V C C. Fixed and Related Fixed Point Theorems for Three Maps in G-Metric Spaces[J]. J Adv Studies Topology, 3, 2012: 12-19.

[8] Zeqing Liu, Xiaoping Zhang, Jeong Sheok Ume and Shin Min Kang. Common fixed point theorems for four mappings satisfying $\psi-$weakly contractive conditions[J]. Fixed Point Theory Appl. 2015: 20(2015).

[9] Rhoades, BE: Some theorems on weakly contractive maps[J]. Nonlinear Anal. 47, 2011:2683-2691.

[10] Dutta, PN, Choudhury, BS: A generalization of contraction principle in metric spaces[J]. Fixed point Theory Appl.2008, Article ID 406368(2008).

[11] Abbas, M, Khan, MA: Common fixed point theorem of two mappings satisfying a generalized weak contractive condition[J]. Int, J. Math. Sci. 2009, Article ID 131068(2009).

[12] Abbas, M, Doric, D: Common fixed point theorem for four mappings satisfying generalized weak contractive condition[J]. Filomat 24, 2010: 1-10.